J. Indones. Math. Soc.
Vol. 25, No. 1 (2019), pp. 24–34.

ON IDEALS OF BI-ALGEBRAS

Sun Shin Ahn[1], Jung Mi Ko[2], and A. Borumand Saeid[3*]

[1]**Department of Mathematics Education, Dongguk University, Seoul 04620, Korea, sunshine@dongguk.edu**
[2]**Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea, jmko@gwnu.ac.kr**
[3]**Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman Kerman, Iran, arsham@uk.ac.kr**

Abstract. In this paper, we discuss normal subalgebras in BI-algebras and obtain the quotient BI-algebra which is useful for the study of structures of BI-algebras. Moreover, we obtain several conditions for obtaining BI-algebras on the non-negative real numbers by using an analytic methods.

Key words and Phrases: BI-algebra, (normal) subalgebra, (normal) ideal.

Abstrak. Dalam artikel ini, didiskusikan tentang sub-aljabar normal di aljabar-BI dan dikonstruksi aljabar kuosien BI yang dapat digunakan untuk mempelajari struktur dari aljabar-BI. Lebih jauh, diberikan beberapa kondisi untuk mendapatkan aljabar-BI pada bilangan real tak-negatif dengan menggunakan metode analitik.

Kata kunci: Aljabar-BI, sub-aljabar (normal), ideal (normal).

1. INTRODUCTION.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim ([7]) introduced the notion of d-algebras, which is another useful generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras, and then investigated other relations between d-algebras and oriented digraphs.

It is known that several generalizations of a B-algebra were extensively investigated by many researchers and properties have been considered systematically.

2000 Mathematics Subject Classification: Primary 06F35, 03G25;

Received: 03-01-2018, revised : 10-10-2018, accepted: 10-10-2018.

The notion of B-algebras was introduced by J. Neggers and H. S. Kim ([5]). They defined a *B-algebra* as an algebra $(X, *, 0)$ of type (2,0) (i.e., a non-empty set with a binary operation "$*$" and a constant 0) satisfying the following axioms:

$(B1)$ $x * x = 0$,
$(B2)$ $x * 0 = x$,
(B) $(x * y) * z = x * [z * (0 * y)]$,

for any $x, y, z \in X$.

C. B. Kim and H. S. Kim ([4]) defined a BG-algebra, which is a generalization of B-algebra. An algebra $(X, *, 0)$ of type (2,0) is called a *BG-algebra* if it satisfies $(B1), (B2)$, and

(BG) $x = (x * y) * (0 * y)$,

for any $x, y \in X$.

Y. B. Jun, E. H. Roh and H. S. Kim ([3]) introduced the notion of a BH-algebra which is a generalization of $BCK/BCI/BCH$-algebras. An algebra $(X, *, 0)$ of type (2,0) is called a *BH-algebra* if it satisfies $(B1), (B2)$, and

(BH) $x * y = y * x = 0$ implies $x = y$,

for any $x, y \in X$.

Moreover, A. Walendziak ([8]) introduced the notion of $BF/BF_1/BF_2$-algebras. An algebra $(X, *, 0)$ of type (2,0) is called a *BF-algebra* if it satisfies $(B1), (B2)$ and

(BF) $0 * (x * y) = y * x$,

for any $x, y \in X$. A BF-algebra is called a *BF_1-algebra* (resp., a *BF_2-algebra*) if it satisfies (BG) (resp., (BH)).

A. Borumand Saeid et al. ([1]) introduced a new algebra, called a BI-algebra, which is a generalization of both a (dual) implication algebra and an implicative BCK-algebra, and they discussed the basic properties of BI-algebras, and investigated some ideals and congruence relations. We will show that every implicative BCK-algebra is a BI-algebra, but the converse need not be true in general. See Proposition 4.7 and Example 4.8.

J. Neggers and H. S. Kim ([7]) gave an analytic method for constructing proper examples of a great variety of non-associative algebra of the BCK-type and generalizations of these. They made several useful (counter-)examples using analytic method.

In this paper, we discuss normal subalgebras in BI-algebras and obtain the quotient BI-algebra which is useful for the study of structures of BI-algebras. Moreover, we obtain several conditions for obtaining BI-algebras on the non-negative real numbers by using an analytic method.

2. PRELIMINARIES.

We recall some definitions and results discussed in [1, 9].

An algebra $(X;*,0)$ of type (2, 0) is called a *BI-algebra* ([1]) if

(B1) $x*x=0$,
(B2) $x*(y*x)=x$,

for all $x,y\in X$.

We introduce a relation "$\le$" on a BI-algebra X by $x\le y$ if and only if $x*y=0$. We note that the relation "$\le$" is not a partial order, since it is only reflexive. A non-empty subset S of a BI-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Since $x*x=0$, for all $x\in X$, it is clear that $0\in S$.

Definition 2.1. *([1]) Let $(X;*,0)$ be a BI-algebra and let I be a non-empty subset of X. Then I is called an ideal of X if*

(I1) $0\in I$,
(I2) $x*y\in I$ *and* $y\in I$ *imply* $x\in I$,

for any $x,y\in X$.

Obviously, $\{0\}$ and X are ideals of X. We call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal I is said to be *proper* if $I\neq X$.

Proposition 2.2. *([1]) Let I be an ideal of a BI-algebra X. If $y\in I$ and $x\le y$, then $x\in I$.*

Proposition 2.3. *([1]) Let X be a BI-algebra. Then*

(i) $x*0=x$,
(ii) $0*x=0$,
(iii) $x*y=(x*y)*y$,
(iv) *if* $y*x=x$, *then* $X=\{0\}$,
(v) *if* $x*(y*z)=y*(x*z)$, *then* $X=\{0\}$,
(vi) *if* $x*y=z$, *then* $z*y=z$ *and* $y*z=y$,
(vii) *if* $(x*y)*(z*u)=(x*z)*(y*u)$, *then* $X=\{0\}$,

for all $x,y,z,u\in X$.

A BI-algebra $(X;*,0)$ is said to be *right distributive* ([1]) (or *left distributive*, resp.) if $(x*y)*z=(x*z)*(y*z)$ $(z*(x*y)=(z*x)*(z*y)$, resp.) for all $x,y,z\in X$.

Proposition 2.4. *([1]) Let X be a right distributive BI-algebra. Then*

(i) $y*x\le y$,
(ii) $(y*x)*x\le y$,
(iii) $(x*z)*(y*z)\le x*y$,
(iv) *if* $x\le y$, *then* $x*z\le y*z$,
(v) *if* $(x*y)*z\le x*(y*z)$,
(vi) *if* $x*y=z*y$, *then* $(x*z)*y=0$,

for all $x,y,z\in X$.

Proposition 2.5. *([1]) Let X be a right distributive BI-algebra. Then the induced relation " $\leq$ " is a transitive relation.*

Example 2.6. *([1]) Let $X := \{0, a, b, c\}$ be a BI-algebra with the following table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then it is easy to check that $I_1 := \{0, a, c\}$ is an ideal of X, but $I_2 := \{0, a, b\}$ is not an ideal of X, since $c * a = b \in I_2$ and $a \in I_2$, but $c \notin I_2$.

Theorem 2.7. *([9]) Let X be a BCK-algebra. Then X is implicative if and only if it is commutative and positive implicative.*

Theorem 2.8. *([9]) Let X be a BCK-algebra. Then the following are equivalent:*

(i) *X is commutative,*
(ii) *$x \leq y \Rightarrow x = y * (y * x)$, for all $x, y \in X$.*

3. NORMAL SUBALGEBRAS

In what follows, let X be a BI-algebra unless otherwise specified.

Definition 3.1. *A non-empty subset N of X is said to be normal (or a normal subalgebra) if $(x * a) * (y * b) \in N$, for any $x * y, a * b \in N$.*

Proposition 3.2. *Let N be a normal subalgebra of X. Then N is a subalgebra of X.*

Proof. Let $x, y \in N$. Then $x * 0, y * 0 \in N$. Since N is a normal subalgebra of X, we have $(x * y) * (0 * 0) = x * y \in N$. Hence N is a subalgebra of X. □

The converse of Proposition 3.2 need not be true in general.

Example 3.3. *([1]) (1) Let $X := \{0, a, b, c\}$ be a BI-algebra with the following table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	b
c	c	0	c	0

Then $\{0, a, b\}$ is a subalgebra of X, but not normal, since $c * c = 0, b * c = b \in \{0, a, b\}$, $(c * b) * (c * c) = c * 0 = c \notin \{0, a, b\}$.
(2) Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then X is a BI-algebra. It is easy to check that $I := \{0, 1\}$ is a normal subalgebra of X. If we consider $J := \{0, 1, 2\}$, then J is a subalgebra of X, but is not a normal subalgebra of X, since $3 * 3 = 0, 2 * 3 = 2 \in J$ and $(3 * 2) * (3 * 3) = 3 * 0 = 3 \notin J$.

Lemma 3.4. *Let N be a normal subalgebra of X. If $x * y \in N$, for all $x, y \in X$, then $y * x \in N$.*

Proof. Let $x * y \in N$, for any $x, y \in X$. Since $y * y = 0 \in N$, we have $y * x = (y * x) * 0 = (y * x) * (y * y) \in N$. This completes the proof.

□

Let N be a normal subalgebra of X. Define a relation " $\sim_N$ " on X by $x \sim_N y$ if and only if $x * y \in N$, for any $x, y \in X$.

Proposition 3.5. *Let N be a normal subalgebra of X. Then $\sim_N$ is a congruence relation on X.*

Proof. By (B1), $\sim_N$ is reflexive. It follows from Lemma 3.4 that $\sim_N$ is symmetric. Let $x \sim_N y$ and $y \sim_N z$, for any $x, y, z \in X$. Then $x * y, y * z \in N$. Using Lemma 3.4, we have $z * y \in N$. Since N is normal, we have $x * z = (x * z) * (y * y) \in N$. Hence $\sim_N$ is an equivalence relation.

Let $x \sim_N y$ and $p \sim_N q$ for any $x, y, p, q \in X$. Then $x * y, p * q \in N$. Since N is normal, we have $(x * p) * (y * q) \in N$. Hence $x * p \sim_N y * q$. Thus $\sim_N$ is a congruence relation on X.

□

Denote $X/N := \{[x]_N | x \in X\}$, where $[x]_N := \{y \in X | x \sim_N y\}$. If we define $[x]_N *' [y]_N := [x * y]_N$, then "$*'$" is well-defined, since $\sim_N$ is a congruence relation.

Theorem 3.6. *Let N be a normal subalgebra of X. Then $(X/N; *', [0]_N)$ is a BI-algebra.*

Proof. Note that $[0]_N = \{x \in X | x \sim_N 0\} = \{x \in X | x * 0 \in N\} = \{x \in X | x \in N\} = N$. Checking two axioms are trivial and we omit the proof.

□

The BI-algebra X/N discussed in Theorem 3.6 is called the *quotient BI-algebra* of X by N. Let X, Y be BI-algebras. A map $f : X \to Y$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$, for any $x, y \in X$.

Proposition 3.7. *Let N be a normal subalgebra of X. Then the mapping $\gamma : X \to X/N$, given by $\gamma(x) = [x]_N$, is a surjective homomorphism and $Ker\gamma = N$.*

Proof. Since $\sim_N$ is a congruence relation, the operation " $*'$ " on X/N defined by $[x]_N *' [y]_N := [x * y]_N$ is well defined. For all $x, y \in X$, we have $\gamma(x * y) = [x * y]_N = [x]_N *' [y]_N = \gamma(x) *' \gamma(y)$. Hence γ is a BI-homomorphism. Since $\gamma(X) = \{\gamma(x)|x \in X\} = \{[x]_N|x \in X\} = X/N$, γ is surjective. Furthermore

$$\begin{aligned} Ker\gamma &= \{x \in X|\gamma(x) = N\} \\ &= \{x \in X|[x]_N = N\} \\ &= \{x \in X|[x]_N = [0]_N\} \\ &= \{x \in X|x \in N\} = N, \end{aligned}$$

proving the proposition. □

The mapping γ discussed in Proposition 3.7 is called the *canonical homomorphism* of X onto X/N.

Proposition 3.8. *Let $f : X \to Y$ be a homomorphism of BI-algebras. If f is injective, then $Kerf = \{0_X\}$.*

Proposition 3.9. *Let $f : X \to Y$ be a homomorphism of BI-algebras. Then $Kerf$ is a subalgebra of X.*

Proof. Let $x, y \in Kerf$. Then $f(x) = 0_Y = f(y)$ and so $f(x * y) = f(x) * f(y) = 0_Y * 0_Y = 0_Y$. Hence $x * y \in Kerf$. □ □

Note that $Ker\phi$ need not be a normal subalgebra of a BI-algebra (see below example).

Example 3.10. *Consider a BI-algebra $X = \{0, a, b, c\}$ as in Example 3.3(1). We define $\phi(x) = 0$, for all $x \in X$. Then $Ker\phi = \{0, a, b, c\}$ is a normal subalgebra of X. If we define $\phi(x) = x$, for all $x \in X$, then $Ker\phi = \{0\}$ is a subalgebra of X, but is not a normal subalgebra of X, since $c * c = 0, b * a = 0$ and $(c * b) * (c * a) = c * 0 = c \notin \{0\}$.*

Definition 3.11. *A BI-algebra X is called a BI_1-algebra if*

(B3) $x * y = 0 = y * x \Rightarrow x = y$, *for all $x, y \in X$.*

Example 3.12. *Consider a BI-algebra $X = \{0, a, b, c\}$ as in Example 2.6. Then $(X, *, 0)$ is a BI_1-algebra.*

Proposition 3.13. *Let X be a BI_1-algebra and Y be a BI-algebra. Let $\phi : X \to Y$ be a homomorphism. Then ϕ is injective if and only if $Ker\phi = \{0_X\}$.*

Proof. Suppose $Ker\phi = \{0_X\}$. If $\phi(x) = \phi(y)$, for any $x, y \in X$, then $\phi(x * y) = \phi(x) * \phi(y) = 0_Y$ and so $x * y \in Ker\phi = \{0_X\}$. Hence $x * y = 0_X$. Similarly, $y * x = 0_X$. Since X is a BI_1-algebra, we obtain $x = y$. Thus ϕ is injective.

The converse is trivial. This completes the proof.

□

Proposition 3.14. *Let A and I be normal subalgebras of X with $I \subseteq A$. Then A/I is a normal subalgebra of a BI-algebra X/I.*

Proof. Let $[x_1]_I *' [x_2]_I, [y_1]_I *' [y_2]_I \in A/I$, for any $[x_1]_I, [x_2]_I, [y_1]_I, [y_2]_I \in A/I$. Then $[x_1 * x_2]_I, [y_1 * y_2]_I \in A/I$ and so $x_1 * x_2, y_1 * y_2 \in A$. Hence $(x_1 * y_1) * (x_2 * y_2) \in A$. It follows that $[(x_1 * y_1) * (x_2 * y_2)]_I$ and $[(x_1 * x_2)_I * (y_1 * y_2)_I] \in A/I$, i.e., $([x_1] *' [y_1]_I) *' ([x_2] *' [y_2]_I) \in A/I$ and $([x_1] *' [x_2]_I) *' ([y_1] *' [y_2]_I) \in A/I$. Thus A/I is a normal subalgebra of a BI-algebra X/I.

□

Definition 3.15. *Let I be an ideal of X. Then I is called a normal ideal of X if it is normal.*

Example 3.16. *Consider a BI-algebra $X = \{0, 1, 2, 3\}$ as in Example 3.3(2). It is easy to show that $I = \{0, 1\}$ is a normal ideal of X, and $J = \{0, 1, 2\}$ is an ideal, but is not a normal ideal of X.*

Proposition 3.17. *Let I be a normal ideal of X. Then I is a subalgebra of X.*

Proof. Let $x, y \in I$. Then $x * x = 0 \in I$ and $y * 0 = y$. Since I is a normal ideal, then $(x * y) * (x * 0) = (x * y) * x \in I$. Since $x \in I$ and I is an ideal, we have $x * y \in I$. This completes the proof. □

Theorem 3.18. *S is a normal subalgebra of X if and only if S is a normal ideal of X.*

Proof. Let S be a normal subalgebra of X. Clearly, $0 \in S$. Suppose that $x * y \in S$ and $y \in S$. By Proposition 2.3(ii), $0 = 0 * y$. Since S is normal, we have $x = (x * 0) * 0 = (x * 0) * (y * y) \in S$. Hence S is an ideal of X.

The converse follows from Proposition 3.17. □

Proposition 3.19. *Let $f : X \to Y$ be a homomorphism of BI-algebras. Then $Kerf$ is an ideal of X.*

Proof. Obviously, $0_X \in Kerf$, i.e., (I1) holds. Let $x * y \in Kerf$ and $y \in Kerf$. Then $0_Y = f(x * y) = f(x) * f(y) = f(x) * 0_Y = f(x)$ and so $x \in Kerf$. Therefore (I2) is satisfied. Thus $Kerf$ is an ideal of X. □

Definition 3.20. *A homomorphism $f : X \to Y$, where X, Y are BI-algebras, is said to be normal if $Kerf$ is a normal ideal of X.*

Example 3.21. *Let $X := \{0, 1, 2, 3, 4\}$ and $Y := \{0, 1, 2, 3\}$ be sets with the following Cayley tables:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

$*'$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

It is easy to show that $(X;*,0)$ *and* $(Y;*',0)$ *are* BI*-algebras. Define functions* $f,g:X\to Y$ *by*

$$f:0\to 0,1\to 0,2\to 2,3\to 2,4\to 1.$$

$$g:0\to 0,1\to 0,2\to 0,3\to 0,4\to 3.$$

It is easy to check that g *is a normal homomorphism. Also* f *is a homomorphism, but not a normal homomorphism. In fact, let* $Kerf := N$*. Then* $N=\{0,1\}$*.* $2*3=0, 1*2=1\in N$ *and* $(2*1)*(3*2)=2*1=2\notin N$*. Hence* $Kerf$ *is not a normal ideal.*

Theorem 3.22. *Let* X,Y *be* BI_1*-algebras. If* $f:X\to Y$ *is a normal homomorphism from* X *onto* Y*, then* $X/Kerf$ *is isomorphic to* Y*.*

Proof. By the definition of a normal homomorphism, $N:=Kerf$ is a normal ideal of X and so N is a normal subalgebra of X. Define a mapping $\phi: X/N\to Y$ by $\phi([x]_N)=f(x)$, for all $x\in X$. Let $[x]_N=[y]_N$. Then $x\sim_N y$, i.e., $x*y\in N$ and $y*x\in N$. Hence $f(x)*f(y)=0_Y=f(y)*f(x)$. Since Y is a BI_1-algebra, we have $f(x)=f(y)$. Therefore $\phi([x]_N)=\phi([y]_N)$. This means that ϕ is well defined. It is easy to check that ϕ is a homomorphism from X/N onto Y. Observe that $Ker\phi=[0]_N$. In fact, $[x]_N\in Ker\phi \Leftrightarrow \phi([x]_N)=0_Y \Leftrightarrow f(x)=0_Y \Leftrightarrow x\in N \Leftrightarrow [x]_N=[0]_N$. It follows from Proposition 3.13 that ϕ is one-to-one. Thus ϕ is an isomorphism from $X/Kerf$ onto Y. □

4. ANALYTIC CONSTRUCTION FOR BI-ALGEBRAS

We apply the analytic method deviced by J. Neggers and H. S. Kim ([6]) for obtaining an example of a BI-algebra. Note that the BI-algebra $(X,*,0)$ in Example 4.6 is not an implicative BCK-algebra. This shows that the notion of BI-algebra is a generalization of an implicative BCK-algebra. Let $X:=[0,\infty)$ be the set of all non-negative real numbers unless otherwise specified. Define a binary operation "$*$" on X as follows:

(a) $x*y=\max\{0,f(x,y)(x-y)\}=\max\{0,\lambda(x,y)x\}$

where $f(x,y)$ and $\lambda(x,y)$ are non-negative real valued functions with

(b) $\lambda(0,y)=0$.

Proposition 4.1. *If* $x,y\in X$ *with* $x>0$*, then*

$$x*y=0 \;\Leftrightarrow\; x\le y \;\Leftrightarrow\; \lambda(x,y)=0.$$

Proof. It follows immediately from (a). □

Proposition 4.2. *The function* $\lambda(x,y)$ *can be described as follows:*

$$\lambda(x,y)=\begin{cases}0 & \text{if } x\le y\\ \dfrac{x-y}{x}f(x,y)>0 & \text{otherwise}\end{cases}$$

Proof. If $x > y$, then, by Proposition 4.1, $\lambda(x, y) > 0$. Since $x > y$, we obtain $x > 0$. By applying (a), we have $x * y = f(x, y)(x - y) = \lambda(x, y)x > 0$, and so we obtain $\lambda(x, y) = \frac{x - y}{x} f(x, y)$. If $x \leq y$ and $x > 0$, then, by Proposition 4.1, we have $\lambda(x, y) = 0$. If $x \leq y$ and $x = 0$, then $\lambda(x, y) = 0$ by the assumption (a). □

Proposition 4.3. *If the function $\lambda(x, y)$ satisfies the condition*

(c) $\lambda(x, x) = 0$,

then the axiom (B1) holds.

Proposition 4.4. *If the function $\lambda(x, y)$ satisfies the condition*

(d) $\lambda(x, 0) = 1$,

then $x * 0 = x$, for all $x \in X$.

Proof. $x * 0 = \max\{0, \lambda(x, 0)x\} = \lambda(x, 0)x = x$. □

Theorem 4.5. *If the function $\lambda(x, y)$ satisfies the conditions (b)∼(d) and*

(e) $\lambda(x, y) < \frac{y}{x}$, *when* $y \leq x$

and

(f) $\lambda(x, \lambda(y, x)y) = 1$, *for all* $x, y \in X$,

then the axiom (B2) holds.

Proof. Consider $x * (y * x) = x$. If $y < x$, then $y * x = 0$. By Proposition 4.1, we obtain $x * (y * x) = x * 0 = x$.

If $x < y$, then $y * x = \lambda(y, x)y$. Let $q := y * x$. If $x < q$, then $\lambda(x, q) = 0$ and hence $x * (y * x) = x * q = \lambda(x, q)x = 0 \neq x$, i.e., (B2) does not hold. If $x > q$, then

$$\begin{aligned} x > q &\Leftrightarrow x > y * x \\ &\Leftrightarrow x > \lambda(y, x)y \\ &\Leftrightarrow \frac{x}{y} > \lambda(y, x). \end{aligned}$$

By using the condition (f), we obtain

$$\begin{aligned} x * (y * x) &= x * q \\ &= \lambda(x, q)x \\ &= \lambda(x, y * x)x \\ &= \lambda(x, \lambda(y, x)y)x \\ &= x. \end{aligned}$$

This proves the theorem. □

Example 4.6. *If we define a binary operation "$*$" on $X = [0,\infty)$ by $x * y := \max\{0, \lambda(x,y)x\}$ where*

$$\lambda(x,y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0, \end{cases}$$

then

$$x * y = \begin{cases} x & \text{if } y = 0 \\ 0 & \text{if } y \neq 0. \end{cases}$$

If $x \neq 0$, then $y * x = 0$ and hence $x * (y * x) = x * 0 = x$. If $x = 0$, then $y * x = y * 0 = y$ and hence $x * (y * x) = x * y = 0 * y = 0 = x$. Hence $(X, *, 0)$ is a BI-algebra. Note that $\lambda(x,y)$ satisfies the conditions (a)∼(f).

Proposition 4.7. *Every implicative BCK-algebra is a BI-algebra.*

The converse of Proposition 4.7 may not be true in general as the following example.

Example 4.8. *Consider the BI-algebra $(X, *, 0)$ discussed in Example 4.6. Assume that $(X; *, 0)$ is an implicative BCK-algebra. By Theorem 2.7, X should be a commutative BCK-algebra. By Theorem 2.8, X satisfies the following property: $x \leq y \Rightarrow x = y * (y * x)$, for all $x, y \in X$. Let $x := 3, y := 5$. Then $5 * (5 * 3) = 5 * 0 = 5 \neq 3$, which is a contradiction. Hence X is a BI-algebra which is not an implicative BCK-algebra.*

A BI-algebra X is said to be *medial* if $(a * b) * (c * d) = (a * c) * (b * d)$, for any $a, b, c, d \in X$.

Theorem 4.9. *There is no non-trivial medial normal BI-algebras.*

Proof. Assume that $(X; *, 0)$ is a medial BI-algebra with $|X| \geq 2$. Then we have

$$\begin{aligned} x &= x * (y * x) \\ &= (x * 0) * (y * x) \\ &= (x * y) * (0 * x) \\ &= (x * y) * 0 \\ &= x * y, \end{aligned}$$

for any $x, y \in X$. It follows that $x = x * x = 0$, i.e., $X = \{0\}$, which is a contradiction. This completes the proof.

□

5. Acknowledgment

We would like to warmly thank the referees for their helpful comments and suggestions for the improvement of this paper.

REFERENCES

[1] A. Borumand Saeid, H. S. Kim and A. Razaei, *On BI-algebras*, An. Şt. Univ. Ovidius Constanţa **25** (2017), 177-194.
[2] K. Iséki, *On BCI-algebras*, Math. Sem. Notes, Kobe Univ. **8**(1980), 125-130.
[3] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Math. **1**(1998), 347-354.
[4] C. B. Kim and H. S. Kim, *On BG-algebras*, Demonstratio Math. **41**(2008), 497-505.
[5] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54**(2002), 21-29.
[6] J. Neggers and H. S. Kim, *On analytic T-algebras*, Sci. Math. Japo. Online **4** (2001), 157-163.
[7] J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca **49**(1999), 19-26.
[8] A. Walendziak, *On BF-algebras*, Math. Slovaca **57**(2007), 119-128.
[9] H. Yisheng, *BCI-algebra*, Science Press, Beijing, 2006.