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ON BEST PROXIMITY POINTS OF GENERALIZED ALMOST-F-CONTRACTION MAPPINGS

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Abstract. We provide some results about best proximity points of generalized almost-F-contraction mappings in metric spaces which generalize and extend recent fixed point theorems. Also, we give an example to illustrate our main result.

Key words and Phrases: Best proximity point, generalized almost-F-contraction, property P.

Abstrak. Pada papaer ini kami menunjukkan beberapa hasil tentang titik-titik kedekatan yang terbaik dari pemetaan konraksi-hampir-F pada ruang metrik yang memperluas dan memperumum teorema titik tetap. Kami juga memberikan sebuah contoh untuk mengilustrasikan hasil utama kami.

Kata kunci: Titik kedekatan terbaik, kontraksi-hampir-F yang diperumum, sifat P.

1. Introduction

Fixed point theory focusses on the strategies for solving non-linear equations of the kind $Tx = x$ in which T is a self mapping defined on a subset of a metric space. But when T is a non-self mapping, it is probable that $Tx = x$ has no solution. So, for non self mapping, we try to find an approximate solution for the equation. On the other hand, best proximity point theorems offer an approximation solution that is optimal. It should be noted that best proximity point theorems furnish an approximate solution to the mentioned equation when T has no fixed point. Thus the best proximity point plays a crucial role in fixed point theory and consequently this research subject has attracted attention of many authors, as confirmed referring to [1, 2, 3, 4, 5]. In [1], the authors introduced the $F-$Suzuki contraction mappings and proved an existence and uniqueness theorem of fixed point. Following this direction of research and motivated by the works of [1, 3],

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we introduce the new class of generalized $F-$Suzuki contractions and prove a best proximity point theorem concerning such contractions. Moreover, an example is given to illustrate the usability of the new theory.

2. Main Results

To introduce our new results, it is fundamental to recall the definition of a best proximity point of a non-self mapping T and the notion of $P-$property. Let A and B be two nonempty subsets of a metric space (X, d). To facilitate the arguments, let

$$A_0 = \{a \in A : d(a, b) = d(A, B), \text{for some } b \in B\}$$

$$B_0 = \{b \in B : d(a, b) = d(A, B), \text{for some } a \in A\}$$

and

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Definition 2.1. *Let A and B be two nonempty subsets of a metric space (X, d). An element $u \in A$ is said to be a best proximity point of the non-self mapping $T : A \to B$ if it satisfies the condition*

$$d(u, Tu) = d(A, B).$$

Definition 2.2. [4] *Let A and B be two nonempty subsets of a metric space (X, d) with A_0 is nonempty. Then pair (A, B) is said to have the (P)-property if for each $x_1, x_2 \in A$ and $y_1, y_2 \in B$, the following implication holds:*

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \;\Rightarrow\; d(x_1, x_2) = d(y_1, y_2).$$

Throughout the article $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ denote the set of natural numbers, positive real numbers and real numbers, respectively.

Definition 2.3. [5] *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*

- **(F1):** *F is strictly increasing, that is, $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$,*
- **(F2):** *for every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$ we have $\lim\limits_{n\to\infty} \alpha_n = 0$ iff $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$,*
- **(F3):** *there exists a number $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0} \alpha^k F(\alpha) = 0$.*

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions $(F1)-(F3)$.

Example 2.4. [1] *The following function* $F\colon \mathbb{R}^+ \to \mathbb{R}$ *belongs to* $\digamma$.

$$F(\alpha) = \ln(\alpha),\ \ F(\alpha) = \ln(\alpha) + \alpha,\ \ F(\alpha) = \frac{-1}{\alpha^{\frac{1}{2}}}.$$

Definition 2.5. [5] *Let* (X, d) *be a metric space. A mapping* $T\colon X \to X$ *is called an* F*-contraction on* X *if there exist* $F \in \digamma$ *and* $\tau > 0$ *such that for all* $x, y \in X$ *with* $d(Tx, Ty) > 0$, *we have*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Definition 2.6. [2] *Let* (X, d) *be a metric space and* $T\colon X \to X$ *be a mapping. Then the mapping* T *is said to be an almost-*F*-contraction if there exist* $F \in \digamma$ *and* $\tau > 0$, $L \geq 0$ *such that*

$$d(Tx, Ty) > 0 \ \Rightarrow\ \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(y, Tx)),$$
$$d(Tx, Ty) > 0 \ \Rightarrow\ \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(x, Ty)).$$

Definition 2.7. *Let* (X, d) *be a complete metric space and* $T\colon X \to X$ *be a mapping. Mapping* T *is said to be a generalized almost-*F*-contraction if there exist* $F \in \digamma$, $\tau > 0$ *and* $L \geq 0$ *such that the following implication holds:*

$$Tx \neq Ty \quad \text{and} \quad \frac{1}{2}d(x, Tx) \leq d(x, y) \ \Rightarrow$$
$$\tau + F(d(Tx, Ty)) \leq F(M(x, y) + Ld(y, Tx))$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

In the sequel, let $d^*(x, y) = d(x, y) - d(A, B)$ for all $x, y \in X$.

Definition 2.8. *Let* (A, B) *be a pair of nonempty subsets of a metric space* (X, d) *and* $T\colon A \to B$ *be a mapping. Mapping* T *is said to be a non-self generalized almost-*F*-contraction if there exist* $F \in \digamma$, $\tau > 0$ *and* $L \geq 0$ *such that the following implication holds:*

$$Tx \neq Ty \quad \text{and} \quad 0 < \frac{1}{2}d^*(x, Tx) \leq d(x, y) \ \Rightarrow$$
$$\tau + F(d(Tx, Ty)) \leq F(M(x, y) - d(A, B) + Ld^*(y, Tx)) \tag{1}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Note that the domain of function F in Definition 2.8 is $(0, +\infty)$. Since
$M(x, y) - d(A, B) > d^*(x, Tx) > 0$ for all $x, y \in A$,
so Definition 2.8 is well defined.

Now we are ready to state and prove our main result.

Theorem 2.9. *Let* (A, B) *be a pair of nonempty closed subsets of a complete metric space* X. *Also assume that* $T\colon A \to B$ *is a generalized almost-*F*-contraction satisfying:*

(i): *the pair* (A, B) *has property* P,
(ii): $T(A_0) \subseteq B_0$,
(iii): $F(\alpha^+) - F(\alpha) < \tau$ *for any* $\alpha > 0$.

Then T *has a best proximity point.*

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, $Tx_0 \in B_0$. So there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Since $Tx_1 \in B_0$, again for some $x_2 \in A_0$, we get $d(x_2, Tx_1) = d(A, B)$. By continuing this process, we can find the sequence $\{x_n\}$ in A_0 as $\{Tx_n\}$ in B_0 and $d(x_{n+1}, Tx_n) = d(A, B)$ for any $n \in \mathbb{N} \cup \{0\}$. Thus for any $n \in \mathbb{N}$, we have $d(x_n, Tx_{n-1}) = d(x_{n+1}, Tx_n) = d(A, B)$. The property P implies

$$d(x_n, x_{n+1}) = d(Tx_n, Tx_{n-1}) \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Note

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) \leq d(x_{n-1}, x_n) + d(A, B). \tag{3}$$

Similarly, $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(A, B)$. Also,

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + d(A, B). \end{aligned} \tag{4}$$

If for some n_0, $d(x_{n_0}, x_{n_0+1}) = 0$, consequently $d(Tx_{n_0-1}, Tx_{n_0}) = 0$. So $Tx_{n_0-1} = Tx_{n_0}$, hence $d(A, B) = d(x_{n_0}, Tx_{n_0})$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(x_n, x_{n+1}) > 0$.
From 3,

$$d^*(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Suppose that $d^*(x_{n-1}, Tx_{n-1}) > 0$ for all $n \geq 1$. Otherwise there is nothing to prove.
By 4 and 5, We get

$$\begin{aligned} \tau + F(d(Tx_{n-1}, Tx_n)) &\leq F(M(x_{n-1}, x_n) - d(A, B) + Ld^*(x_n, Tx_{n-1})) \\ &= F(M(x_{n-1}, x_n)) - d(A, B)) \\ &\leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned} \tag{6}$$

By 2 and 6, We derive

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) &= \tau + F(d(Tx_{n-1}, Tx_n)) \\ &\leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$, then

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1}))$$

which is a contradiction and hence for any $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{7}$$

Consequently the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below. We are going to show that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.
From 7, we deduce that

$$F(d(x_n, x_{n+1})) < F(d(x_n, x_{n-1})) - \tau, \quad \text{for all} \quad n \in \mathbb{N}$$

By repeating this process for any n, we get

$$F(d(x_n, x_{n+1})) < F(d(x_0, x_1)) - n\tau \tag{8}$$

by tending n to infinity, it follows that $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ and so

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

We claim that $\{x_n\}$ is cauchy. By assumption $F3$ in Definition 2.3 and 9, for some $k \in (0, 1)$,

$$\lim_{n\to\infty} d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) = 0.$$

Note that by 8,

$$d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) \le d(x_n, x_{n+1})^k F(d(x_0, x_1) - n\tau).$$

If $n \to \infty$, we get $nd(x_n, x_{n+1})^k = 0$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \frac{1}{n^{\frac{1}{k}}}$ for any $n > n_0$. If $m > n > n_0$, then

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{n\ge n_0} \frac{1}{n^{\frac{1}{k}}}.$$

So $\lim_{n\to\infty} d(x_n, x_m) = 0$. From the completeness of X and closedness of A, there exists $x^* \in A$ such that $x_n \to x^*$. Consider

$$\begin{aligned} d(x^*, Tx^*) &\le d(x^*, Tx_n) + d(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + d(A, B) + d(Tx_n, Tx^*). \end{aligned}$$

By taking $n \to \infty$,

$$d^*(x^*, Tx^*) \le \limsup_{n\to\infty} d(Tx_n, Tx^*). \tag{10}$$

On the other hand, $d(x_n, Tx_n) \le d(x_n, x_{n+1}) + d(A, B)$, then

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B).$$

Since $M(x_n, x^*) = \max\{d(x^*, Tx^*), d(x_n, Tx_n), d(x_n, x^*)\}$, we deduce easily

$$\lim_{n\to\infty} M(x_n, x^*) = d(x^*, Tx^*). \tag{11}$$

By using the triangular inequality

$$\begin{aligned} &d^*(x_n, Tx_n) = d(x_n, Tx_n) - d(A, B) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B) \\ &= d(x_n, x_{n+1}), \end{aligned}$$

$$\begin{aligned} &d^*(x_{n+1}, Tx_{n+1}) = d(x_{n+1}, Tx_{n+1}) - d(A, B) \\ &\le d(x_{n+1}, Tx_n) + d(Tx_{n+1}, Tx_n) - d(A, B) \\ &= d(Tx_n, Tx_{n+1}) = d(x_{n+1}, x_{n+2}) \\ &< d(x_n, x_{n+1}). \end{aligned}$$

Above inequalities imply that

$$\frac{1}{2}(d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})) \leq d(x_n, x_{n+1}).$$

Suppose that the following inequalities hold for some $n \in \mathbb{N}$:

$$\frac{1}{2}d^*(x_n, Tx_n) > d(x_n, x^*), \quad \frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, x^*).$$

Then $d(x_n, x_{n+1}) \leq d(x_n, x^*) + d(x_{n+1}, x^*) < d(x_n, x_{n+1})$ which is a contradiction. Then for any $n \in \mathbb{N} \cup \{0\}$, either

$$\frac{1}{2}d^*(x_n, Tx_n) \quad \leq \quad d(x_n, x^*) \quad \text{or} \tag{12}$$

$$\frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) \quad \leq \quad d(x_{n+1}, x^*). \tag{13}$$

If 12 holds, then

$$\tau + F(d(Tx_n, Tx^*)) \leq F(M(x_n, x^*) - d(A, B) + Ld^*(Tx_{n+1}, x^*)). \tag{14}$$

In addition

$$\begin{aligned} d^*(Tx_{n+1}, x^*) &= d(Tx_{n+1}, x^*) - d(A, B) \\ &\leq d(Tx_{n+1}, x_{n+2}) + d(x_{n+2}, x^*) - d(A, B). \end{aligned}$$

We conclude that $\lim_{n\to\infty} d^*(Tx_{n+1}, x^*) = 0$. Then by 10, 11 and relation 14 and that F is increasing, we deduce that

$$\begin{aligned} &\tau + F(d^*(x^*, Tx^*)) \leq \tau + \limsup_{n\to\infty} F(d(Tx_n, Tx^*)) \\ &\leq \limsup_{n\to\infty} F(M(x_n, x^*) - d(A, B) + Ld^*(Tx_{n+1}, x^*)). \end{aligned} \tag{15}$$

Put $d^*(x^*, Tx^*) = \alpha$. Note that the sequence $M(x_n, x^*) - d(A, B) + Ld^*(Tx_{n+1}, x^*)$ tends to α from right. If $\alpha > 0$, then by taking $n \to \infty$ in 14, we obtain $\tau + F(\alpha) \leq F(\alpha^+)$, that is, $F(\alpha^+) - F(\alpha) \geq \tau$ which is a contradiction with the assumption. So $\alpha = 0$, that is $d(x^*, Tx^*) = d(A, B)$. If 13 holds, the proof is similar.

Theorem 2.10. *In addition to the hypothesis of Theorem 2.9, assume that T also satisfies the following condition: there exist $G \in \digamma$ and some $l \geq 0$, $\tau > 0$ such that for all $x, y \in A$ with $Tx \neq Ty$,*

$$\tau + G(d(Tx, Ty)) \leq G(d(x, y) + ld^*(x, Tx)),$$

then best proximity point of T is unique.

Proof. Let x_0, y_0 be disjoint best proximity points of T. Then by property P, we obtain $d(x_0, y_0) = d(Tx_0, Ty_0)$. So $Tx_0 \neq Ty_0$,

$$\tau + G(d(x_0, y_0)) = \tau + G(d(Tx_0, Ty_0)) \leq G(d(x_0, y_0)),$$

which is a contradiction. Hence best proximity point of T is unique.

Example 2.11. *Let* $X = [0,4]$, $A = [0,1] \cup [3,4]$ *and* $B = [\frac{3}{2}, 2]$. *Define*

$$Tx = \begin{cases} \frac{3}{2}, & x \in [0,1] \\ 2, & x \in [3,4] \end{cases}$$

One can consider $d(A,B) = \frac{1}{2}$, $A_0 = \{1\}$ *and* $B_0 = \{\frac{3}{2}\}$. *We show that* T *satisfies in Theorem 2.9. First let* $x \in [0,1]$ *and* $y \in [3,4]$. *we can observe that*

$$\frac{1}{2} d^*(x, Tx) \le d(x,y)$$

for any $x \in [0,1]$ *and* $y \in [3,4]$ *and also for any* $x \in [3,4]$ *and* $y \in [0,1]$. *Now, we are going to prove that* T *satisfies in contraction condition 1 For this, take* $x \in [0,1]$ *and* $y \in [3,4]$. *Let* $F \in \digamma$ *is continuous. Replacing the values of* x, y *in relation 1, we have*

$$\begin{aligned} \tau + F(\frac{1}{2}) &\le F(\frac{3}{2}) \le (F(\max\{\frac{3}{2} - x, y-2, y-x\} - \frac{1}{2} \\ &+ L(y - \frac{3}{2})) \end{aligned} \tag{16}$$

Note that when $x \in [3,4]$ *and* $y \in [0,1]$, *then we obtain*

$$\tau + F(\frac{1}{2}) \le F(\max\{x-2, \frac{3}{2} - y, x-y\} - \frac{1}{2} + L(2-y)).$$

Put $L = 0$ *and* $\tau = F(1) - F(\frac{1}{2})$. *Then all condition of Theorem 2.9 are satisfied. It is easy to see that* T *satisfies in Theorem 2.10 with* $F = G$. *Note that* $x = 1$ *is the unique best proximity point of* T.

Theorem 2.12. *Let* (A,B) *be a pair of nonempty closed subsets of a complete metric space* X. *Also assume that* $T\colon A \to B$ *is a generalized almost-*F*-contraction satisfying:*

- **(i):** *the pair* (A,B) *has property* P,
- **(ii):** $T(A_0) \subseteq B_0$,
- **(iii):** F *is continuous.*

Then T *has a best proximity point.*

PROOF. Since F is continuous, then F satisfies in assumption (iii) of Theorem 2.9. So by Theorem 2.9, T has a best proximity point.

Theorem 2.13. *Let* (X,d) *be a complete metric space and* $T\colon X \to X$ *be a generalized almost-*F*-contraction such that* $F(\alpha^+) - F(\alpha) < \tau$ *for any* $\alpha > 0$. *Then* T *has a fixed point.*

PROOF. Put $A = B = X$ in Theorem 2.9. Then by Theorem 2.9, T has a fixed point.

Corollary 2.14. *Let (X, d) be a complete metric space and $T\colon X \to X$ be a generalized almost-F-contraction. Also assume that F is continuous, Then T has a fixed point.*

Corollary 2.15. *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a F-contraction such that $F(\alpha^+) - F(\alpha) < \tau$ for any $\alpha > 0$. Then T has a fixed point.*

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