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MEASURE OF NONCOMPACTNESS IN THE STUDY OF SOLUTIONS FOR A SYSTEM OF INTEGRAL EQUATIONS

Vatan Karakaya[1], Mohammad Mursaleen[2], Nour El Houda Bouzara[3], and Derya Sekman [4]

[1]**Department of Mathematical Engineering, Yildiz Technical University, Davutpasa Campus, Esenler, 34210 Istanbul, Turkey, vkkaya@yahoo.com**
[2]**Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India, mursaleenm@gmail.com**
[3] **Faculty of Mathematics, University of Science and Technology Houari Boumediène, Bab-Ezzouar, 16111 Algies, Algeria, bzr.nour@gmail.com**
[4] **Faculty of Arts and Sciences, Department of Mathematics, Ahi Evran University, 40100 Kirsehir, Turkey, deryasekman@gmail.com**

Abstract. In this work, we prove the existence of solutions for a tripled system of integral equations using some new results of fixed point theory associated with measure of noncompactness. These results extend the results in some previous works. Also, the condition under which the operator admits fixed points is more general than the others in literature.

Key words and Phrases: Measure of noncompactness, Fixed point, System of integral equations.

Abstrak. Dalam paper ini, dibuktikan eksistensi solusi dari suatu sistem *tripled* dari persamaan integral menggunakan suatu hasil terbaru dalam teori titik tetap yang berasosiasi dengan ukuran *noncompactness*. Hasil-hasil ini memperluas hasil-hasil yang sudah ada sebelumnya. Lebih jauh, kondisi operator yang memenuhi syarat titik tetap di dalam paper ini juga lebih umum dibandingkan dengan yang sudah ada di dalam literatur-literatur sebelumnya.

Kata kunci: Ukuran *noncompactness*, titik tetap, sistem persamaan integral.

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1. INTRODUCTION

In recent years, measure of noncompactness which was introduced by Kuratowski [18] in 1930 and has provided powerful tools for obtaining the solutions of a large variety of integral equations and systems. One can find related references in studies involving Aghajani et al. [2], [3], [4], [5], Banas [9], Banas and Rzepka [13], Mursaleen and Mohiuddine [19], Mursaleen and Rizvi [20], Araba et al. [8], Deepmala and Pathak [16], Shaochun and Gan [21], Sikorska [22], Alotaibi et al. [7], and many others.

In this paper, we study the solvability of the following system of integral equations

$$\begin{cases} x(t) = g_1(t) + f_1\left(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \psi\left(\int_0^{q_1(t)} h(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right)\right) \\ y(t) = g_2(t) + f_2\left(t, x(\xi_2(t)), y(\xi_2(t)), z(\xi_2(t)), \psi\left(\int_0^{q_2(t)} h(t,s,x(\eta_2(s)), y(\eta_2(s)), z(\eta_2(s)))\,ds\right)\right) \\ z(t) = g_3(t) + f_3\left(t, x(\xi_3(t)), y(\xi_3(t)), z(\xi_3(t)), \psi\left(\int_0^{q_3(t)} h(t,s,x(\eta_3(s)), y(\eta_3(s)), z(\eta_3(s)))\,ds\right)\right) \end{cases},$$

by establishing some results of existence for fixed points of condensing operators in Banach spaces.

Throughout this paper, X is assumed to be a Banach space. The family of bounded subset, closure and closed convex hull of X are denoted by $\mathcal{B}_X$, $\overline{X}$ and $ConvX$, respectively.

We now gather some well-known definitions and results from the literature which will be used throughout this paper.

Definition 1.1 ([10]). *Let X be a Banach space and $\mathcal{B}_X$ the family of bounded subset of X. A map*

$$\mu : \mathcal{B}_X \to [0, \infty)$$

which satisfies the following:

(1) $\mu(A) = 0 \Leftrightarrow A$ *is a precompact set,*
(2) $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$,
(3) $\mu(A) = \mu(\overline{A})$, $\forall A \in \mathcal{B}_X$,
(4) $\mu(ConvA) = \mu(A)$,
(5) $\mu(\lambda A + (1-\lambda)B) \leqslant \lambda\mu(A) + (1-\lambda)\mu(B)$, *for* $\lambda \in [0,1]$,
(6) Let (A_n) *be a sequence of closed sets from* $\mathcal{B}_X$ *such that* $A_{n+1} \subseteq A_n$, $(n \geqslant 1)$ *and* $\lim_{n\to\infty} \mu(A_n) = 0$. *Then, the intersection set* $A_\infty = \bigcap_{n=1}^{\infty} A_n$ *is nonempty and* A_∞ *is precompact.*

The functional μ is called measure of noncompactness defined on the Banach space X.

Theorem 1.2 ([11]). *Let C be a nonempty closed, bounded and convex subset of X. If $T : C \to C$ is a continuous mapping*

$$\mu(TA) \leqslant k\mu(A), \quad k \in [0,1),$$

then T has a fixed point.

The following theorem is considered as a generalization of Darbo fixed point theorem.

Theorem 1.3 ([1]). *Let C be a nonempty closed, bounded and convex subset of X and $T : C \to C$ be a continuous mapping such that for any subset A of C*

$$\mu(TA) \leqslant \beta(\mu(A))\,\mu(A),$$

where $\beta : \mathbb{R}_+ \to [0,1)$ that is $\beta(t_n) \to 1$ implies $t_n \to 0$. Then, T has at least one fixed point.

Corollary 1.4 ([1]). *Let C be a nonempty closed, bounded and convex subset of X and $T : C \to C$ be a continuous mapping such that for any subset A of C*

$$\mu(TA) \leqslant \varphi(\mu(A)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and upper semicontinuous functions, that is, for every $t > 0$, $\varphi(t) < t$. Then, T has at least one fixed point.

Theorem 1.5 ([6]). *Let μ_1, μ_2, ...,μ_n be measures of noncompactness in Banach spaces E_1, E_2, ...E_n, (respectively).*

Then the function

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), ..., \mu_n(X_n)),$$

defines a measure of noncompactness in $E_1 \times E_2 \times \ ... \times E_n$ where X_i is the natural projection of X on E_i, for $i = 1, 2, ..., n$, and F be a convex function defined by

$$F : [0,\infty)^n \to [0,\infty),$$

such that,

$$F(x_1, x_2, ..., x_n) = 0 \Leftrightarrow x_i = 0, \ \text{for } i = 1, 2, ..., n.$$

Example 1.6 ([17]). *We can notice that by taking*

$$F(x,y,z) = \max\{x,y,z\} \ \text{for any} \ (x,y,z) \in [0,\infty)^3,$$

or

$$F(x,y,z) = x + y + z \ \text{for any} \ (x,y,z) \in [0,\infty)^3.$$

Then, F satisfies the conditions of Theorem 1.5. Thus, for a measure of noncompactness μ_i $(i = 1, 2, 3)$, we have that

$$\widetilde{\mu}(X) = \max(\mu_1(X_1), \mu_2(X_2), \mu_3(X_3)),$$

or

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) + \mu_3(X_3),$$

defines a measure of noncompactness in the space $E \times E \times E$ where X_i, $i = 1, 2, 3$ are the natural projections of X on E_i.

2. MAIN RESULTS

Theorem 2.1. *Let A be a nonempty, bounded ,closed and convex subset of a Banach space X and let $\varphi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ be a nondecreasing and upper semicontinous function such that $\varphi(t) < t$ for all $t > 0$. Then for any measure of noncompactness μ, and continuous operators $T_i : A \times A \times A \to A$ $(i = 1, 2, 3)$ satisfying*

$$\mu\left(T_i\left(X_1 \times X_2 \times X_3\right)\right) \leqslant \varphi\left(\max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\right), \quad X_1, X_2, X_3 \in A, \quad (1)$$

there exist $x^*, y^*, z^* \in A$ such that $\begin{cases} T_1\left(x^*, y^*, z^*\right) = x^* \\ T_2\left(x^*, y^*, z^*\right) = y^* \\ T_3\left(x^*, y^*, z^*\right) = z^* \end{cases}$.

Proof. Consider the following measure of noncompactness

$$\widetilde{\mu}\left(A \times A \times A\right) = \max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right),$$

where $X_1, X_2, X_3 \in A$ and the mapping $T : A \times A \times A \to A$,

$$T\left(x, y, z\right) = \left(T_1\left(x, y, z\right), T_2\left(x, y, z\right), T_3\left(x, y, z\right)\right).$$

We have,

$$\begin{aligned} \widetilde{\mu}\left(T\left(A \times A \times A\right)\right) &= \widetilde{\mu}\left(\left(T_1\left(X_1 \times X_2 \times X_3\right)\right), T_2\left(X_1 \times X_2 \times X_3\right), T_3\left(X_1 \times X_2 \times X_3\right)\right) \\ &= \max\left\{\mu\left(T_1\left(X_1 \times X_2 \times X_3\right)\right), \mu\left(T_2\left(X_1 \times X_2 \times X_3\right)\right), \mu\left(T_3\left(X_1 \times X_2 \times X_3\right)\right)\right\} \\ &\leqslant \max\left\{\varphi\left(\max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\right), \varphi\left(\max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\right), \right. \\ &\qquad \left. \varphi\left(\max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\right)\right\}. \end{aligned}$$

By hypothesis φ is a non-decreasing function, then

$$\begin{aligned} \widetilde{\mu}\left(T\left(A \times A \times A\right)\right) &\leqslant \varphi\left[\max\left\{\max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right), \max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right), \right.\right. \\ &\qquad \left.\left. \max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\right\}\right]. \end{aligned}$$

Consequently,

$$\widetilde{\mu}\left(T\left(A \times A \times A\right)\right) \leqslant \varphi\left(\widetilde{\mu}\left(A \times A \times A\right)\right).$$

So,

$$\mu\left(T_1\left(x, y, z\right), T_2\left(x, y, z\right), T_3\left(x, y, z\right)\right) \leqslant \varphi\left(\max\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\right).$$

By Corollary 1.4, we conclude that there exist $x^*, y^*, z^* \in A$ such that

$$T\left(x^*, y^*, z^*\right) = \left(x^*, y^*, z^*\right).$$

In the other hand,

$$T\left(x^*, y^*, z^*\right) = \left(T_1\left(x^*, y^*, z^*\right), T_2\left(x^*, y^*, z^*\right), T_3\left(x^*, y^*, z^*\right)\right).$$

Hence,

$$\begin{cases} T_1\left(x^*, y^*, z^*\right) = x^* \\ T_2\left(x^*, y^*, z^*\right) = y^* \\ T_3\left(x^*, y^*, z^*\right) = z^* \end{cases}.$$

Definition 2.2 ([17])**.** *A tripled (x, y, z) of a mapping $T : A \times A \times A \to A$, is called a tripled fixed point if*

$$T\left(x, y, z\right) = x, \quad T\left(y, x, z\right) = y \quad \textit{and} \quad T\left(z, y, x\right) = z.$$

Remark 2.3. *Let $T : A \times A \times A \to A$ be a continuous mapping. If we define $T_1(x, y, z) = T(x, y, z)$, $T_2(x, y, z) = T(y, x, z)$ and $T_3(x, y, z) = T(z, y, x)$, then main results of* [17] *can be considered as a result of Theorem 2.1.*

It is very natural to extend the above result from three dimensions to multidimensional fixed point and in the same way we can prove the following theorem.

Theorem 2.4. *Let A be a nonempty, bounded ,closed and convex subset of a Banach space X and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing and upper semicontinous function such that $\varphi(t) < t$ for all $t > 0$. Then for any measure of noncompactness μ and for continuous operators $T_i : A^n \to \Omega \ \ (i = 1, ..., n)$ satifying*

$$\mu(T_i(X_1 \times ... \times X_n)) \leqslant \varphi(\max(\mu(X_1), ..., \mu(X_n))), \ X_i \in A, \ i = \overline{1, n},$$

there exist $x_1^*, ..., x_n^*$ such that

$$\begin{cases} T_1(x_1^*, ..., x_n^*) = x_1^* \\ \vdots \\ T_n(x_1^*, ..., x_n^*) = x_n^* \end{cases}.$$

As a particular case we get the following corollary:

Corollary 2.5 ([3]). *Let A be a nonempty, bounded ,closed and convex subset of a Banach space X and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing and upper semicontinous function such that $\varphi(t) < t$ for all $t > 0$. Then for any measure of noncompactness μ, the continuous operator $G : A^n \to A$ satisfying*

$$\mu(G(X_1 \times ... \times X_n)) \leqslant k \max(\mu(X_1), ..., \mu(X_n)), \ X_1, ..., X_n \in A.$$

And for the case $n = 2$, we have the following result.

Corollary 2.6 ([3]). *Let A be a nonempty, bounded ,closed and convex subset of a Banach space X and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing and upper semicontinous function such that $\varphi(t) < t$ for all $t > 0$. Then for any measure of noncompactness μ, the continuous operator $G : A \times A \to A$ satisfying*

$$\mu(G(X_1 \times X_2)) \leqslant k \max(\mu(X_1), \mu(X_2)), \ X_1, X_2 \in A.$$

In the following we choose for the space X the space $BC(\mathbb{R}^+)$, i.e., the space of all real functions defined, bounded and continuous on $\mathbb{R}^+$. Then, we get the following theorem.

Theorem 2.7. *Let A be a nonempty, bounded, closed and convex subset of $BC(\mathbb{R}^+)$ and $T_i : A \times A \times A \to A$ be a continuous operator such that for every $x, y, z, u, v, w \in A$.*

$$\|T_i(x, y, z) - T_i(u, v, w)\|_\infty \leqslant \varphi(\max\{\|x - u\|_\infty, \|y - v\|_\infty, \|z - w\|_\infty\}), \quad (2)$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and upper semicontinuous function such that $\varphi(t) < t$ for all $t > 0$. Then there exist $x^*, y^*, z^* \in A$ such that, $\begin{cases} T_1(x^*, y^*, z^*) = x^* \\ T_2(x^*, y^*, z^*) = y^* \\ T_3(x^*, y^*, z^*) = z^* \end{cases}.$

Proof. To verify that the operator $T_i : A\times A\times A\to A$ satisfy the condition (1) we recall the following notions.

The measure of noncompactness on $BC\left(\mathbb{R}^{+}\right)$ for a positive fixed t on $\mathcal{B}_{BC(\mathbb{R}^{+})}$ is defined as follows:

$$\mu\left(X\right)=\omega_0\left(X\right)+\limsup_{t\to\infty} diamX\left(t\right),$$

that is, $diamX\left(t\right)=\sup\left\{\left|x\left(t\right)-y\left(t\right)\right| : x,y\in X\right\},\ X\left(t\right)=\left\{x\left(t\right):x\in X\right\}$ .and

$$\omega_0\left(X\right)=\lim_{K\to\infty}\omega_0^K\left(X\right),$$

$$\omega_0^K\left(X\right)=\lim_{\epsilon\to 0}\omega^K\left(X,\epsilon\right),$$

$$\omega^K\left(X,\epsilon\right)=\sup\left\{\omega^K\left(x,\epsilon\right):x\in X\right\},$$

$$\omega^K\left(x,\epsilon\right)=\sup\left\{\left|x\left(t\right)-x\left(s\right)\right|:t,s\in[0,K],\ \ |t-s|\leqslant\epsilon\right\},\text{ for } K>0,$$

where $\omega^K\left(x,\epsilon\right)$ for $x\in X$ and $\epsilon>0$, is the modulus of continuity of x on the compact $[0,K]$, where K is a positive number.

We have

$$\left\|T_i\left(x,y,z\right)\left(t\right)-T_i\left(x,y,z\right)\left(s\right)\right\|\leqslant\varphi\left(\max\left\{\left\|x\left(t\right)-x\left(s\right)\right\|,\left\|y\left(t\right)-y\left(s\right)\right\|,\left\|z\left(t\right)-z\left(s\right)\right\|\right\}\right),$$

by taking the supremum and using the fact that φ is nondecreasing, we get

$$\omega^K\left(T_i\left(x,y,z\right),\epsilon\right)\leqslant\varphi\left(\max\left\{\omega^K\left(x,\epsilon\right),\omega^K\left(y,\epsilon\right),\omega^K\left(z,\epsilon\right)\right\}\right).$$

Thus,

$$\omega_0\left(T_i\left(X_1\times X_2\times X_3\right)\right)\leqslant\varphi\left(\max\left\{\omega_0\left(X_1\right),\omega_0\left(X_2\right),\omega_0\left(X_3\right)\right\}\right). \tag{3}$$

Since in (2) x,y and z are arbitrary and φ is non-decreasing,

$$DiamT_i\left(X_1\times X_2\times X_3\right)\left(t\right)\leqslant\varphi\left(\max\left\{DiamX_1\left(t\right),\ DiamX_2\left(t\right),DiamX_3\left(t\right)\right\}\right).$$

In further, $X_1\left(t\right),X_2\left(t\right),X_3\left(t\right)$ are subspaces of $BC\left(\mathbb{R}_+\right)$. Then,

$$\limsup_{t\to\infty} DiamT_i\left(X_1\times X_2\times X_3\right)\left(t\right)\leqslant\limsup_{t\to\infty}\varphi\left(\max\left\{DiamX_1\left(t\right),\ DiamX_2\left(t\right),DiamX_3\left(t\right)\right\}\right)+\Phi\left(\epsilon\right)$$

$$\leqslant\varphi\left(\max\left\{\limsup_{t\to\infty} DiamX_1\left(t\right),\limsup_{t\to\infty}\ DiamX_2\left(t\right),\limsup_{t\to\infty} DiamX_3\left(t\right)\right\}\right).$$

Using $\varphi\left(t\right)<t$ for all $t>0$ and from (3) and the above inequality, we get

$$\mu\left(T_i\left(X_1\times X_2\times X_3\right)\right)\leqslant\varphi\left(\max\left(\mu\left(X_1\right),\mu\left(X_2\right),\mu\left(X_3\right)\right)\right),\ X_1,X_2,X_3\in A.$$

Consequently, there exist $x^*,y^*,z^*\in A$ such that

$$\begin{aligned}T\left(x^*,y^*,z^*\right) &= \left(T_1\left(x^*,y^*,z^*\right),T_2\left(x^*,y^*,z^*\right),T_3\left(x^*,y^*,z^*\right)\right)\\ &= \left(x^*,y^*,z^*\right).\end{aligned}$$

Thus,

$$\left\{\begin{array}{l}T_1\left(x^*,y^*,z^*\right)=x^*\\ T_2\left(x^*,y^*,z^*\right)=y^*\\ T_3\left(x^*,y^*,z^*\right)=z^*\end{array}\right..$$

3. APPLICATION

Now, we will use the results of the previous section to resolve the following system

$$\begin{cases} x(t) = g_1(t) + f_1\left(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \psi\left(\int_0^{q_1(t)} h(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\, ds\right)\right) \\ y(t) = g_2(t) + f_2\left(t, x(\xi_2(t)), y(\xi_2(t)), z(\xi_2(t)), \psi\left(\int_0^{q_2(t)} h(t,s,x(\eta_2(s)), y(\eta_2(s)), z(\eta_2(s)))\, ds\right)\right) \\ z(t) = g_3(t) + f_3\left(t, x(\xi_3(t)), y(\xi_3(t)), z(\xi_3(t)), \psi\left(\int_0^{q_3(t)} h(t,s,x(\eta_3(s)), y(\eta_3(s)), z(\eta_3(s)))\, ds\right)\right) \end{cases}. \tag{4}$$

We study system (4) under the following assumptions:

(i) $\xi_i, \eta_i, q_i : \mathbb{R}_+ \to \mathbb{R}_+$, $(i = 1, 2, 3)$, are continuous and $\xi_i(t) \to \infty$ as $t \to \infty$.

(ii) The function $\psi_i : \mathbb{R} \to \mathbb{R}$, $(i = 1, 2, 3)$, is continuous and there exist positive δ_i, α_i such that

$$|\psi_i(t_1) - \psi_i(t_2)| \leqslant \delta_i |t_1 - t_2|^{\alpha_i},$$

for $i = 1, 2, 3$ and any $t_1, t_2 \in \mathbb{R}_+$.

(iii) $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, $g_i : \mathbb{R}_+ \to \mathbb{R}$ are bounded and there exists nondecreasing continuous function $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Phi_i(0) = 0$, $i = 1, 2, 3$, such that

$$|f_i(t, x_1, x_2, x_3, x_4) - f_i(t, y_1, y_2, y_3, y_4)| \leqslant (\varphi_i(\max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\})) + \Phi_i(|x_4 - y_4|).$$

(iv) The functions defined by $|f_i(t, 0, 0, 0, 0)|$, $i = 1, 2, 3$ are bounded on $\mathbb{R}_+$, i.e.,

$$M_i = \sup\{f_i(t, 0, 0, 0, 0) : t \in \mathbb{R}_+\} < \infty. \tag{5}$$

(v) $h_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are continuous functions and there exists a positive constant D such that $i = 1, 2, 3$,

$$\sup\left\{\left|\int_0^{q_i(t)} h_i(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\, ds\right| : \ t, s \in \mathbb{R}_+, \ x, y, z \in BC(\mathbb{R}_+)\right\} < D, \tag{6}$$

and

$$\lim_{t \to \infty} \int_0^{q_i(t)} [h_i(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s))) - h_i(t, s, u(\eta(s)), v(\eta(s)), w(\eta(s)))]\, ds = 0, \tag{7}$$

with respect to $x, y, z, u, v, w \in BC(\mathbb{R}_+)$.

Consider the following operator,

$$T_i(x, y, z) = g_i(t) + f_i\left(t, x(\xi_i(t)), y(\xi_i(t)), z(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t, s, x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\, ds\right)\right).$$

Solving the system (4) is equivalent to find the fixed points of the operator T_i. Then let verify the conditions of Theorem 2.7.

First, since g_i and f_i $(i = 1, 2, 3)$ are continuous then the operators T_i are continuous.

In further, for $x, y, z \in B_r(r)$ (for $r > 0$) let,

$$\begin{aligned} & \left\| T_i(x,y,z)(t) \right\| \\ = & \left\| g_i(t) + f_i\left(t, x(\xi_i(t)), y(\xi_i(t)), z(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\,ds\right)\right)\right\| \\ \leqslant & \left\| f_i\left(t, x(\xi_i(t)), y(\xi_i(t)), z(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\,ds\right)\right)\right. \\ & \left. - f(t,0,0,0) + f(t,0,0,0) \right\| + \left\| g_i(t) \right\| \\ \leqslant & \left\| g_i(t) \right\| + \left\| f(t,0,0,0) \right\| \\ & + \varphi_i(\max\{|x(\xi_i(t))|, |y(\xi_i(t))|, |z(\xi_i(t))|\}) \\ & + \Phi_i\left(\psi\left(\int_0^{q_i(t)} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\,ds\right)\right). \end{aligned}$$

Since, g_i are bounded, f_i are continuous functions and using hypothesis (iv)-(v), we get

$$\begin{aligned} \left\| T_i(x,y,z) \right\|_\infty & \leqslant \varphi_i\left(\max\left\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty\right\}\right) + G + M_i + \Phi_i\left(\delta_i D^{\alpha_i}\right) \\ & \leqslant \varphi_i(r) + G + M_i + \Phi_i\left(\delta_i D^{\alpha_i}\right), \end{aligned}$$

for some $r_0 \geqslant 0$, we obtain $T_i(B_{r_0} \times B_{r_0} \times B_{r_0}) \subset B_{r_0}$.

Moreover,

$$\begin{aligned} & \left\| T_i(x,y,z) - T_i(u,v,w) \right\|_\infty \\ = & \sup_t \left\| g_i(t) + f_i\left(t, x(\xi_i(t)), y(\xi_i(t)), z(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\,ds\right)\right)\right. \\ & \left. - g_i(t) - f_i\left(t, u(\xi_i(t)), v(\xi_i(t)), w(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t,s,u(\eta_i(s)), v(\eta_i(s)), w(\eta_i(s)))\,ds\right)\right)\right\| \\ = & \sup_t \left\| f_i\left(t, x(\xi_i(t)), y(\xi_i(t)), z(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\,ds\right)\right)\right. \\ & \left. - f_i\left(t, u(\xi_i(t)), v(\xi_i(t)), w(\xi_i(t)), \psi\left(\int_0^{q_i(t)} h(t,s,u(\eta_i(s)), v(\eta_i(s)), w(\eta_i(s)))\,ds\right)\right)\right\| \\ \leqslant & \sup_t \left\{ \varphi_i\left(\max\left\{|x(\xi_i(t)) - u(\xi_i(t))|, |y(\xi_i(t)) - v(\xi_i(t))|, |z(\xi_i(t)) - w(\xi_i(t))|\right\}\right)\right. \\ & \left. + \Phi_i\left(\left|\begin{array}{c} \psi\left(\int_0^{q_i(t)} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s)))\,ds\right) \\ -\psi\left(\int_0^{q_i(t)} h(t,s,u(\eta_i(s)), v(\eta_i(s)), w(\eta_i(s)))\,ds\right) \end{array}\right|\right)\right\} \\ \leqslant & \varphi_i\left(\max\left\{\|x-u\|_\infty, \|y-v\|_\infty, \|z-w\|_\infty\right\}\right) \\ & + \sup_t \Phi_i\left(\delta_i \left|\int_0^{q_i(t)} \left\{\begin{array}{c} h(t,s,x(\eta_i(s)), y(\eta_i(s)), z(\eta_i(s))) \\ -h(t,s,u(\eta_i(s)), v(\eta_i(s)), w(\eta_i(s))) \end{array}\right\} ds\right|^{\alpha_i}\right). \end{aligned}$$

Consider,

$$\left|\int_0^{q_i(t)}\left\{h\left(t,s,x\left(\eta_i\left(s\right)\right),y\left(\eta_i\left(s\right)\right),z\left(\eta_i\left(s\right)\right)\right)-h\left(t,s,u\left(\eta_i\left(s\right)\right),v\left(\eta_i\left(s\right)\right),w\left(\eta_i\left(s\right)\right)\right)\right\}ds\right|.$$

Using the condition (7), we get

$$\left|\int_0^{q_i(t)}\left\{h\left(t,s,x\left(\eta_i\left(s\right)\right),y\left(\eta_i\left(s\right)\right),z\left(\eta_i\left(s\right)\right)\right)-h\left(t,s,u\left(\eta_i\left(s\right)\right),v\left(\eta_i\left(s\right)\right),w\left(\eta_i\left(s\right)\right)\right)\right\}ds\right|\leqslant\epsilon$$

and

$$\delta_i\left|\int_0^{q_i(t)}\left\{h\left(t,s,x\left(\eta_i\left(s\right)\right),y\left(\eta_i\left(s\right)\right),z\left(\eta_i\left(s\right)\right)\right)-h\left(t,s,u\left(\eta_i\left(s\right)\right),v\left(\eta_i\left(s\right)\right),w\left(\eta_i\left(s\right)\right)\right)\right\}ds\right|^{\alpha_i}\leqslant\delta_i\epsilon^{\alpha_i}.$$

Thus,

$$\Phi_i\left(\delta_i\left|\int_0^{q_i(t)}\left\{\begin{array}{c}h\left(t,s,x\left(\eta_i\left(s\right)\right),y\left(\eta_i\left(s\right)\right),z\left(\eta_i\left(s\right)\right)\right)\\-h\left(t,s,u\left(\eta_i\left(s\right)\right),v\left(\eta_i\left(s\right)\right),w\left(\eta_i\left(s\right)\right)\right)\end{array}\right\}ds\right|^{\alpha_i}\right)\leqslant\Phi_i\left(\delta_i\epsilon^{\alpha_i}\right).$$

On the other hand Φ_i is continuous function and $\Phi_i(0)=0$, ϵ is arbitrary, then for $\epsilon\to 0$, we get

$$\left\|T_i\left(x,y,z\right)-T_i\left(u,v,w\right)\right\|_\infty\leqslant\varphi_i\left(\max\left\{\left\|x-u\right\|_\infty,\left\|y-v\right\|_\infty,\left\|z-w\right\|_\infty\right\}\right).$$

Consequently by Theorem 2.7, there exist x^*,y^*,z^* such that

$$\left\{\begin{array}{l}T_1\left(x^*,y^*,z^*\right)=x^*\\T_2\left(x^*,y^*,z^*\right)=y^*\\T_3\left(x^*,y^*,z^*\right)=z^*\end{array}\right..$$

Then, we had proved the following theorem.

Theorem 3.1. *Under the conditions $(i)-(v)$ the system of integral equations (4) has at least one solution in the space $BC\left(\mathbb{R}_+\right)\times BC\left(\mathbb{R}_+\right)\times BC\left(\mathbb{R}_+\right)$.*

Example 3.2. *Let the system of integral equations*

$$\left\{\begin{array}{l}x\left(t\right)=\frac{t^2}{2+2t^4}+\frac{x\left(\sqrt{t}\right)+y\left(\sqrt{t}\right)+z\left(\sqrt{t}\right)}{3t^2+3}+\arctan\int_0^{\sqrt{t}}\frac{x\left(s^2\right)s\left|\sin y\left(s^2\right)\right|\left|\cos z\left(s^2\right)\right|}{e^t(1+x^2(s^2))(1+\sin^2y(s^2))(1+\cos^2z(s^2))}ds\\y\left(t\right)=\frac{1}{2}e^{-t^2}+\frac{t^2(x(t)+y(t)+z(t))}{3t^4+3}+\sin\int_0^t\frac{e^sy^2(s)\left(1+\cos^2x(s)\right)\left(1+\sin^2z(s)\right)}{e^{t^2}(1+y^2(s))(1+\sin^2x(s))(1+\cos^2z(s))}ds\\z\left(t\right)=\frac{1}{2\sqrt{1+t^4}}+\frac{t^3(x(t)+y(t)+z(t))}{3t^5+3}+\cos\int_0^{t^2}\frac{s^2|\cos z(s)|+\sqrt{e^s(1+z^2(s))(1+\sin^2y(s))(1+\cos^2x(s))}}{e^t(1+z^2(s))(1+\sin^2y(s))(1+\cos^2x(s))}ds\end{array}\right..$$

We notice that by taking

$$g_1\left(t\right)=\frac{t^2}{2+2t^4},\ g_2\left(t\right)=\frac{1}{2}e^{-t^2},\ g_3\left(t\right)=\frac{1}{2\sqrt{1+t^4}},$$

$$\begin{array}{l}f_1\left(t,x,y,z,p\right)=\frac{x+y+z}{3t^2+3}+p\\f_2\left(t,x,y,z,p\right)=\frac{t^2(x+y+z)}{3t^4+3}+p\\f_3\left(t,x,y,z,p\right)=\frac{t^3(x+y+z)}{3t^5+3}+p\end{array},$$

$$\begin{aligned} h_1(t,s,x,y,z) &= \frac{xs\left|\sin y\right|\left|\cos z\right|}{e^t\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)} \\ h_2(t,s,x,y,z) &= \frac{e^s\left(1+y^2\right)\left(1+\sin^2 x\right)\left(1+\cos^2 z\right)}{e^{t^2}\left(1+y^2\right)\left(1+\sin^2 x\right)\left(1+\cos^2 z\right)} \\ h_3(t,s,x,y,z) &= \frac{s^2\left|\cos z\right|+\sqrt{e^s\left(1+z^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 x\right)}}{e^t\left(1+z^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 x\right)} \end{aligned}$$

and

$$\begin{aligned} \eta_1(t) &= t^2,\ \eta_2(t)=\eta_3(t)=t \\ \xi_1(t) &= \sqrt{t},\ \xi_2(t)=\xi_3(t)=t \\ q_1(t) &= \sqrt{t},\ q_2(t)=t,\ q_3(t)=t^2 \\ \Psi_1(t) &= \arctan t,\ \Psi_2(t)=\sin t,\ \Psi_3(t)=\cos t, \end{aligned}$$

we get the system of integral equations (4).

To solve this system we need to verify the conditions $(i)-(v)$.

Obviously, $\xi_i, \eta_i, q_i : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous and* $\xi^i \to \infty$ *as* $t \to \infty$. *In further, the functions* $\psi_i : \mathbb{R} \to \mathbb{R}$ *are continuous for* $\delta_i = \alpha_i = 1$, *we have*

$$\left|\psi_i(t_1)-\psi_i(t_2)\right| \leqslant \delta_i\left|t_1-t_2\right|^{\alpha_i},$$

for any $t_1, t_2 \in \mathbb{R}_+$. *The conditions* (i) *and* (ii) *hold.*

Now, let

$$\begin{aligned} \left|f_1(t,x,y,z,p)-f_1(t,u,v,w,\rho)\right| &= \left|\frac{x+y+z}{3t^2+3}+p-\left(\frac{u+v+w}{3t^2+3}+\rho\right)\right| \\ &\leqslant \frac{1}{3t^2+3}\left[\left|x-u\right|+\left|y-v\right|+\left|z-w\right|\right]+\left|p-\rho\right| \\ &\leqslant \frac{3}{3t^2+3}\max\left\{\left|x-u\right|,\left|y-v\right|,\left|z-w\right|\right\}+\left|p-\rho\right| \\ &\leqslant \frac{1}{t^2+1}\max\left\{\left|x-u\right|,\left|y-v\right|,\left|z-w\right|\right\}+\left|p-\rho\right| \\ &= \varphi_1\left(\max\left\{\left|x-u\right|,\left|y-v\right|,\left|z-w\right|\right\}\right)+\Phi\left(\left|p-\rho\right|\right). \end{aligned}$$

Similarly, we prove that

$$\left|f_2(t,x,y,z,p)-f_2(t,u,v,w,\rho)\right| \leqslant \varphi_2\left(\max\left\{\left|x-u\right|,\left|y-v\right|,\left|z-w\right|\right\}\right)+\Phi\left(\left|p-\rho\right|\right)$$

and

$$\left|f_3(t,x,y,z,p)-f_3(t,u,v,w,\rho)\right| \leqslant \varphi_3\left(\max\left\{\left|x-u\right|,\left|y-v\right|,\left|z-w\right|\right\}\right)+\Phi\left(\left|p-\rho\right|\right).$$

Then, (iii) *also holds.*

In further (iv) *is valid. Indeed,*

$$M_i = \sup\left|\left\{f_i(t,0,0,0,0) : t \in \mathbb{R}_+\right\}\right| = 0, i = 1,2,3.$$

Let us verify the last condition (v). *First, note that*

$$\begin{aligned}
&\left|h_1(t,s,x,y,z)-h_1(t,s,u,v,w)\right|\\
=&\left|\frac{xs\left|\sin y\right|\left|\cos z\right|}{e^t\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)}-\frac{us\left|\sin v\right|\left|\cos w\right|}{e^t\left(1+u^2\right)\left(1+\sin^2 v\right)\left(1+\cos^2 w\right)}\right|\\
\leqslant&\left|\frac{x}{1+x^2}\frac{s}{e^t}-\frac{u}{1+u^2}\frac{s}{e^t}\right|\leqslant\frac{1}{2}\frac{s}{e^t}+\frac{1}{2}\frac{s}{e^t}\\
=&\frac{s}{e^t}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t\left|h_1(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))-h_1(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\right|ds\\
\leqslant&\lim_{t\to\infty}\int_0^t\frac{s}{e^t}ds=0.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\left|h_2(t,s,x,y,z)-h_2(t,s,u,v,w)\right|\\
=&\left|\frac{e^s\left(y^2\right)\left(1+\cos^2 x\right)\left(1+\sin^2 z\right)}{e^{t^2}\left(1+y^2\right)\left(1+\sin^2 x\right)\left(1+\cos^2 z\right)}-\frac{e^s\left(v^2\right)\left(1+\cos^2 u\right)\left(1+\sin^2 w\right)}{e^{t^2}\left(1+v^2\right)\left(1+\sin^2 u\right)\left(1+\cos^2 w\right)}\right|\\
\leqslant&\left|\frac{y^2}{1+y^2}\frac{e^s}{e^{t^2}}-\frac{v^2}{1+v^2}\frac{e^s}{e^{t^2}}\right|\leqslant 2\frac{e^s}{e^{t^2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t\left|h_2(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))-h_2(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\right|ds\\
\leqslant&\lim_{t\to\infty}\int_0^t 2\frac{e^s}{e^{t^2}}ds=0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|h_3(t,s,x,y,z)-h_3(t,s,u,v,w)\right|\\
=&\left|\frac{s^2\left|\cos z\right|+\sqrt{e^s\left(1+z^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 x\right)}}{e^t\left(1+z^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 x\right)}-\frac{s^2\left|\cos w\right|+\sqrt{e^s\left(1+w^2\right)\left(1+\sin^2 v\right)\left(1+\cos^2 u\right)}}{e^t\left(1+w^2\right)\left(1+\sin^2 v\right)\left(1+\cos^2 u\right)}\right|\\
\leqslant&\left|\frac{s^2}{e^t}\left(\cos z-\cos w\right)\right|\leqslant\frac{s^2}{e^t}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t\left|h_2(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))-h_2(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\right|ds\\
\leqslant&\lim_{t\to\infty}\int_0^t\frac{s^2}{e^t}ds=0.
\end{aligned}$$

Furthermore, for any $x, y, z \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$,

$$\sup\left\{\left|\int_0^t h_i\left(t, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right), z\left(\eta\left(s\right)\right)\right) ds\right|,\ t, s \in \mathbb{R}_+\right\} < D.$$

It is easy to see that for an $r_0 > 0$, *we have*

$$\varphi\left(r_0\right) + \frac{1}{2} + \Phi\left(D\right) \leqslant r_0,$$

holds and the condition (v) *is valid.*

Finally, the system has at least one solution in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

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