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TREES WITH FOUR AND FIVE DISTINCT SIGNLESS LAPLACIAN EIGENVALUES

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Abstract. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The signless Laplacian matrix of G is the matrix $Q = D + A$, such that D is a diagonal matrix and A is the adjacency matrix of G. The eigenvalues of Q is called the signless Laplacian eigenvalues of G and denoted by q_1, q_2, $\ldots$, q_n in a graph with n vertices. In this paper all trees with four and five distinct signless Laplacian eigenvalues are characterized.

Key words and Phrases: Tree, eigenvalue, signless Laplacian matrix, semi-edge walk.

Abstrak. Diberikan graf sederhana G dengan himpunan titik $V(G) = \{v_1, v_2, \cdots, v_n\}$ dan himpunan sisi $E(G)$. Matriks Laplace tak bertanda G adalah matriks $Q = D + A$ dengan D merupakan matriks diagonal yang terindeks himpunan verteks graf G dengan D_{ii} adalah derajat verteks v_i dan A adalah matriks ketetanggaan G, dengan $A_{ij} = 1$ jika ada sisi dari i to j di G dan $A_{ij} = 0$ untuk kasus yang lain. Jika G adalah graf dengan n verteks, nilai eigen Q dikatakan Laplace tak bertanda G dan dinotasikan sebagai q_1, q_2, $\cdots$, q_n. Dalam paper ini dibuktikan karakterisasi semua pohon dengan nilai eigen Laplace tak bertanda sebanyak empat dan lima.

Kata kunci: Pohon, matriks Laplace tak bertanda, walk semi-edge.

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G is the $\{0, 1\}$-matrix A indexed by the vertex set $V(G)$, where $A_{ij} = 1$ when there is an edge from i to j in G and $A_{ij} = 0$ otherwise. The adjacency spectrum of G, denoted by $Spec(G)$, is the

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multiset of eigenvalues of $A(G)$. The degree matrix of G is defined by $D(G) = diag(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, where $d_G(v)$ or simply $d(v)$ is the degree of a vertex v in G. The signless Laplacian matrix of G is the matrix $Q(G) = D(G) + A(G)$. It is known that $Q(G)$ is nonnegative, symmetric and positive semidefinite, so its eigenvalues are real and can be arranged as $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$. We call the eigenvalues of $A(G)$ and $Q(G)$ as the $A-$eigenvalues and $Q-$eigenvalues of G, respectively. For additional results on graphs with few distinct A-eigenvalues, we refer the reader to [3, 4, 6, 19, 20]. Recently many studies have been done on the signless Laplacian eigenvalues, the papers [7, 8, 9] give a survey on this work. Also some bounds for the signless Laplacian eigenvalue can be found in [10, 11]. The authors [15, 16] computed the signless Laplacian spectral moments of some graphs. Let us recall some definitions and notations to be used throughout the paper. The distance between two vertices in a graph is the number of edges in a shortest path connecting them. Denote by $d(u, v)$ the distance between two vertices u and v. The diameter d of a graph is the greatest distance between any pair of its vertices.

The line graph of a simple graph G, $L(G)$, is a graph that represents the adjacencies between edges of G. In other words the line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are incident") in G.

Let G_1 and G_2 be two graphs. The corona product G_1 and G_2, denote by $G_1 \circ G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 by joining any vertex of the j-th copy of G_2 to the j-th vertex of G_1, where $1 \leq j \leq |V(G_1)|$, see [2, 18] for more details.

Throughout this paper we denote a star, a path and a complete graph of order n by $K_{1,n-1}$, P_n, K_n, respectively. Let $S_{a,b}$ be the double star graph obtained from the stars $K_{1,a}$ and $K_{1,b}$ by joining the vertex of degree a in $K_{1,a}$ and the vertex of degree b in $K_{1,b}$, see Figure 1. In this paper all trees with four and five distinct signless Laplacian eigenvalues are characterized.

2. PRELIMINARIES

In this section, we present some useful facts on graphs by given number of distinct Q-eigenvalues. The authors, [8] defined a semi-edge walk in a graph as in the following:

A semi-edge walk W of length k in an (undirected) graph G is an alternating sequence v_1, e_1, v_2, e_2, $\ldots$, v_k, e_k, v_{k+1} of vertices v_1, v_2, $\ldots$, v_{k+1} and edges e_1, e_2, $\ldots$, e_k such that for any $i = 1, 2, \ldots, k$ the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i. If $v_1 = v_{k+1}$, then we say that W is a closed semi-edge walk. If end-vertices of the edge e_i are not distinct, then W is called a walk in G. Additional results about walk and semi-edge walk can be found in [13, 14, 17].

Theorem 2.1. *([5]) Let Q be the signless Laplacian matrix of a graph G. Then the (i,j)-entry of the matrix Q^k is equal to the number of semi-edge walks of length k starting at vertex v_i and terminating at vertex v_j.*

Theorem 2.2. *([5]) Let G be a connected graph with diameter d. Then G has at least $d+1$ distinct signless Laplacian eigenvalues.*

Theorem 2.3. *([5]) The multiplicity of 0 as a signless Laplacian eigenvalue of an undirected graph G equals the number of bipartite connected components of G.*

Theorem 2.4. *([5] Proposition (1.3.10)) A graph G is bipartite if and only if the Laplacian spectrum and the signless Laplacian spectrum of G are equal.*

Theorem 2.5. *([2]) Let G_1 be any graph, G_2 be an $r-$regular graph, and $G = G_1 \circ G_2$. Let $Spec(G_1) = (\mu_1, \mu_2, \ldots, \mu_n)$ and $Spec(G_2) = (\eta_1, \eta_2, \ldots, \eta_m = r)$. Then*

(a) $\dfrac{\mu_i + r \pm \sqrt{(r-\mu_i)^2 + 4m}}{2} \in Spec(G)$ *with multiplicity 1 for* $i = 1, \ldots, n$.

(b) $\eta_j \in Spec(G)$ *with multiplicity n for* $j = 1, \ldots, m-1$.

In the following we have an explicit formula for the eigenvalues of $L(G)$ in terms of the signless Laplacian eigenvalues of G.

Theorem 2.6. *([5]) Suppose G has m edges, and let q_1, q_2, …, q_r be the positive signless Laplacian eigenvalues of G. Then the eigenvalues of $L(G)$ are $\mu_i = q_i - 2$ for $i = 1, \ldots, r$ and $\mu_i = -2$ if $r < i \leq m$.*

Corollary 2.7. *Let P_n be a path with n vertices. Since the path P_n has line graph P_{n-1} and is bipartite, the signless Laplacian eigenvalues of P_n are $2 + 2\cos\frac{\pi i}{n}$, $i = 1, \cdots, n$.*

Lemma 2.8. *([21] Lemma 4.3.) Let $L(S_{a,b})$ be the line graph of double star graph $S_{a,b}$, where the degree of the central vertices is equal to a and b. Then for $b \geq a > 1$, $L(S_{a,b})$ has exactly four distinct eigenvalues.*

In [6], the authors discussed on graphs with three distinct Q-eigenvalues and they shown that the largest Q-eigenvalue of a connected graph G is non integer if and only if $G = K_n - e$, for $n \geq 4$. Also the authors in [12], characterized all graphs with four Laplacian eigenvalues and they presented some families of graphs with four distinct Laplacian eigenvalues.

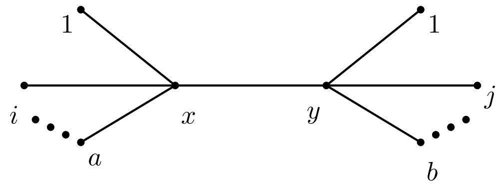

FIGURE 1. A double star graph.

3. MAIN RESULTS

In this section, all trees with four and five distinct signless Laplacian eigenvalues are characterized. To do this we need the following lemma.

Lemma 3.1. *The double star graph $S_{a,b}$ with $b \geq a > 1$ has exactly five distinct signless Laplacian eigenvalues.*

PROOF. By Lemma 2.8., the line graph of double star graph $S_{a,b}$ has exactly four distinct eigenvalues. Thus Theorem 2.6. states that the graph $S_{a,b}$ has four positive signless Laplacian eigenvalues. On the other hand by Theorem 2.3, the graph $S_{a,b}$ has 0 as signless Laplacian eigenvalues. So the double star graph $S_{a,b}$ has exactly five signless Laplacian eigenvalues.

In the paper [12], the authors characterized all bipartite graphs with four distinct Laplacian eigenvalues. By Theorem 2.4., all graphs introduced in that paper have four signless Laplacian eigenvalues. In the following we prove that among all trees the path P_4 is only tree with four signless Laplacian eigenvalues where this graph is the graph $\mathcal{G}(r, s)$ of the paper [12], with $r = s = 1$.

Theorem 3.2. *Suppose that G is a tree.Then G has exactly four distinct signless Laplacian eigenvalues if and only if $G \cong P_4$.*

PROOF. If $G \cong P_4$, then Corollary 2.7 states that G has exactly four distinct signless Laplacian eigenvalues. Conversely let G be a tree with four distinct signless Laplacian eigenvalues. Then by Theorem 2.2., $4 \geq d+1$ and so $d \leq 3$. If $d = 2$, then $G \cong S_n$. By [1], the star graph S_n has three signless Laplacian eigenvalues. Now suppose that $d = 3$. Let i and j be the vertices where $d(i, j) = 3$ and $i - x - y - j$ be a path between i and j. If G just has the same number of vertices, then $G \cong P_4$. Now assume that $|V(G)| = n > 4$, then the other vertices of G must be connected to x or y. Thus the resulting graph is double star graph $S_{a,b}$. By Lemma 3.1. the double star graph $S_{a,b}$ has five distinct signless Laplacian eigenvalues. Therefore only tree with four distinct signless Laplacian eigenvalues is the path P_4.

We now obtain all trees with exactly five signless Laplacian eigenvalues. For this purpose we consider the following Lemma and then we conclude that only trees with five signless Laplacian eigenvalues are the path P_5 and the double star graph

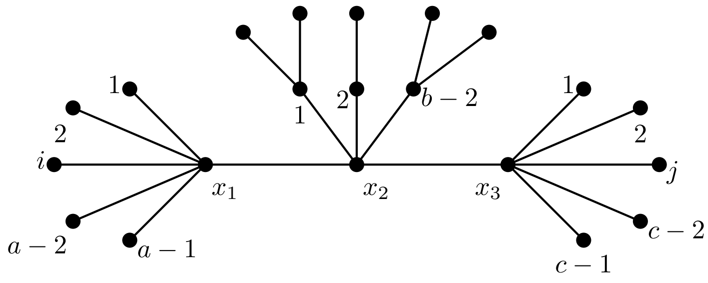

FIGURE 2. The graph H.

$S_{a,b}$. Let H be the graph where is shown in Figure 2. In following when we say that there is a semi-edge on a vertex, it means that we have a semi-edge walk on the edges connected to that vertex.

Lemma 3.3. *For the graph H we have:*

- $[Q^5(H)]_{x_1,x_3} = 2a^2 - a + 2b^2 - 2b + 2c^2 - c + \sum_{i=1}^{b-2} r_i + 2ab + 2ac + 2bc + a^3 + a^2b + a^2c + b^2a + c^2a + b^3 + b^2c + c^2b + c^3 + abc - 2,$
- $[Q^4(H)]_{x_1,x_3} = a^2 + b^2 + c^2 + a + b + c + ab + ac + bc - 2,$
- $[Q^3(H)]_{x_1,x_3} = a + b + c,$
- $[Q^2(H)]_{x_1,x_3} = 1,$
- $[Q(H)]_{x_1,x_3} = 0.$

PROOF. Consider the graph H such that $d(x_1) = a$, $d(x_2) = b$ and $d(x_3) = c$. Denote by w and $s.w$ an edge and semi-edge, respectively. To calculate the (x_1, x_3)-entry of $Q^5(H)$, we should compute the number of semi-edge walks of length five from the vertex x_1 to the vertex x_3. To do this, we have two cases: either there is one semi-edge over the some vertices or there are three semi-edge over the some vertices. In the first case, if the semi-edge is on the vertex x_1, then for any adjacent vertex to x_1 such as the vertex 1, we have two types of semi-edges walks as $x_1 \overset{s.w}{-} x_1 \overset{w}{-} 1 \overset{w}{-} x_1 \overset{w}{-} x_2 \overset{w}{-} x_3$ and $x_1 \overset{w}{-} 1 \overset{w}{-} x_1 \overset{s.w}{-} x_1 \overset{w}{-} x_2 \overset{w}{-} x_3$. The number of each such semi-edge walks is equal to a^2. Also for any vertex that is adjacent to the vertices x_2 and x_3 such as the vertex 1, we have two types of semi-edges walks as $x_1 \overset{s.w}{-} x_1 \overset{w}{-} x_2 \overset{w}{-} 1 \overset{w}{-} x_2 \overset{w}{-} x_3$ and $x_1 \overset{s.w}{-} x_1 \overset{w}{-} x_2 \overset{w}{-} x_3 \overset{w}{-} 1 \overset{w}{-} x_3$. By a simple check one can see that the number of each such semi-edge walks are equal to $a(b-1)$ and $a(c-1)$, respectively. In total if one semi-edge is on the vertex x_1, then there is $2a^2 + a(b-1) + a(c-1)$ semi-edge walk of length five from x_1 to x_3. If the semi-edge is over the adjacent vertices to x_1, then for any adjacent vertex to x_1 such as the vertex 1, we have a semi-edge walk of length five as $x_1 \overset{w}{-} 1 \overset{s.w}{-} 1 \overset{w}{-} x_1 \overset{w}{-} x_2 \overset{w}{-} x_3$. The number of such semi-edge walks is equal to $a - 1$.

Now assume that the semi-edge is on the vertex x_2, then for any adjacent vertex to x_2 such as the vertex 1, we have two types of semi-edge walks as $x_1 \overset{w}{-}$

$x_2 \overset{s.w}{-} x_2 \overset{w}{-} 1 \overset{w}{-} x_2 \overset{w}{-} x_3$ and $x_1 \overset{w}{-} x_2 \overset{w}{-} 1 \overset{w}{-} x_2 \overset{s.w}{-} x_2 \overset{w}{-} x_3$. It is easy to see that the number of each such semi-edge walks is equal to b^2. By a simple calculation one can see that for any vertex that is adjacent to the vertices x_1 and x_3, except x_2, there are $b(a-1)$ and $b(c-1)$ semi-edge walks, respectively. Hence the number of semi-edge walks where have one semi-edge on the vertex x_2 is equal to $2b^2 + b(a-1) + b(c-1)$.

Next let the semi-edge be on the adjacent vertices to the vertex x_2, then for each of $b-2$ vertices that are adjacent to x_2, such as 1, there is a semi-edge walk as $x_1 \overset{w}{-} x_2 \overset{w}{-} 1 \overset{s.w}{-} 1 \overset{w}{-} x_2 \overset{w}{-} x_3$. Here, the number of such semi-edge walks is equal to $\sum_{i=1}^{b-2} r_i$ in which r_i is the degree of the $i-$th vertex that is adjacent to x_2.

Notice that if the semi-edge is on the vertex x_3, then by a similar calculation there is $2c^2$ semi-edge walk of length five. Also for any vertex that is adjacent to the vertex x_1 except x_2 and for any vertex that is adjacent to the vertex x_2 except x_3, there are $c(a-1)$ and $c(b-1)$ semi-edge walks, respectively. So in the case that one semi-edge is on the vertex x_3 there is $2c^2 + c(a-1) + c(b-1)$ semi-edge walk of length five. If the semi-edge is on the adjacent vertices to x_3, then it is easy to see that the number of such semi-edge walks is equal to $c-1$. Therefore in the first case the number of semi-edge walks of length five from x_1 to x_3 is equlal to:

$$a-1+2a^2+a(b-1)+a(c-1)+\sum_{i=1}^{b-2} r_i+2b^2+b(a-1)+b(c-1)+2c^2+c(a-1)+c(b-1)+c-1.$$

In the case that there are three semi-edges on some vertices, Table 1 shows the number of semi-edge walks of length five from x_1 to x_3.

Finally we have:

$$\begin{aligned}[Q^5(H)]_{x_1,x_3} &= 2a^2 - a + 2b^2 - 2b + 2c^2 - c + \sum_{i=1}^{b-2} r_i + 2ab + 2ac + 2bc + a^3 \\ &+ a^2b + a^2c + b^2a + c^2a + b^3 + b^2c + c^2b + c^3 + abc - 2.\end{aligned}$$

In the following we calculate the $[Q^4(H)]_{x_1,x_3}$. To do this, we should compute the number of semi-edge walks of length four from the vertex x_1 to the vertex x_3. We have two cases: either there is no semi-edge on every vertex or there are two semi-edges on some vertices. Consider the case that there is no semi-edge on the vertices. Then it is easy to check that there are a, $b-1$ and $c-1$ semi-edge walks of length four, for the adjacent vertices to the vertices x_1, x_2 and x_3, respectively.

In the second case, we have two semi-edges on some vertices. Assume that every two semi-edges are on the vertex x_1, then a simlpe calculation states that there is $a + a(a-1)$ semi-edge walk of length four from x_1 to x_3. While if two semi-edges are on each of the vertices x_2 and x_3, then there are $b + b(b-1)$ and $c + c(c-1)$ semi-edge walks of length four, respectively. Now if one semi-edge is on the vertex x_1 and another is on the vertices x_2 or x_3, then by a simple calculation one can see that the number of such semi-edge walks are ab and ac, respectively.

TABLE 1. Some of semi-edge walks of length five and their number.

Three semi-edges on vertices	**The number of semi-edge walks**
two semi-edge on x_1 and one semi-edge on x_2	$ab + ab(a-1)$
two semi-edge on x_1 and one semi-edge on x_3	$ac + ac(a-1)$
one semi-edge on x_1 and two semi-edge on x_2	$ab + ab(b-1)$
one semi-edge on x_1 and two semi-edge on x_3	$ac + ac(c-1)$
two semi-edge on x_2 and one semi-edge on x_3	$bc + bc(b-1)$
one semi-edge on x_2 and two semi-edge on x_3	$bc + bc(c-1)$
three semi-edge on x_1	$a + 3a(a-1) + a(a-1)(a-2)$
three semi-edge on x_2	$b + 3b(b-1) + b(b-1)(b-2)$
three semi-edge on x_3	$c + 3c(c-1) + c(c-1)(c-2)$
one semi-edge on x_1, one semi-edge on x_2 and one semi-edge on x_3	abc

Next if one semi-edge is on the vertex x_2 and another is on the vertex x_3, then there is bc semi-edge walk of length four. Finally we have:

$$[Q^4(H)]_{x_1,x_3} = a^2 + b^2 + c^2 + a + b + c + ab + ac + bc - 2.$$

To compute the $[Q^3(H)]_{x_1,x_3}$, we should calculate the number of semi-edge walks of length three from x_1 to x_3. Such semi-edge walks occur in the case that we have one semi-edge on some vertices. By a simple check one can see that if the semi-edge is on the vertices x_1, x_2 and x_3, then there are a, b and c semi-edge walks of length three, respectively. Therefore $[Q^3(H)]_{x_1,x_3} = a + b + c$. It is easy to see that there is one walk of length two from x_1 to x_3 and there is no semi-edge walk of length 1 from x_1 to x_3. This implies that $[Q^2(H)]_{x_1,x_3} = 1$ and $[Q(H)]_{x_1,x_3} = 0$. This completes the proof.

Lemma 3.4. *The number of closed semi-edge walks of length k, $1 \leq k \leq 5$, on the vertex i in H are as the following:*

- $[Q^5(H)]_{i,i} = a^3 + 4a^2 + 6a + b + 4,$
- $[Q^4(H)]_{i,i} = a^2 + 3a + 4,$
- $[Q^3(H)]_{i,i} = a + 3,$
- $[Q^2(H)]_{i,i} = 2,$
- $[Q(H)]_{i,i} = 1.$

PROOF. First we compute the number of closed semi-edge walks of length 5 on the vertex i. There are three cases: either there is one semi-edge on some vertices, or

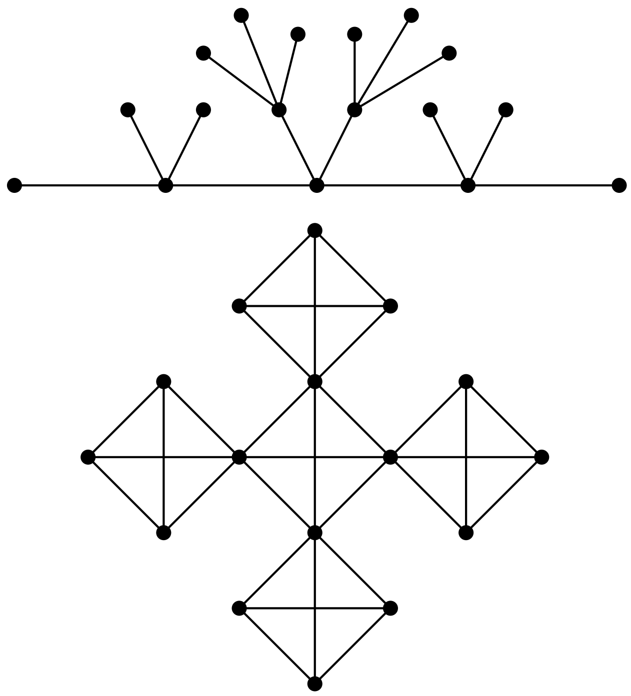

FIGURE 3. A tree of order 17 and its line graph that is corona product of $K_4 \circ K_3$.

there are three semi-edges on some vertices or there are five semi-edges on vertices. In the first case, if the semi-edge is on the vertex i, then a simple calculation states that there is $2a$ closed semi-edge walk of length five. While if the semi-edge is on the vertex x_1, there is $2a^2$ closed semi-edge walk of length five. Notice that if the semi-edge is on the adjacent vertices to x_1, then by a simple check one can see that there is $a + b - 1$ closed semi-edge walk.

In the second case, if three semi-edges are on the vertex i, then it is easy to see that there are four closed semi-edge walks as following:

$$i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i \overset{w}{-} x_1 \overset{w}{-} i,$$

$$i \overset{s.w}{-} i \overset{s.w}{-} i \overset{w}{-} x_1 \overset{w}{-} i \overset{s.w}{-} i,$$

$$i \overset{s.w}{-} i \overset{w}{-} x_1 \overset{w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i,$$

$$i \overset{w}{-} x_1 \overset{w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i.$$

Now let two semi-edges be on the vertex i and the third semi-edge be on the vertex x_1. Then it is easy to see that there is $3a$ closed semi-edge walk on the vertex i. If two semi-edges are on the vertex x_1 and another is on the vertex i, then it is easy to check that the number of such closed semi-edge walks is equal to $2a+2a(a-1)$. If three semi-edges are on the vertex x_1, then a simple calculation show that there is $a+3a(a-1)+a(a-1)(a-2)$ closed semi-edge walk.

In the third case, there is one closed semi-edge walk of length five as $i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i$. Therefore the number of closed semi-edge walk of length five on the vertex i is equal to $a^3+4a^2+6a+b+4$.

Now we calculate the number of closed semi-edge walks of length four on the vertex i. For any adjacent vertex to x_1 such as the vertex 1, there is one semi-edge walk of length four of this form, $i \overset{w}{-} x_1 \overset{w}{-} 1 \overset{w}{-} x_1 \overset{w}{-} i$. The number of such walk is equal to a. In the case that we have two semi-edges on the vertex x_1, there is $a+a(a-1)$ closed semi-edge walk of length four. Also if two semi-edges are on the vertex i, then a simple calculation states that there are three closed semi-edge walks of length four. While if one semi-edge is on the vertex i and another is on the vertex x_1, then there is $2a$ closed semi-edge walk of length four. Now assume that there are four semi-edges on the vertex i. Then there is one closed semi-edge walk of the form $i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i$. In total the number of closed semi-edge walks of length four is equal to a^2+3a+4.

To calculate the $[Q^3(H)]_{i,i}$, it is easy to see that there is one closed semi-edge walk of length three such as $i \overset{s.w}{-} i \overset{s.w}{-} i \overset{s.w}{-} i$. Also if one semi-edge is on the vertex i, then there are two types semi-edges of the form $i \overset{w}{-} x_1 \overset{w}{-} i \overset{s.w}{-} i$ and $i \overset{s.w}{-} i \overset{w}{-} x_1 \overset{w}{-} i$. Let the semi-edge be on the vertex x_1. Then by a simple check one can see that there is a closed semi-edge walk of length 3. Therefore $[Q^3(H)]_{i,i} = a+3$. It is easy to see that there are two closed semi-edge walks of length two and one closed semi-edge walk of length one on the vertex i. This completes the proof.

Lemma 3.5. *The following relations are satisfied:*

- $[Q^5(H)]_{j,j} = c^3+4c^2+6c+b+4$, $[Q^4(H)]_{j,j} = c^2+3c+4$, $[Q^3(H)]_{j,j} = c+3$, $[Q^2(H)]_{j,j} = 2$ *and* $[Q(H)]_{j,j} = 1$.
- $[Q^5(H)]_{i,j} = a+b+c+2$ *and* $[Q^4(H)]_{i,j} = [Q^3(H)]_{i,j} = [Q^2(H)]_{i,j} = [Q(H)]_{i,j} = 0$.
- $[Q^5(H)]_{i,x_3} = a^2+b^2+c^2+2a+2b+2c+ab+ac+bc-1$, $[Q^4(H)]_{i,x_3} = a+b+c+1$, $[Q^3(H)]_{i,x_3} = 1$ *and* $[Q^2(H)]_{i,x_3} = [Q(H)]_{i,x_3} = 0$.
- $[Q^5(H)]_{x_2,j} = 6b^2+6c^2+3bc+\sum_{i=1}^{b-2} r_i + a - b + b^3 + b^2c + c^2b + c^3 - 3b^2 + 2b - 3c^2 + 2c - 1$, $[Q^4(H)]_{x_2,j} = b^2+c^2+bc+2b+2c$, $[Q^3(H)]_{x_2,j} = b+c+1$, $[Q^2(H)]_{x_2,j} = 1$ *and* $[Q(H)]_{x_2,j} = 0$.

PROOF. The proof of the first part is similar to Lemma 3.4 and the proof of other parts are similar to Lemma 3.3.

What follows, we consider the set all trees and obtain trees with exactly five distinct signless Laplacian eigenvalues.

Theorem 3.6. *A tree G has exactly five distinct signless Laplacian eigenvalues if and only if $G \cong P_5$ or $G \cong S_{a,b}$.*

Proof. If $G \cong P_5$ or $G \cong S_{a,b}$, then Corollary 2.7 and Lemma 3.1 state that G has exactly five distinct signless Laplacian eigenvalues. Conversely suppose that G has q_1, q_2, q_3, q_4 and q_5 as the signless Laplacian eigenvalues. Then the minimal polynomial of the matrix Q is $f(x) = (x - q_1)(x - q_2) \ldots (x - q_5)$. So we have the following relation:

$$\begin{aligned} Q^5 &- (\sum_{i=1}^{5} q_i)Q^4 + (\sum_{1\leq i<j\leq 5} q_iq_j)Q^3 - (\sum_{1\leq i<j<r\leq 5} q_iq_jq_r)Q^2 \\ &+ (\sum_{1\leq i<j<r<t\leq 5} q_iq_jq_rq_t)Q - (q_1q_2q_3q_4q_5)I = 0. \end{aligned} \tag{1}$$

Since G is bipartite, one of these eigenvalues is equal to 0. Thus the last sentence of the equation (1) is 0. On the other hand, by Theorem 2.2 we have $d \leq 4$. If $d = 2$, then $G \cong S_n$ which is a contradiction. If $d = 3$, then $G \cong S_{a,b}$ and Lemma 3.1. states that G has five signless Laplacian eigenvalues.

Now assume that $d = 4$, i and j are two vertices such that $d(i, j) = 4$ and $i - x_1 - x_2 - x_3 - j$ is a path of length four between i and j. If G has only these five vertices, then $G \cong P_5$. Otherwise let $|V(G)| = n \geq 6$, then the other vertices must be connected to the vertices x_1, x_2 or x_3, and so $G = H$. We first claim that in this graph $d(x_1) = d(x_2) = d(x_3)$. By considering equation (1), we have the following system:

$$\begin{cases} a + b + c + 2 = \sum_{i=1}^{5} q_i, \\[2mm] a^3 + 4a^2 + 6a + b + 4 = (a^2 + 3a + 4)\sum_{i=1}^{5} q_i + (-a - 3)\sum_{1\leq i<j\leq 5} q_iq_j \\ +2\sum_{1\leq i<j<r\leq 5} q_iq_jq_r - \sum_{1\leq i<j<r<t\leq 5} q_iq_jq_rq_t, \\[2mm] c^3 + 4c^2 + 6c + b + 4 = (c^2 + 3c + 4)\sum_{i=1}^{5} q_i + (-c - 3)\sum_{i=1}^{5}\sum_{1\leq i<j\leq 5} q_iq_j \\ +2\sum_{i=1}^{5}\sum_{1\leq i<j<r\leq 5} q_iq_jq_r - \sum_{1\leq i<j<r<t\leq 5} q_iq_jq_rq_t, \\[2mm] a^2 + b^2 + c^2 + 2a + 2b + +2c + ab + ac + bc - 1 = (a + b + c + 1)\sum_{i=1}^{5} q_i - \sum_{1\leq i<j\leq 5} q_iq_j. \end{cases}$$

In above system, we subtract the third equation from the second one, next the fourth relation multiplying in $c - a$ and finally we sum both of the resulting relations with above relation. At the rest by substituting the value of $\sum_{i=1}^{5} q_i$ of the first equation, we obtain

$$a = c. \tag{2}$$

Now by considerin the (i,j), (x_1,x_3), (i,x_3) and (x_2,j)-entries of equation (1), we have the following new system:

$$\begin{cases} a+b+c+2=\sum_{i=1}^{5} q_i, \\ \\ 2a^2+2b^2-a-2b+2c^2-c+\sum_{i=1}^{b-2} r_i+2ab+2ac+2bc+a^3+a^2b+a^2c+b^2a \\ +c^2a+b^3+b^2c+c^2b+c^3+abc-2=(a^2+b^2+c^2+ab+ac+bc+a+b+c-2)\sum_{i=1}^{5} q_i \\ -(a+b+c)\sum_{1\le i<j\le 5} q_iq_j+\sum_{1\le i<j<r\le 5} q_iq_jq_r, \\ \\ a^2+b^2+c^2+2a+2b+2c+ab+ac+bc-1=(a+b+c+1)\sum_{i=1}^{5} q_i-\sum_{1\le i<j\le 5} q_iq_j, \\ \\ 6a^2+6c^2+3bc+\sum_{i=1}^{b-2} r_i+a-b+b^3+b^2c+c^2b+c^3-3b^2+2b-3c^2+2c-1 \\ =(2b+2c+b^2+c^2+bc)\sum_{i=1}^{5} q_i-(b+c+1)\sum_{1\le i<j\le 5} q_iq_j+\sum_{1\le i<j<r\le 5} q_iq_jq_r. \end{cases}$$

In recent system, first we subtract the fourth equation from the second equation, next the third relation multiplying in $1-a$ and then we sum both of this relation by the above relation. At the rest we replace the value of $\sum_{i=1}^{5} q_i$ from the first equation in the resulting equation and so we have

$$a=b. \tag{3}$$

By (2) and (3) we have $a=b=c$. By a simple check one can see that the degree of $(b-2)$ vertices attached to x_2, can not be equal to one. Thus the degree of these vertices is at least two and so the degree oh these vertices is equal to a. If $a=b=c=2$, then $G\cong P_5$ and so G has five signless Laplacian eigenvalues. Let $a=b=c\ge 3$, then $G=H$ with $d(x_1)=d(x_2)=d(x_3)=a\ge 3$. We claim that this graph has at least six signless Laplacian eigenvalues. To proves, it is sufficient to show that the line graph $L(H)$ has at least five eigenvalues. Put $F=L(H)$. One can see that the graph F is the corona product of K_a and K_{a-1}, Figure 3. Notice that Theorem 2.5. states that -1, $\frac{a-3\pm\sqrt{a^2+2a-3}}{2}$ and $\frac{2a-3\pm\sqrt{4a-3}}{2}$ are members of the spectrum of $K_a\circ K_{a-1}$. This implies that the graph $L(H)=K_a\circ K_{a-1}$ has at least five eigenvalues. On the other hand, Theorem 2.6. states that the graph H has at least six signless Laplacian eigenvalues. Overall for any a, b and c, the graph H has at least six signless Laplacian eigenvalues. Finally we conclude that the trees with exactly five signless Laplacian eigenvalues are the graphs P_5 and $S_{a,b}$. This completes the proof.

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