

CAYLEY GRAPHS VERSUS ALGEBRAIC GRAPHS

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Abstract. Let Γ be a finite group and let $S \subseteq \Gamma$ be a subset. The *Cayley graph*, denoted by $Cay(\Gamma, S)$ has vertex set Γ and two distinct vertices $x, y \in \Gamma$ are joined by a directed edge from x to y if and only if there exists $s \in S$ such that $x = sy$. In this manuscript, we characterize the generating sets S for which $Cay(\Gamma, S)$ is isomorphic to some *algebraic graphs*, namely, unit graphs, co-unit graphs, total graph and co-total graphs.

Key words and Phrases: Cayley Graphs; Finite group, Generating sets.

1. PREAMBLE

The research in algebraic combinatorics is aims at exposing the relationship between algebra and graph theory and at advancing applications of one to the other. In the last decade, many authors have studied algebraic graphs (Viz., zero-divisor graph, total graph, unit graph, co-total graph, maximal graph, co-unit graphs, prime graph, etc) associated to algebraic structures. Perhaps the best known one is the Cayley graph [2] of a *group*.

Let Γ be an abelian group. For \mathbb{Z}_n , the group of integer modulo n , the sets $Z(\mathbb{Z}_n)$ and $U(\mathbb{Z}_n)$ are defined as; $U(\mathbb{Z}_n) = \{x : gcd(x, n) = 1\}$ and $Z(\mathbb{Z}_n) = \{y : gcd(y, n) \neq 1\}$. For generating set S of Γ we define the *Cayley graph* of Γ to be the (simple) graph $Cay(\Gamma, S)$ with vertices Γ , the set of elements of Γ , and two distinct vertices x and y are adjacent if and only if there exists $s \in S$ such that $x = sy$. Thus $Cay(\Gamma, S)$ is connected graph when $S \subseteq U(\Gamma)$, and $Cay(\Gamma, S)$ is disconnected

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when $S \subseteq Z(\Gamma)$. With this motivation in Section 2, we characterize the generating set S for which $Cay(\Gamma, S)$ is isomorphic to one of the graph, namely, unit graphs, co-unit graphs, total graph and co-total graphs. However, for convenience of the reader, we shall also gather some definitions and results for Cayley graphs which will be used in the sequel.

In Section 3, we give several basic results about Cayley graphs associated with some specific generating sets. We have also established the results that gives the relation between Cayley graph and unit graphs, co-unit graphs, total graph and co-total graphs. Further, in view of Theorem 3.3 we have shown that the study on the notion of unit graph, co-unit graphs, co-total graphs and total graph for the group \mathbb{Z}_n is a particular case for the study on the $Cay(\mathbb{Z}_n, S)$ associated with specify generating sets, viz., $U(\mathbb{Z}_n)$ and $Z^0(\mathbb{Z}_n)$. Throughout, many examples are given to illustrate the theory, and we pose several questions. For detail study about co-total graph, and co-unit graphs, we refers to reader [5].

In what follows, all consider graphs are *simple*, i.e., undirected graphs in which any two vertices are joined by at most one edge and without loops. Throughout, \mathbb{Z}_n will denote the group of integers modulo n ; and C_n, K_n , and will denote an n -cycle, and the complete graph on n vertices, respectively. To avoid trivialities, we implicitly assume when necessary that graphs are nonempty. For terminology and notation from group theory or graph theory not defined in this paper, we refer the reader to [4] and [3], respectively.

2. GENERATING SETS

In general, given any group Γ , a generating set S of Γ is merely a nonempty set such that $e \notin S$ and if $a \in S$, then $a^{-1} \in S$, where ‘ e ’ is the identity element of Γ . Consequently, $S \neq \phi$ and if all the elements of Γ are self inverse, then all possible generating sets are $2^n - 2$. In view of this definition, the following result recognize the structure of such generating set. In 2011, Beny and Rakhmonov [1] have derived a formula for number of Cayley graphs on \mathbb{Z}_n .

Theorem 2.1. *Let $\Gamma \cong \mathbb{Z}_n$ and S be a generating set of Γ and $N(S)$ be the collection of all generating sets. Then*

- i) *If n is odd, then $|N(S)| = 2^{\frac{n-1}{2}} - 1$;*
- ii) *If n is even, then $|N(S)| = 2^{\frac{n}{2}} - 1$.*

Example 2.2. Consider the group \mathbb{Z}_4 , then there are precisely three generating sets $\{1, 3\}$, $\{2\}$ and $\{1, 2, 3\}$. If we take $n = 4$ in Theorem 2.1, then we get $|N(S)| = 3$. In the similar vein, consider the group \mathbb{Z}_5 , then there are precisely three generating sets $\{1, 4\}$, $\{2, 3\}$ and $\{1, 2, 3, 4\}$. If we take $n = 5$ in Theorem 2.1, then we get $|N(S)| = 3$.

Theorem 2.3. *Let $\Gamma \cong \mathbb{Z}_2^t \times \mathbb{Z}_{2n}$. Then the number of generating sets consisting of only nonzero self-inverse elements are $2^{(2^{t+1}-1)} - 1$.*

Proof. Let us consider $\Gamma \cong \mathbb{Z}_2^t \times \mathbb{Z}_{2n}$ and our aim is to show that number of generating sets consisting of only self-inverse elements are $2^{(2^{t+1}-1)} - 1$. Note that

each element of \mathbb{Z}_2^t is self inverse and there are precisely two self inverse elements namely '0' and 'n' in \mathbb{Z}_{2n} . By fixing exactly one from '0' and 'n' at $(t+1)^{th}$ position all self inverse elements can be obtained. Firstly, by fixing zero at $(t+1)^{th}$ position in $(t+1)$ tuple, i.e. $(a_1, a_2, \dots, a_t, 0)$, there are $2^t - 1$ choices for a_i and hence $2^t - 1$ nonzero self inverse elements. In the similar way, by fixing 'n' at $(t+1)^{th}$ position in $(t+1)$ tuple, i.e., $(a_1, a_2, \dots, a_t, n)$ there are 2^t choices for a_i and hence for self inverse elements. Therefore the total number of non-zero self inverse elements in Γ are $2^{t+1}-1$. Now it is easy to see that the number of generating sets consisting of only self-inverse elements are $2^{(2^{t+1}-1)} - 1$. \square

Remark 2.4. Let $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_{2n}$. Then there are exactly three nonzero self-inverse elements and hence six generating set consisting of nonzero self-inverse elements.

3. CAYLEY GRAPHS AND ALGEBRAIC GRAPHS

Let $(A, *)$ be an algebraic structure. A graph $G := (V, E)$ is called an *algebraic graph*, if $V \subseteq A$ and the adjacency rule is due to *binary operation* ' $*$ ' of algebraic structure. In this section, we establish the results that gives the relation between Cayley graph and algebraic graphs, namely, unit graph, co-unit graph, total graphs and co-total graphs.

Theorem 3.1. Let $\tilde{G}(\mathbb{Z}_n)$ and $\tilde{T}(\Gamma(\mathbb{Z}_n))$ be co-unit graph, and co-total graph, respectively. Then for a positive integer 'n' the following holds:

- (1) $Cay(\mathbb{Z}_n, S) \cong \tilde{G}(\mathbb{Z}_n)$, when $S = U(\mathbb{Z}_n)$;
- (2) $Cay(\mathbb{Z}_n, S) \cong \tilde{T}(\Gamma(\mathbb{Z}_n))$, when $S = Z^0(\mathbb{Z}_n)$.

Proof. It is clear that the vertex set of both $Cay(\mathbb{Z}_n, S)$ and $\tilde{G}(\mathbb{Z}_n)$ are same. Let $v_i, v_j \in V(Cay(\mathbb{Z}_n, S))$. Then v_i is adjacent to v_j if and only if $v_i - v_j \in S$.

First, when $S = U(\mathbb{Z}_n)$, this implies that $v_i - v_j \in U(\mathbb{Z}_n)$. This shows that $(v_i, v_j) \in E(Cay(\mathbb{Z}_n, S))$ if and only if $(v_i, v_j) \in E(\tilde{G}(\mathbb{Z}_n))$. Therefore, $Cay(\mathbb{Z}_n, S) \cong \tilde{G}(\mathbb{Z}_n)$.

On the other hand, when $S = Z^0(\mathbb{Z}_n)$. This implies that $v_i - v_j \in Z^0(\mathbb{Z}_n)$. This shows that $(v_i, v_j) \in E(Cay(\mathbb{Z}_n, S))$ if and only if $(v_i, v_j) \in E(\tilde{T}(\Gamma(\mathbb{Z}_n)))$. Therefore, $Cay(\mathbb{Z}_n, S) \cong \tilde{T}(\Gamma(\mathbb{Z}_n))$. \square

Lemma 3.2. Let n be an even positive integer. Then for all $a, b \in \mathbb{Z}_n$ the following holds:

- (i) For $S = U(\mathbb{Z}_n)$; $a - b \in S$ if and only if $a + b \in S$;
- (ii) For $S = Z^0(\mathbb{Z}_n)$; $a - b \in S$ if and only if $a + b \in S \cup \{0\}$.

Proof. For the case (i); (\Rightarrow) Let n be an even positive integer and $S = U(\mathbb{Z}_n)$. Then $U(\mathbb{Z}_n)$ consists of all odd positive integer less than n . If $a - b \in S = U(\mathbb{Z}_n)$, then $a - b$ must be an odd number. Note that $a + b = a - b + 2b$, which is an odd number, and hence $a + b \in S$.

(\Leftarrow) If $a + b \in S = U(\mathbb{Z}_n)$, then $a + b$ must be an odd number. Note that in this

case exactly one a or b must be odd and hence $a - b$ is an odd number. Therefore $a - b \in S$.

For the case (ii); (\Rightarrow) if $S = Z^0(\mathbb{Z}_n)$. Then $Z^0(\mathbb{Z}_n)$ consists of all even positive integer less than n . If $a - b \in S = Z^0(\mathbb{Z}_n)$, then $a - b$ must be an even number. Note that $a + b = a - b + 2b$, which is an even number may be zero, and hence $a + b \in S \cup \{0\}$.

(\Leftarrow) If $a + b \in S = Z^0(\mathbb{Z}_n) \cup \{0\}$, then $a + b$ must be an even number. Note that in this case either both a and b are even or both are odd. Then $a - b$ is an even number, and hence $a - b \in S$. \square

Theorem 3.3. *Let $G(\mathbb{Z}_n)$ and $T(\Gamma(G(\mathbb{Z}_n)))$ be unit graph and total graph, respectively. Then for an even positive integer 'n' the following holds:*

- (1) $Cay(\mathbb{Z}_n, S) \cong G(\mathbb{Z}_n)$, when $S = U(\mathbb{Z}_n)$;
- (2) $Cay(\mathbb{Z}_n, S) \cong T(\Gamma(\mathbb{Z}_n))$, when $S = Z^0(\mathbb{Z}_n)$.

Proof. To show the result it suffices to show the isomorphism between the given graphs. Let 'n' be an even positive integer. To tackle the case (1), define a function $f : V(Cay(\mathbb{Z}_n, S)) \rightarrow V(G(\mathbb{Z}_n))$ such that

$$f(v) = \begin{cases} v, & \text{if } v \text{ is odd;} \\ n - v, & \text{otherwise.} \end{cases}$$

Clearly f is a bijective function. Now we want to show that under this f adjacency is preserved. To do this, let v_i and v_j be two vertices of $Cay(\mathbb{Z}_n, S)$. Then v_i is adjacent to v_j , if $v_i - v_j \in S = U(\mathbb{Z}_n)$ which is odd and this is possible only when either both $v_i, v_j \in Z^0(\mathbb{Z}_n)$ or exactly one of them v_i or v_j belong to $U(\mathbb{Z}_n)$.

For the first possibility, one of v_i or v_j must be odd although belong to $Z^0(\mathbb{Z}_n)$, then $f(v_i) = v_i$ and $f(v_j) = n - v_j$, this implies that $f(v_i) + f(v_j) = v_i + n - v_j = n + (v_i - v_j) \in U(\mathbb{Z}_n)$. Next, if $v_i \in U(\mathbb{Z}_n)$ and $v_j \in Z^0(\mathbb{Z}_n)$, then $f(v_i) = v_i$ and $f(v_j) = n - v_j$ and the sum $f(v_i) + f(v_j) = v_i + n - v_j = n + (v_i - v_j) \in U(\mathbb{Z}_n)$. This implies that $f(v_i) + f(v_j) \in U(\mathbb{Z}_n)$. This shows that $(v_i, v_j) \in E(Cay(\mathbb{Z}_n, S)) \Leftrightarrow (f(v_i), f(v_j)) \in E(G(\mathbb{Z}_n))$. Therefore $Cay(\mathbb{Z}_n, S) \cong G(\mathbb{Z}_n)$, when $S = U(\mathbb{Z}_n)$.

For the case (2), again define a function $f : V(Cay(\mathbb{Z}_n, S)) \rightarrow V(T(\Gamma(\mathbb{Z}_n)))$ such that

$$f(v) = \begin{cases} v, & \text{if } v \text{ is even} \\ n - v, & \text{otherwise.} \end{cases}$$

Clearly f is a bijective function. Now we want to show that under this f adjacency is preserved. To do this, let v_i and v_j be two vertices of $Cay(\mathbb{Z}_n, S)$. Then v_i is adjacent to v_j , if $v_i - v_j \in S = Z^0(\mathbb{Z}_n)$. Then there are two possibilities, either $v_i - v_j$ is even or odd.

First, if $v_i - v_j$ is even, then either both v_i, v_j are even or odd. Let us suppose both v_i and v_j are even. Then the sum $f(v_i) + f(v_j) = v_i + v_j$ which is even and belong to $Z(\mathbb{Z}_n)$ (using Lemma 3.2). Now, if both v_i, v_j are odd, then the sum $f(v_i) + f(v_j) = (n - v_i) + (n - v_j) = 2n - (v_i + v_j)$ again belong to $Z(\mathbb{Z}_n)$. Hence in the first possibility, $v_i - v_j \in Z^0(\mathbb{Z}_n)$ if and only if $f(v_i) + f(v_j) \in Z(\mathbb{Z}_n)$. Next, in the second possibility, if $v_i - v_j \in Z^0(\mathbb{Z}_n)$ and it is odd, then note that at least one

of v_i or v_j must be odd. Let us suppose v_i be odd. Then the sum $f(v_i) + f(v_j) = (n - v_i) + v_j = n - (v_i - v_j)$. Consequently, $f(v_i) + f(v_j) \in Z(\mathbb{Z}_n)$. Hence in both the possibilities adjacency is preserved under f . Therefore $\text{Cay}(\mathbb{Z}_n, S) \cong T(\Gamma(\mathbb{Z}_n))$, when $S = Z^0(\mathbb{Z}_n)$. \square

Remark 3.4. It may hence be observed from Theorem 3.3 that the study on the notion of unit graph and total graph for \mathbb{Z}_n is a particular case for the study on the $\text{Cay}(\mathbb{Z}_n, S)$ associated with specify generating sets, viz., $U(\mathbb{Z}_n)$ and $Z^0(\mathbb{Z}_n)$.

Theorem 3.5. *If $S_1 \subseteq S_2$, then $\text{Cay}(\mathbb{Z}_n, S_1)$ is a subgraph of $\text{Cay}(\mathbb{Z}_n, S_2)$.*

Proof. Let $S_1 \subseteq S_2$. Clearly, the vertex set of $\text{Cay}(\mathbb{Z}_n, S_1)$ and $\text{Cay}(\mathbb{Z}_n, S_2)$ is same and is equal to $\{0, 1, 2, \dots, n - 1\}$. Let v_i and v_j be two arbitrary vertices in $\text{Cay}(\mathbb{Z}_n, S_1)$. If v_i and v_j are adjacent in $\text{Cay}(\mathbb{Z}_n, S_1)$, then $v_i - v_j \in S_1$. This implies that $v_i - v_j \in S_2$ as $S_1 \subseteq S_2$. Therefore each edge of $\text{Cay}(\mathbb{Z}_n, S_1)$ is also an edge of $\text{Cay}(\mathbb{Z}_n, S_2)$. Hence, $\text{Cay}(\mathbb{Z}_n, S_1)$ is a subgraph of $\text{Cay}(\mathbb{Z}_n, S_2)$. \square

Theorem 3.6. *Let $\Gamma \cong \mathbb{Z}_2^t \times \mathbb{Z}_{2n}$ and S be the collection of all self inverse elements of Γ .*

- (1) *If $S' \subseteq S$ and $|S'| = 1$, then $\text{Cay}(\Gamma, S') \cong \underbrace{K_2 \cup K_2 \cup \dots \cup K_2}_{(2^t n) \text{-times}}$;*
- (2) *If $S' = S$, then $\text{Cay}(\Gamma, S) \cong \underbrace{K_{|S+1|} \cup K_{|S+1|} \cup \dots \cup K_{|S+1|}}_{n \text{-times}}$.*

Proof. Case (1): Let $S' \subseteq S$ and $|S'| = 1$. Then $\text{Cay}(\Gamma, S')$ is 1-regular graph. The only possible 1-regular graph is either K_2 or copies of K_2 . Since there are $2^{t+1} \cdot n$ vertices in $\text{Cay}(\Gamma, S')$, this indicates that $\text{Cay}(\Gamma, S')$ is isomorphic to copies of K_2 . Also to form K_2 only two vertices are required. Hence, there are $2^t \cdot n$ copies of K_2 . Therefore $\text{Cay}(\Gamma, S') \cong \underbrace{K_2 \cup K_2 \cup \dots \cup K_2}_{(2^t n) \text{-times}}$.

Case (2): From the Theorem 2.3 clearly, $(2^{t+1} - 1)$ non-zero self inverse elements in Γ . This implies that $|S| = 2^{(t+1)} - 1$. Then, $\text{Cay}(\Gamma, S)$ is $2^{(t+1)} - 1$ regular graph. Let $V' = \{(a_1, a_2, \dots, a_t, a_{t+1}) : a_i \in \{0, 1\} (1 \leq i \leq t) \text{ and } a_{t+1} \in \{0, n\}\}$. The set V' is subset of $V(\text{Cay}(\Gamma, S))$ and specifically it is the collection of all self inverse elements, i.e., $V' = S \cup (0, 0, \dots, 0, 0)$. Let v_i and v_j be two arbitrary vertices of V' . Then $v_i - v_j = (c_1, c_2, \dots, c_t, c_{t+1}) \setminus (0, 0, \dots, 0)$: $c_i \in \{0, 1\} (1 \leq i \leq t)$ and $c_{t+1} \in \{0, n\}$. This implies that $v_i - v_j \in S$, also v_i and v_j are arbitrary which indicates that for all $v_i, v_j \in V'$, $v_i - v_j \in S$. Therefore all the vertices of V' are mutually adjacent and each of them have degree $2^{(t+1)} - 1$. Also no vertex of V' is adjacent to other vertices of $\text{Cay}(\Gamma, S)$ as $\text{Cay}(\Gamma, S)$ is $2^{(t+1)} - 1$ regular. Let $V'' = \{(a_1, a_2, \dots, a_t, a_{t+1} + 1) : a_i \in \{0, 1\} (1 \leq i \leq t) \text{ and } a_{t+1} \in \{0, n\}\}$. Let v'_i and v'_j be two arbitrary vertices of V'' . Then $v'_i - v'_j = (d_1, d_2, \dots, d_t, d_{t+1}) \setminus (0, 0, \dots, 0)$: $d_i \in \{0, 1\} (1 \leq i \leq t)$ and $d_{t+1} \in \{0, n\}$. This implies that $v'_i - v'_j \in S$, also v'_i and v'_j are arbitrary which indicates that for all $v'_i, v'_j \in V''$, $v'_i - v'_j \in S$. Therefore all the vertices of V'' are again

mutually adjacent and each of them have degree $2^{(t+1)} - 1$. Also no vertex of V'' is adjacent to other vertices of $\text{Cay}(\Gamma, S)$ as $\text{Cay}(\Gamma, S)$ is $2^{(t+1)} - 1$ regular. Following the same procedure, one get the copies of $K_{2^{t+1}}$. Since there are $n \cdot 2^{(t+1)}$ vertices in $\text{Cay}(\Gamma, S)$, so there will be precisely n copies of $K_{2^{t+1}}$ are obtained. Therefore, $\text{Cay}(\Gamma, S) \cong \underbrace{K_{|S+1|} \cup K_{|S+1|} \cup \dots \cup K_{|S+1|}}_{n\text{-times}}$. \square

Corollary 3.7. *Let $\Gamma \cong \mathbb{Z}_2^t$ and S be the collection of all self inverse elements of Γ . Then the followings holds:*

- (1) *If $S' \subseteq S$ and $|S'| = 1$, then $\text{Cay}(\Gamma, S') \cong \underbrace{K_2 \cup K_2 \cup \dots \cup K_2}_{(2^{t-1})\text{-times}}$;*
- (2) *If $S' = S$, then $\text{Cay}(\Gamma, S') \cong K_{2^t}$.*

Proof. By taking $n = 1$ in Theorem 3.6, the proof follows. \square

Theorem 3.8. *Let $\Gamma \cong \mathbb{Z}_2^t \times \mathbb{Z}_{2n}$ and S be the collection of all self inverse elements of Γ . Then for $n = 1$, $\text{Cay}(\Gamma, S)$ is connected.*

Proof. Invoking Theorem 3.6 and Corollary 3.7, $\text{Cay}(\Gamma, S') \cong K_{2^{t+1}}$, and hence connected. \square

Theorem 3.9. *Let $S = \{(1, 1, \dots, 1, a_t), a_t \in U(\mathbb{Z}_{2n})\}$. Then $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}) \cong \tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$.*

Proof. Let us consider \mathbb{Z}_{2n} and we want to show that $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}) \cong \tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$. For this first we will show $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$ and then $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong \tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$. To do this let us suppose on contrary that

$$\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \not\cong G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}) \tag{1}$$

Although $V(\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)) = V(G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}))$. Let $v_i = (a_1, a_2, \dots, a_t, a_{t+1})$ and $v_j = (b_1, b_2, \dots, b_t, b_{t+1})$ be two vertices of $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$. Then v_i and v_j are adjacent in $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$ whenever $v_i - v_j \in S$, i.e., $(a_1 - b_1, a_2 - b_2, \dots, a_t - b_t, a_{t+1} - b_{t+1}), a_i - b_i = 1$ for all $1 \leq i \leq t$ and $a_{t+1} - b_{t+1} \in U(\mathbb{Z}_{2n})$. Since $a_i, b_i \in \mathbb{Z}_2$, $a_i - b_i = 1$ gives us that one of a_i or b_i must be '0'. This implies that $a_i + b_i = 1$ for all $1 \leq i \leq t$. As $a_{t+1} - b_{t+1} \in U(\mathbb{Z}_{2n})$, this implies that $a_{t+1} + b_{t+1} \in U(\mathbb{Z}_{2n})$. Hence $(v_i, v_j) \in E(\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S))$. This implies that $(v_i, v_j) \in E(G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}))$. This shows that each edge of $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$ is also an edge of $G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$. Therefore only possibility is to be (1) is that two vertices v'_i, v'_j are adjacent in $G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$ but not in $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$, i.e., $v'_i + v'_j \in U(\mathbb{Z}_{2n})$, but $v'_i - v'_j \notin S$. From Lemma 3.2, $v'_i + v'_j \in U(\mathbb{Z}_{2n})$ implies that $v'_i - v'_j \in U(\mathbb{Z}_{2n})$. This gives us $S \subset U(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$, a contradiction. Hence our assumption is wrong. Therefore $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong G(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$.

Now to show that $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong \tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$. Let $v_i = (a_1, a_2, \dots, a_t, a_{t+1})$ and $v_j = (b_1, b_2, \dots, b_t, b_{t+1})$ be two vertices of $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$, although $V(\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)) = V(\tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}))$. Then v_i and v_j are adjacent in $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$.

\mathbb{Z}_{2n}, S) whenever $v_i - v_j \in S$. Since $S = \{(1, 1, \dots, 1, a_k), a_k \in U(\mathbb{Z}_{2n})\}$, which indicates that $(v_i, v_j) \in E(\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S))$ if and only if $(v_i, v_j) \in E(\tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}))$. This shows that each edge of $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S)$ is also an edge of $\tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$. Therefore $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong \tilde{G}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})$. \square

Theorem 3.10. *Let $S = \{(a_1, a_2, \dots, a_{t-1}, a_t)\}$, with either at least one $a_i = 0$ for $1 \leq i \leq t-1$ and $a_t \in \mathbb{Z}_{2n} \setminus \{0\}$ or $a_i = 1$ for all i , $1 \leq i \leq t-1$ and $a_t \in Z(\mathbb{Z}_{2n}) \cup \{0\}$. Then $\text{Cay}(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}, S) \cong T(\Gamma(\mathbb{Z}_2^t \times \mathbb{Z}_{2n})) \cong \tilde{T}(\Gamma(\mathbb{Z}_2^t \times \mathbb{Z}_{2n}))$.*

Proof. The proof can be obtained by the arguments that are analogous to those given in the proof of Theorem 3.9. \square

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