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TRANSITIVITY OF THE δ^n-RELATION IN HYPERGROUPS

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Abstract. The δ^n-relation was introduced by Leoreanu-Fotea et. al. [13]. In this article, we introduce the concept of δ^n-heart of a hypergroup and we determine necessary and sufficient conditions for the relation δ^n to be transitive. Moreover, we determine a family $P_\sigma(H)$ of subsets of a hypergroup H and we give sufficient conditions such that the geometric space $(H, P_\sigma(H))$ is strongly transitive and the relation δ^n is transitive.
Key words and Phrases: Geometric spaces, Hypergroup, strongly regular relation.

Abstrak. Konsep relasi-δ^n telah diperkenalkan oleh Leoreanu-Fotea et. al. [13]. Dalam artikel ini, diperkenalkan konsep δ^n-heart dari suatu hipergrup dan ditentukan syarat perlu dan cukup bagi relasi-δ^n yang transitif. Lebih jauh, ditentukan juga suatu famili subset $P_\sigma(H)$ dari suatu hipergrup H dan diberikan syarat cukup bagi *geometric space* $(H, P_\sigma(H))$ yang transitif kuat dan relasi-δ^n yang transitif.

Kata kunci:*Geometric spaces*, Hipergrup, relasi reguler kuat

1. Introduction

The concept of a hyperstructure first was introduced by Marty in [14], and then it studied by many authors, for example see [3, 5, 6, 15, 16]. The notion of fundamental relation on hypergroups was introduced by Koskas [11], and then studied by Corsini [2], Freni [7, 9] and Gutan [10], Vougiouklis [18, 19], Davvaz et. al. [6] and Leoreanu-Fotea et. al. [13]. In [9], Freni firstly proved that the relation β is transitive in every hypergroup. The relation γ and γ^* were firstly introduced and analyzed by Freni [7]. He proved that the relation γ on hypergroup is transitive and $\gamma = \gamma^*$. Also, Freni [8] determined a family $P_\sigma(H)$ of subsets of a hypergroup H such that the geometric space $(H, P_\sigma(H))$ is strongly transitive. Anavariyeh and

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Davvaz [1] used the notion of strongly transitive geometric space on hypermodules. Mirvakili and Davvaz [17] used the notion of strongly transitive geometric space on arbitrary hyperring and obtained new result in this respect.

Let us recall now some basic notions and results of hypergroup theory. A *hyperstructure* is a set H together with a function $\cdot : H \times H \longrightarrow \wp^*(H)$ called hyperoperation, where $\wp^*(H)$ denotes the set of all non-empty subsets of H. If $A, B \subseteq H,\ x \in H$ then we define

$$A \cdot B = \bigcup_{a\in A,\ b\in B} a \cdot b,\ x \cdot B = \{x\} \cdot B,\ A \cdot x = A \cdot \{x\}.$$

The structure $(H, \cdot)$ is called a *semihypergroup* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, and is called a *hypergroup* if it is a semihypergroup and $a \cdot H = H \cdot a = H$ for all $a \in H$. A non-empty subset K of a hypergroup H is called *left invertible* if for all $(a, b) \in H^2$, the implication $y \in K \circ x \Rightarrow x \in K \circ y$ holds. K is *invertible* if K is left and right invertible. Suppose that $(H, \cdot)$ and $(H', \circ)$ are two semihypergroup. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H. We say that f is a *good homomorphism* if for all a and b in H, $f(a \cdot b) = f(a) \circ f(b)$. A non-empty subset K of a hypergroup $(H, \cdot)$ is called a *subhypergroup* if it is a hypergroup, that is for all $k \in K$, $K \cdot k = k \cdot K = K$. A non-empty subset of a hypergroup $(H, \cdot)$ is called a *complete part* of H if the following implication holds:

$$A \cap \prod_{i=1}^{n} x_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A.$$

If $(H, \cdot)$ is a hypergroup and $R \subseteq H \times H$ is an equivalence relation, we set

$$A \bar{\bar{R}} B \Leftrightarrow a\ R\ b,\ \forall a \in A,\ \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H. The relation R is called *strongly regular on the left (on the right)* if $x\ R\ y \Rightarrow a \cdot x\ \bar{\bar{R}}\ a \cdot y$ ($x\ R\ y \Rightarrow x \cdot a\ \bar{\bar{R}}\ y \cdot a$, respectively), for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is strongly regular on the right and on the left. Strongly regular equivalence play in semi-hypergroup theory a role analogous to congruences in semigroup theory. If R is a strongly regular equivalence on a hypergroup H, then we can define a binary operation $\otimes$ on the quotient set H/R such that $(H/R, \otimes)$ is a group.

Definition 1.1. *(See* [13]*) For any natural number n, we define the relation δ^n on the hypergroup $(H, \cdot)$, as follows: $\delta^n = \cup_{m\geq 1}\delta^n_m$, where for every integer $m \geq 1$, δ^n_m is the relation defined as follows:*

$$x\delta^n_m y \Leftrightarrow \exists (x_1, \cdots, x_m) \in H^m,\quad \exists \tau \in \mathbb{S}_m,$$

$$x \in \prod_{i=1}^{n} x_i,\ y \in \prod_{i=1}^{n} x_{\tau(i)}^{j_{\tau(i)}} \quad or \quad y \in \prod_{i=1}^{n} x_{\tau(i)},\ x \in \prod_{i=1}^{n} x_i^{j_i}$$

where $\forall i \in \{1, 2, \cdots, m\}, j_i \in \{1, n+1\}$ and $x_i^{j_i} = x_i \cdot x_i \cdot \cdots \cdot x_i$, ($j_i$ times).

Denote by δ^{n*} the transitive closure of δ^n. The relation δ^{n*} is a strongly regular relation. The relation δ^{n*} is the smallest equivalence relation on hypergroup

H, such that the quotient H/δ^{n*} is an abelian group. Moreover, for all $x \in H$, $[\delta^{n*}(x)]^{n+1} = \delta^{n*}(x)$ hold, which means that $[\delta^{n*}(x)]^n = e$, the identity of the abelian group H/δ^{n*}.

Moreover, we recall the following relation on H, which is included in δ^n_m:

$$x\rho^n_m y \Leftrightarrow \exists (x_1, \cdots, x_m) \in H^m: \ x \in \prod_{i=1}^{n} x_i, \ y \in \prod_{i=1}^{n} x_i^{j_i} \quad or \quad y \in \prod_{i=1}^{n} x_i, \ x \in \prod_{i=1}^{n} x_i^{j_i}$$

where $\forall i \in \{1, 2, \cdots, m\}, j_i \in \{1, n+1\}$ and $x_i^{j_i} = x_i \cdot x_i \cdot \cdots \cdot x_i$, ($j_i$ times).

Set $\rho^n = \bigcup_{m \in \mathbb{N}^*} \rho^n_m$ and let ρ^{n*} be the transitive closure of ρ^n. The relation ρ^{n*} is the smallest equivalence relation on hypergroup H, such that the quotient H/ρ^{n*} is a group and $[\rho^{n*}(x)]^n = e$ which is the identity of the group H/ρ^{n*}.

Example 1.2. *(See [13]) Let $n = 2$ and $H = \mathbb{S}_3 \times \mathbb{S}_3$, where $\mathbb{S}_3$ be the permutation group of order 3, i.e.,*

$$\mathbb{S}_3 = \{(1), (12), (13), (23), (123), (132)\}.$$

Then $H/\delta^{n*} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

2. Transitivity conditions of δ^n

Definition 2.1. *Let M be a non-empty subset of H. Then, we say that M is a δ^n-part of H if for every $m \in \mathbb{N}$, $(x_1, \cdots, x_m) \in H^m$ and for every $\sigma \in \mathbb{S}_m$ and $j_i \in \{1, n+1\}$, if $\left(\prod_{i=1}^m x_i \bigcup \prod_{i=1}^m x_i^{j_i}\right) \bigcap M \neq \emptyset$ implies that*

- (F_1) $\prod_{i=1}^m x_i \cap M \neq \emptyset \Rightarrow \ \prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}} \subseteq M$,
- (F_2) $\prod_{i=1}^m x_i^{j_i} \cap M \neq \emptyset \Rightarrow \ \prod_{i=1}^m x_{\sigma(i)} \subseteq M$.

Proposition 2.2. *Let M be a non-empty subset of a hypergroup H. Then, the following conditions are equivalent:*

1. *M is a δ^n-part of H.*
2. *$x \in M, x\delta^n y \Rightarrow y \in M$.*
3. *$x \in M, x\delta^{n*} y \Rightarrow y \in M$.*

Proof. ($1 \Rightarrow 2$) If $(x, y) \in H^2$ is a pair such that $x \in M$ and $x\delta^n y$, then there exist $m \in \mathbb{N}^*, (z_1, z_2, \cdots, z_m) \in H$ and $\sigma \in S_m$ such that (i) $x \in \prod_{i=1}^m x_i, \ y \in \prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}}$ or (ii) $x \in \prod_{i=1}^m x_i^{j_i}, \ y \in \prod_{i=1}^m x_{\sigma(i)}$, where $j_i \in \{1, n+1\}$. Since M is a δ^n-part of H, if $x \in \prod_{i=1}^m x_i \bigcap M$, then we get $\prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}} \subseteq M$ by Definition 2.1(F_1). Thus $y \in M$. If $x \in \prod_{i=1}^m x_i^{j_i} \bigcap M$, then we have $\prod_{i=1}^m x_{\sigma(i)} \subseteq M$ by Definition 2.1(F_2). Thus $y \in M$.

($2 \Rightarrow 3$) Let $(x, y) \in H$ such that $x \in M$ and $x\delta^{n*} y$. Obviously, there exist $k \in \mathbb{N}$ and $(w_0 = x, w_1, \cdots, w_{k-1}, w_k = y) \in H^k$ such that $x = w_0 \delta^n w_1 \delta^n \cdots \delta^n w_{k-1}, w_k = y$. Since $x \in M$, applying (2) k-times, we obtain $y \in M$.

($3 \Rightarrow 1$) Let $(\prod_{i=1}^m x_i \bigcup \prod_{i=1}^m x_i^{j_i}) \bigcap M \neq \emptyset$ and $x \in (\prod_{i=1}^m x_i \bigcup \prod_{i=1}^m x_i^{j_i}) \bigcap M$. If $x \in \prod_{i=1}^m x_i$ then for every $\sigma \in S_m$ and for every $y \in \prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}}$ where

$j_i \in \{1, n+1\}$, we have $x\delta^n y$, thus $x \in M$ and $x\delta^{n*}y$. We obtain $y \in M$ by (3), whence $\prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}} \subset M$. If $x \in \prod_{i=1}^m x_i^{j_i}$ then for every $\sigma \in S_m$ and for every $y \in \prod_{i=1}^m x_{\sigma(i)}$ where $j_i \in \{1, n+1\}$, we have $x\delta^n y$, thus $x \in M$ and $x\delta^{n*}y$. So $y \in M$ by (3), whence $\prod_{i=1}^m x_{\sigma(i)} \subset M$. □

Definition 2.3. *The intersection of all δ^n-parts which contain M is called δ^n-closure of M in H and it will be denoted by $K(M)$.*

Before proving the next theorem, we introduce the following notations:

For every element x of a hypergroup H, set:
$T_m(x) = \bigcup \left\{ \prod_{i=1}^m x_{\sigma(i)} | \ \sigma \in \mathbb{S}_m, j_i = \{1, n+1\}, x \in \prod_{i=1}^m x_i^{j_i} \right\}$;
$P_m(x) = \bigcup \left\{ \prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}} \ | \ \sigma \in \mathbb{S}_m, j_i = \{1, n+1\}, x \in \prod_{i=1}^m x_i \right\}$
$P_\sigma(x) = \bigcup_{m \geq 1} (T_m(x) \cup P_m(x))$.

For the preceding notations and definitions, it follows at once the following:

Lemma 2.4. *For every $x \in H$, $P_\sigma(x) = \{y \in H \mid x \ \delta^n \ y\}$.*

Proof. For every pair (x, y) of elements of H we have:

$$x\delta^n y \Leftrightarrow \exists (x_1, \cdots, x_m) \in H^m, \ \ \exists \sigma \in \mathbb{S}_m, x \in \prod_{i=1}^n x_i, \ y \in \prod_{i=1}^n x_{\tau(i)}^{j_{\sigma(i)}}$$

$$\text{or } y \in \textstyle\prod_{i=1}^n x_{\sigma(i)}, \ x \in \prod_{i=1}^n x_i^{j_i} \Leftrightarrow \exists m \in \mathbb{N}^* : y \in P_m(x)$$

$$\text{or } y \in T_m(x) \Leftrightarrow y \in P_\sigma(x).$$

□

Lemma 2.5. *Let $(H, \circ)$ be a hypergroup and let M be a δ^{n*}-part of H. If $x \in M$, then $P_\sigma(x) \subseteq M$.*

Proof. If $y \in P_\sigma(x)$, then $x\delta^n y$. Thus there exists $m \geq 1$ such that $x\delta_m^n y$, whence there exists $(x_1, x_2, \cdots, x_m) \in H^m$ and $\sigma \in S_m$, such that (i) $x \in \prod_{i=1}^m x_i$, $y \in \prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}}$ or (ii) $x \in \prod_{i=1}^m x_i^{j_i}$, $y \in \prod_{i=1}^m x_{\sigma(i)}$, where $j_i \in \{1, n+1\}$. If (i) holds, since $x \in \prod_{i=1}^m x_i \bigcap M$ and M is a δ^{n*}-part, it follows that $y \in \prod_{i=1}^m x_{\sigma(i)}^{j_{\sigma(i)}} \subseteq M$ by Definition 2.1(F_1), and thus $y \in M$. If (ii) holds, since $x \in \prod_{i=1}^m x_i^{j_i} \bigcap M$ and M is a δ^{n*}-part, it follows that $y \in \prod_{i=1}^m x_{\sigma(i)} \subseteq M$ by Definition 2.1(F_2), and so $y \in M$. Therefore, in any case we have $P_\sigma(x) \subseteq M$. □

Theorem 2.6. *Let H be a hypergroup. The following conditions are equivalent:*

(1) *δ^n is transitive;*
(2) *for every $x \in H$, $\delta^{n*}(x) = P_\sigma(x)$;*
(3) *for every $x \in H$, $P_\sigma(x)$ is a δ^n-part of H.*

Proof. $(1 \Rightarrow 2)$ By Lemma 2.5, for every pair (x, y) of elements of H we have:

$$y \in \delta^{n*}(x) \Leftrightarrow x\delta^{n*}y \Leftrightarrow x\delta^n y \Leftrightarrow y \in P_\sigma(x).$$

$(2 \Rightarrow 3)$ By Proposition 2.2, if M is a non-empty subset of H, then M is a δ^n-part of H if and only if it is union of equivalence classes modulo δ^{n*}. Particularly, every equivalence class modulo δ^{n*} is a δ^n-part of H.

$(3 \Rightarrow 1)$ Let $x\delta^n y$ and $y\delta^n z$. Thus, $x \in P_\sigma(y)$ and $y \in P_\sigma(z)$ by Lemma 2.4. Since $P_\sigma(z)$ is a δ^{n*}-part, by Lemma 2.5, we have $P_\sigma(y) \subseteq P_\sigma(z)$ and hence $x \in P_\sigma(z)$. Therefore, $x\delta^n y$ by Lemma 2.4 and the proof is complete. □

Definition 2.7. *Let $(H, \circ)$ be a hypergroup and $\phi : H \longrightarrow H/\delta^n$ be the canonical projection. We denote by $e = [\delta^{n*}(x)]^n$ for all $x \in H$ the identity of the group H/δ^n. The set $\phi^{-1}(e)$ is called the δ^n-heart of H and it is denoted by D_{δ^n}.*

Theorem 2.8. *D_{δ^n} is the smallest subhypergroup of H, which is also a δ^n-part of H.*

Proposition 2.9. *For every non-empty subset M of a hypergroup H, we have:*

(1) *$\varphi^{-1}(\varphi(M)) = D(H)M = MD(H)$;*
(2) *M is a δ^n-part if and only if $\varphi^{-1}(\varphi(M)) = M$.*

Proof. 1) For every $x \in D(H)M$, there exists a pair $(a, b) \in D(H) \times M$ such that $x \in ab$. Then $\varphi(x) = \varphi(a) \otimes \varphi(b) = e \otimes \varphi(b) = \varphi(b)$. Therefore $x \in \varphi^{-1}(\varphi(b)) \subset \varphi^{-1}(\varphi(M))$.

Conversely, for every $x \in \varphi^{-1}(\varphi(M))$, there exists an element $a \in M$ such that $\varphi(x) = \varphi(a)$. By the reproducibility, $b \in H$ exists such that $x \in ba$, so $\varphi(a) = \varphi(x) = \varphi(b) \otimes \varphi(a)$, hence $\varphi(b) = e$ and $a \in \varphi^{-1}(e) = D(H)$. Therefore $x \in ba \subset D(H)M$. This proves that $\varphi^{-1}(\varphi(M)) = D(H)M$.

In the same way, we can prove that $\varphi^{-1}(\varphi(M)) = MD(H)$.

For the proof of the sufficiency suppose that $m\delta^{n*}x$ and $m \in M$. Thus $\varphi(x) = \varphi(m) \in \varphi(M)$ and so $x \in \varphi^{-1}(\varphi(M)) = M$. Therefore by Proposition 2.2 it follows that M is a δ^n-part of H. □

Definition 2.10. *Let z be some element of H. A hypergroup H is called δ^{n*}-strong whenever*

(i) *For all $x, y \in H$ if $x\delta^{n*}y$, then $xz \cap yz \neq \emptyset$ and $zx \cap zy \neq \emptyset$ and*
(ii) *$\{z\}$ is invertible.*

Theorem 2.11. *If H is a δ^{n*}-strong hypergroup for some element $z \in H$, then δ^n is transitive.*

Proof. By Theorem 2.6, it is enough to show that for all $x \in H, P(x)$ is a δ^{n*}-part of H. According to Proposition 2.9, we have to check that $\varphi^{-1}(\varphi(P(x))) = P(x)$. Let $t \in \varphi^{-1}(\varphi(P(x)))$, thus there exists $h \in P(x)$ such that $\varphi(t) = \varphi(h)$ and hence $\delta^{n*}(t) = \delta^{n*}(h)$. Since $h \in P(x)$, $h\delta^n x$ by Lemma 2.4. Thus $\delta^{n*}(x) = \delta^{n*}(h)$ and so $\delta^{n*}(t) = \delta^{n*}(x)$. Since H is a δ^{n*}-strong hypergroup, we have $xz \cap tz \neq \emptyset$ and hence there exists $s \in xz \cap tz$. Therefore $x \in tzz$ and $t \in xzz$, because $\{z\}$ is invertible and so $t \in tzzzz$. Since

$$(tzz, t\underbrace{zz\cdots z}_{j_2 \text{ times}}\underbrace{zz\cdots z}_{j_3 \text{ times}}) \in \delta^n$$

where $j_1 = 1, j_2 = n+1$ and $j_3 = n+1$, we have $x\delta^n t$ and hence $t \in P(x)$. So we have $\varphi^{-1}(\varphi(P(x))) \subseteq P(x)$; it is obvious that $P(x) \subseteq \varphi^{-1}(\varphi(P(x)))$. Therefore $\varphi^{-1}(\varphi(P(x))) = P(x)$ and the proof is complete. □

3. Strongly transitive geometric spaces associated to hypergroups

According to [8], a geometric space is a pair $(S, \mathcal{B})$ such that S is a non-empty set, whose elements we call points, and $\mathcal{B}$ is a non-empty family of subsets of S, whose elements we call blocks. $\mathcal{B}$ is a covering of S if for every point $y \in S$, there exists a block $B \in \mathcal{B}$ such that $y \in B$. If C is a subset of S, we say that C is a $\mathcal{B}$-part or $\mathcal{B}$-subset of S if for every $B \in \mathcal{B}$,

$$B \cap C \neq \emptyset \Rightarrow B \subseteq C.$$

If $B_1, B_2, \cdots, B_n$ are n blocks of geometric space $(S, \mathcal{B})$ such that $B_i \cap B_{i+1} \neq \emptyset$, for any $i \in \{1, 2, \cdots, n-1\}$, then the n-tuple $B_1, B_2, \cdots, B_n$ is called a polygonal of $(S, \mathcal{B})$. The concept of polygonal allows us to define on S the following relation.

$$x \approx y \Leftrightarrow x = y \text{ or a polygonal } (B_1, B_2, \cdots, B_n) \text{ exists such that } x \in B_1 \text{ and } y \in B_n.$$

The relation $\approx$ is an equivalence and it is easy to see that it coincides with the transitive closure of the following relation:

$$x \approx y \Leftrightarrow x = y \text{ or there exists } B \in \mathcal{B} \text{ such that } \{x, y\} \in B,$$

so $\approx$ is equal to $\bigcup_{n \geq 1} \sim^n$, where $\sim^n = \sim \circ \sim \circ \cdots \circ \sim$ n times.

Theorem 3.1. [8] *For every pair (A, B) of blocks of a geometric space $(S, \mathcal{B})$ and for any integer $n \in \mathbb{N}$, the following conditions are equivalent:*

(1) $A \cap B \neq \emptyset, x \in B \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subseteq C.$
(2) $A \cap B \neq \emptyset, x \in \Gamma(B) \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subseteq C.$
(3) $A \cap \Gamma(B) \neq \emptyset, x \in \Gamma(B) \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subseteq C.$

Theorem 3.2. *If $(S, \mathcal{B})$ is a strongly transitive geometric space, then the relation $\sim$ on S is transitive. Hence $\approx = \sim$.*

Let H be a hypergroup and let $P_\delta(H)$ be the family of subsets of H defined as follows: for every integer $m \geqslant 1$ and for every m-tuple $(z_1, z_2, \cdots, z_m) \in H^m$, we set

(1) $B_\delta(z_1) = \{z_1, z_1^{n+1}\}$.
(2) $B_\delta(z_1, z_2, \cdots, z_m) = \bigcup \left\{ \prod_{i=1}^m z_{\tau(i)}^{j_{\tau(i)}} \mid \tau \in S_m, j_i \in \{1, n+1\} \right\}$, if $m \geq 2$.

where S_m is the symmetric group of all permutations of the set $\{1, 2, \cdots, m\}$.

Also, we can consider another geometric space $(H, P_\rho(H))$ that defined as follows: for every integer $m \geqslant 1$ and for every m-tuple $(z_1, z_2, \cdots, z_m) \in H^m$, we set

(1) $B_\rho(z_1) = \{z_1, z_1^{n+1}\}$.
(2) $B_\rho(z_1, z_2, \cdots, z_m) = \bigcup \left\{ \prod_{i=1}^{m} z_i^{j_i} \mid j_i \in \{1, n+1\} \right\}$, if $m \geq 2$.

Note that if $z_1 = z_2 = \cdots = z_m = z$ then

$$B_\delta(z_1, z_2, \cdots, z_m) = B_\rho(z_1, z_2, \cdots, z_m) = \bigcup \left\{ z^{j_1+j_2+\cdots+j_m} \mid j_i \in \{1, n+1\} \right\} = B_m^z$$

Corollary 3.3. *If for all $x \in H, x^{n+1} = x$ then*

$$B_\delta(z_1, z_2, \cdots, z_m) = \bigcup \left\{ \prod_{i=1}^{m} z_{\tau(i)} \mid \tau \in S_m \right\}$$

Corollary 3.4. *If $(H, \circ)$ is a commutative hypergroup then two geometric spaces $(H, P_\rho(H))$ and $(H, P_\delta(H))$ are equal.*

Lemma 3.5. *Let $(H, \circ)$ be a hypergroup. Then*

(1) $B_\rho(z_1, z_2, \cdots, z_m) \subseteq B_\delta(z_1, z_2, \cdots, z_m)$.
(2) $B_\delta(z_1, z_2, \cdots, z_m) = \bigcup \left\{ B_\rho(z_{\tau(1)}, z_{\tau(2)}, \cdots, z_{\tau(m)}) \mid \tau \in S_m \right\}$.

Proof. It is straightforward. □

Lemma 3.6. *If $(z_1, z_2, \cdots, z_m)$ is a m-tuple of elements of a hypergroup $(H, \circ)$, Then:*

(1) *For every $\sigma \in S_m$ we have*
$B_\delta(z_1, z_2, \cdots, z_m) = B_\delta(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(m)})$.
(2) *For every $z \in H$, we have*
$[B_\delta(z_1, z_2, \cdots, z_m)] \circ z \subset B_\delta(z_1, z_2, \cdots, z_m, z)$.
$z \circ [B_\delta(z_1, z_2, \cdots, z_m)] \subset B_\delta(z, z_1, z_2, \cdots, z_m)$.
(3) *For every $(m+k)$-tuple of elements of a hypergroup $(H, \circ)$, we have*
$B_\delta(z_1, z_2, \cdots, z_m) \circ B_\delta(x_1, x_2, \cdots, x_k) \subset B_\delta(z_1, z_2, \cdots z_m, x_1, x_2, \cdots, x_k)$

Proof. (1) For every permutation $\sigma \in S_m$, we have

$$\begin{aligned} x \in B_\delta(z_{\sigma_{(1)}}, z_{\sigma_{(2)}}, \cdots, z_{\sigma_{(m)}}) &\Leftrightarrow \exists \tau \in S_m : x \in \textstyle\prod_{i=1}^{m} z_{\tau(\sigma(i))}^{j_{\tau(\sigma(i))}} \\ &\Leftrightarrow \exists \tau \in S_m : x \in \textstyle\prod_{i=1}^{m} z_{\tau\circ\sigma(i)}^{j_{\tau\circ\sigma(i)}} \\ &\Leftrightarrow x \in B_\delta(z_1, z_2, \cdots, z_m). \end{aligned}$$

(2) If $w \in [B_\delta(z_1, z_2, \cdots, z_m)] \circ z$, then an element $y \in B_\delta(z_1, z_2, \cdots, z_m)$ and a $\tau \in S_m$ exist such that $w \in y \circ z$ and $y \in \prod_{i=1}^{m} z_{\tau(i)}^{j_{\tau(i)}}$. Setting $z = z_{m+1}$ and $j_{m+1} = 1$, if σ is the permutation of S_{m+1} such that:

$$\begin{cases} \sigma(i) = \tau(i), & \forall i \in \{1, 2, \cdots, m\}; \\ \sigma(m+1) = m+1 \end{cases}$$

we have $w \in \left(\prod_{i=1}^{m} z_{\sigma(i)}^{j_{\sigma(i)}}\right) \circ z_{m+1} = \prod_{i=1}^{m+1} z_{\sigma(i)}^{j_{\sigma(i)}}$.

(3) For every $y \in B_\delta(z_1, z_2, \cdots, z_m) \circ B_\delta(x_1, x_2, \cdots, x_k)$, there exist elements $a \in B_\delta(z_1, z_2, \cdots, z_m)$ and $b \in B_\delta(x_1, x_2, \cdots, x_k)$ such that $y \in a \circ b$. If $a \in B_\delta(z_1, z_2, \cdots, z_m)$, a permutation $\sigma \in S_m$ exists such that $a \in \prod_{i=1}^{m} z_{\sigma(i)}^{j_{\sigma(i)}}$ where $j_i \in \{1, n+1\}$ and if $b \in B_\delta(x_1, x_2, \cdots, x_k)$, a permutation $\theta \in S_k$ exists such that $b \in \prod_{i=1}^{k} x_{\theta(i)}^{j_{\theta(i)}}$ where $j_i \in \{1, n+1\}$. Thus

$$y \in a \circ b \subset \left(\prod_{i=1}^{m} z_{\sigma(i)}^{j_{\sigma(i)}}\right) \circ \left(\prod_{i=1}^{k} x_{\theta(i)}^{j_{\theta(i)}}\right).$$

Supposing that $x_1 = z_{m+1}, x_2 = z_{m+2}, \cdots, x_k = z_{m+k}$, a permutation $\tau \in S_{m+k}$ exists such that

$$y \in \left(\prod_{i=1}^{m} z_{\sigma(i)}^{j_{\sigma(i)}}\right) \circ \left(\prod_{i=1}^{k} x_{\theta(i)}^{j_{\theta(i)}}\right) = \prod_{i=1}^{m+k} z_{\tau(i)}^{j_{\tau(i)}},$$

thus

$$y \in B_\delta(z_1, z_2, \cdots, z_m, z_{m+1}, \cdots, z_{m+k}) = B_\delta(z_1, z_2, \cdots, z_m, x_1, x_2, \cdots, x_k).$$

Notice that the permutation τ is defined as follows:

$$\begin{cases} \tau(i) = \sigma(i), & \text{if } 1 \leq i \leq m; \\ \tau(i) = \theta(i). & \text{if } m+1 \leq i \leq m+k \end{cases}$$

□

Lemma 3.7. *Let $(H, \circ)$ be a hypergroup. Then*

(1) *If $z_k \in a \cdot b$ then*

$$B_\delta(z_1, z_2, \cdots, z_m) \subseteq B_\delta(z_1, z_2, \cdots, z_{k-1}, \underbrace{a, a, \cdots, a}_{j_k \textit{ times}}, \underbrace{b, b, \cdots, b}_{j_k \textit{ times}}, z_{k+1} \cdots, z_m).$$

(2) *If $z_k^{j_k} \subset a^{j_k} \cdot b^{j_k}$ then*

$$B_\delta(z_1, z_2, \cdots, z_m) \subseteq B_\delta(z_1, z_2, \cdots, z_{k-1}, a, b, z_{k+1} \cdots, z_m).$$

Proof. (1) Let $z_k \in a \cdot b$ and $y \in B_\delta(z_1, z_2, \cdots, z_m)$. Then there exists $\tau \in S_m$ such that $y \in \prod_{i=1}^{m} z_{\tau(i)}^{j_{\tau(i)}}$ and $j_i \in \{1, n+1\}$. Setting $\tau(h) = k$, we have

$$y \in \prod_{i=1}^{m} z_{\tau(i)}^{j_{\tau(i)}} = \prod_{i=1}^{h-1} z_{\tau(i)}^{j_{\tau(i)}} \circ z_k^{j_k} \circ \prod_{i=h+1}^{m} z_{\tau(i)}^{j_{\tau(i)}} \subset \prod_{i=1}^{h-1} z_{\tau(i)}^{j_{\tau(i)}} \circ (a \circ b)^{j_k} \circ \prod_{i=h+1}^{m} z_{\tau(i)}^{j_{\tau(i)}}$$

$$\subset \prod_{i=1}^{h-1} z_{\tau(i)}^{j_{\tau(i)}} \circ \underbrace{(a \circ b) \circ (a \circ b) \circ \cdots \circ (a \circ b)}_{j_k \text{ times}} \circ \prod_{h+1}^{m} z_{\tau(i)}^{j_{\tau(i)}}.$$

Setting that $z'_k = a$ and $z'_{m+1} = b, z'_{m+2} = a, z'_{m+3} = b, z'_{m+3} = a, \cdots, z'_{m+2j_k-2} = a, z'_{m+2j_k-1} = b$, and a permutation $\sigma \in S_{m+1}$ exists such that

$$y \in \left(\prod_{i=1}^{h-1} z_{\sigma(i)}^{j_{\sigma(i)}}\right) \circ z'_{\sigma(h)} \circ z'_{\sigma(h+1)} \circ z'_{\sigma(h+2)} \circ \cdots \circ z'_{\sigma(h+2j_i-1)} \circ \left(\prod_{i=h+2j_i}^{m+2j_i-1} z_{\sigma(i)}^{j_{\sigma(i)}}\right),$$

thus $y \in B_\delta(z_1, \cdots, z_{k-1}, z'_k, z_{k+1}, \cdots, z_m, z'_{m+1}, z'_{m+2}, \cdots, z'_{m+2j_i-1})$. Moreover,

$$\begin{aligned} &B_\delta(z_1, \cdots, z_{k-1}, z'_k, z_{k+1}, \cdots, z_m, z'_{m+1}, z'_{m+2}, \cdots, z'_{m+2j_i-1}) \\ &\quad = B_\delta(z_1, \cdots, z_{k-1}, z'_k, z'_{m+1}, z'_{m+2}, \cdots, z'_{m+2j_i-1}, z_{k+1}, \cdots, z_m) \\ &\quad = B_\delta(z_1, \cdots, z_{k-1}, a, b, a, b, \cdots, a, b, z_{k+1}, \cdots, z_m) \\ &\quad = B_\delta(z_1, \cdots, z_{k-1}, \underbrace{a, a, \cdots, a}_{j_k \text{ times}}, \underbrace{b, b, \cdots, b}_{j_k \text{ times}}, z_{k+1}, \cdots, z_m). \end{aligned}$$

Therefore, we have

$$B_\delta(z_1, z_2, \cdots, z_m) \subseteq B_\delta(z_1, z_2, \cdots, z_{k-1}, \underbrace{a, a, \cdots, a}_{j_k \text{ times}}, \underbrace{b, b, \cdots, b}_{j_k \text{ times}}, z_{k+1} \cdots, z_m).$$

We notice that, if $h = m$ then the permutation σ is defined as follows:

$$\begin{cases} \sigma(i) = \tau(i), \quad \forall i \in \{1, 2, \cdots, m\}; \\ \sigma(m+1) = m+1, \ \sigma(m+2) = m+2, \cdots, \sigma(m+2j_i-1) = m+2j_i-1. \end{cases}$$

while, if $1 \leqslant h < m$, then σ is such that:

$$\begin{cases} \sigma(i) = \tau(i), \quad \text{if } 1 \le i \le h; \\ \sigma(m+1) = m+1; \\ \sigma(i) = \tau(i-2j_i). \quad \text{if } h+1 \le i \le m+2j_i-1. \end{cases}$$

(2) The proof follows the same argument exploited in Lemma 3.1 of [8]. □

Corollary 3.8. *If $(H, \circ)$ is a hypergroup and $z_k \in a \cdot b$ then*

$$B_\delta(z_1, z_2, \cdots, z_m) \subseteq B_\delta(z_1, z_2, \cdots, z_{k-1}, B^a_{j_k}, B^b_{j_k}, z_{k+1} \cdots, z_m).$$

Corollary 3.9. *Let $(z_1, z_2, \cdots, z_m)$ be a m-tuple of elements of a hypergroup $(H, \circ)$. If an integer $k \geq 1$, a k-tuple $(x_1, x_2, \cdots, x_k) \in H^k$ and an element $k' \in \{1, 2, \cdots, m\}$ exist such that $z_{k'}^{j_{k'}} \in B_\delta(x_1, x_2, \cdots, x_k)$, then*

$$B_\delta(z_1, z_2, \cdots, z_m) \subseteq B_\delta(z_1, z_2, \cdots, z_{k'-1}, x_1, x_2, \cdots, x_k, z_{k'+1} \cdots, z_m).$$

Lemma 3.10. *Let $(H, \circ)$ be a commutative hypergroup. If there exists an integer $k \geq 1$, a k-tuple $(x_1, x_2, \cdots, x_k) \in H^m$ and element $k' \in \{1, 2, \cdots, m\}$ such that $z_{k'} \in \prod_{i=1}^k x_i$, then*

$$B_\rho(z_1, z_2, \cdots, z_m) \subseteq B_\rho(z_1, z_2, \cdots, z_{k'-1}, x_1, x_2, \cdots, x_k, z_{k'+1} \cdots, z_m).$$

Theorem 3.11. *If $(H,\circ)$ is a hypergroup and for every $t_k,(x_1,x_2,\cdots,x_{k'})\in H^{k'}$ and $t_k\in\prod_{i=1}^{k'}x_i$ we have $t_k^{j_k}\subset\prod_{i=1}^{k'}x_i^{j_k}$ where $j_k\in\{1,n+1\}$, Then the geometric space $(H,P_\delta(H))$ is strongly transitive.*

Proof. Let $B_\delta(z_1,z_2,\cdots,z_m)$ and $B_\delta(y_1,y_2,\cdots,y_s)$ be two block of $P_\delta(H)$ such that

$$B_\delta(z_1,z_2,\cdots,z_m)\cap B_\delta(y_1,y_2,\cdots,y_s)\neq\emptyset \text{ and } y\in B_\delta(y_1,y_2,\cdots,y_s).$$

Let $b\in B_\delta(z_1,z_2,\cdots,z_m)\cap B_\delta(y_1,y_2,\cdots,y_s)$. A pair $(a,c)\in H$ of elements of H exists such that $z_m\in a\circ y$ and $y\in b\circ c$. Since $y\in B_\delta(y_1,y_2,\cdots,y_s)$, by Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned} y\in b\circ c\subset[B_\delta(z_1,z_2,\cdots,z_m)]\circ c &\subset B_\delta(z_1,z_2,\cdots,z_m,c)\\ &\subset B_\delta(z_1,z_2,\cdots,z_{m-1},a,y,c)\\ &\subset B_\delta(z_1,z_2,\cdots,z_{m-1},a,y_1^{j_y},y_2^{j_y},\cdots,y_s^{j_y},c)\end{aligned}$$

where $j_y\in\{1,n+1\}$.
Moreover, since $b\in B_\delta(y_1,y_2,\cdots,y_s)$, we obtain

$$\begin{aligned} B_\delta(z_1,z_2,\cdots,z_m) &\subset B_\delta(z_1,z_2,\cdots,z_{m-1},a,y)\\ &\subset B_\delta(z_1,z_2,\cdots,z_{m-1},a,b,c)\\ &\subset B_\delta(z_1,z_2,\cdots,z_{m-1},a,y_1^{j_y},y_2^{j_y},\cdots,y_s^{j_y},c).\end{aligned}$$

Therefore $B_\delta(z_1,z_2,\cdots,z_m)\cup\{y\}\subset B_\delta(z_1,z_2,\cdots,z_{m-1},a,y_1^{j_y},y_2^{j_y},\cdots,y_s^{j_y},c)$ and the geometric space $(H,P_\delta(H))$ is strongly transitive. □

Corollary 3.12. *If $(H,\circ)$ is a hypergroup and for every $x\in H,x^{n+1}=x$, then the geometric space $(H,P_\delta(H))$ is strongly transitive.*

Proof. Since for every $x\in H,x^{n+1}=x$, thus by Corollary 3.3 we have

$$B_\delta(z_1,z_2,\cdots,z_m)=\bigcup\left\{\prod_{i=1}^{m}z_{\tau(i)}\mid\tau\in S_m\right\}.$$

Hence the proof follows the same argument exploited in Theorem 3.4 of [8]. □

Theorem 3.13. *If $(H,\circ)$ is a commutative hypergroup, Then the geometric space $(H,P_\rho(H))$ is strongly transitive.*

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