J. Indones. Math. Soc.
Vol. 17, No. 2 (2011), pp. 85–96.

OPTIMAL GENERALIZED LOGARITHMIC MEAN BOUNDS FOR THE GEOMETRIC COMBINATION OF ARITHMETIC AND HARMONIC MEANS

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Abstract. In this paper, we answer the question: for $\alpha \in (0,1)$, what are the greatest value $p = p(\alpha)$ and least value $q = q(\alpha)$, such that the double inequality $L_p(a,b) \le A^{\alpha}(a,b)H^{1-\alpha}(a,b) \le L_q(a,b)$ holds for all a, $b > 0$? where $L_p(a,b)$, $A(a,b)$, and $H(a,b)$ are the p-th generalized logarithmic, arithmetic, and harmonic means of a and b, respectively.

Key words: Generalized logarithmic mean, arithmetic mean, harmonic mean.

Abstrak. Dalam paper ini, kami menjawab pertanyaan: untuk $\alpha \in (0,1)$, berapa nilai terbesar $p = p(\alpha)$ dan nilai terkecil $q = q(\alpha)$, sehingga ketidaksamaan ganda $L_p(a,b) \le A^{\alpha}(a,b)H^{1-\alpha}(a,b) \le L_q(a,b)$ dipenuhi untuk semua a, $b > 0$? dengan $L_p(a,b)$, $A(a,b)$, dan $H(a,b)$ secara berturut-turut adalah rata-rata logaritmik yang diperumum, aritmatik, dan harmonik ke-p dari a and b.

Kata kunci: Rata-rata logaritmik yang diperumum, rata-rata aritmatik, rata-rata harmonik.

2000 Mathematics Subject Classification: 26E60.
Received: 05-05-2011, revised: 12-10-2011, accepted: 12-10-2011.

1. **Introduction**

For $p \in \mathbb{R}$ the generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b is defined by

$$L_p(a,b) = \begin{cases} a, & a = b, \\ \left[\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, & p \neq 0, p \neq -1, a \neq b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0, a \neq b, \\ \frac{b-a}{\ln b - \ln a}, & p = -1, a \neq b. \end{cases}$$

It is well-known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. In the recent past, the generalized logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for L_p can be found in the literature [1, 8, 9, 13, 14, 19, 20, 22, 23]. It might be surprising that the generalized logarithmic mean has applications in economics, physics and even in meteorology [10, 17, 18]. In [10] the authors study a variant of Jensen's functional equation involving L_p, which appear in a heat conduction problem.

Let $A(a,b) = (a+b)/2$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $L(a,b) = (b-a)(\ln b - \ln a)$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then

$$\begin{aligned} \min\{a,b\} &< H(a,b) < G(a,b) = L_{-2}(a,b) < L(a,b) = L_{-1}(a,b) \\ &< I(a,b) = L_0(a,b) < A(a,b) = L_1(a,b) < \max\{a,b\}. \end{aligned}$$

In [7, 11, 21] the authors present bounds for $L(a,b)$ and $I(a,b)$ in terms of $G(a,b)$ and $A(a,b)$.

Theorem 1.1. *For all positive real numbers a and b with $a \neq b$ we have*

$$A^{\frac{1}{3}}(a,b)G^{\frac{2}{3}}(a,b) < L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b)$$

and

$$\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b).$$

The proof of the following Theorem 1.2 can be found in [5].

Theorem 1.2. *For all positive real numbers a and b with $a \neq b$ we have*

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b) + I(a,b)) < \frac{1}{2}(G(a,b) + A(a,b)).$$

For $p \in \mathbb{R}$, the p-th power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

The main properties of these means are given in [5]. Several authors discussed the relationship of certain means to $M_p(a,b)$. The following sharp bounds for L, I, $(IL)^{1/2}$ and $(I+L)/2$ in terms of power means are proved in [2, 3, 6, 12, 15, 16].

Theorem 1.3. *For all positive real numbers a and b with $a \neq b$ we have*

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b), \quad M_{2/3}(a,b) < I(a,b) < M_{\ln 2}(a,b),$$

$$M_0(a,b) < I^{1/2}(a,b)L^{1/2}(a,b) < M_{1/2}(a,b)$$

and

$$\frac{1}{2}[I(a,b) + L(a,b)] < M_{1/2}(a,b).$$

The following Theorems 1.4-1.6 were established by Alzer and Qiu in [4].

Theorem 1.4. *The inequalities*

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b)$$

hold for all positive real numbers a and b with $a \neq b$ if and only if

$$\alpha \leq 2/3 \quad \text{and} \quad \beta \geq 2/e = 0.73575\cdots.$$

Theorem 1.5. *Let a and b be real numbers with $a \neq b$. If $0 < a,\ b \leq e$, then*

$$[G(a,b)]^{A(a,b)} < [L(a,b)]^{I(a,b)} < [A(a,b)]^{G(a,b)}.$$

And, if $a, b \geq e$, then

$$[A(a,b)]^{G(a,b)} < [I(a,b)]^{L(a,b)} < [G(a,b)]^{A(a,b)}.$$

Theorem 1.6. *For all positive real numbers a and b with $a \neq b$ we have*

$$M_c(a,b) < \frac{1}{2}(L(a,b) + I(a,b))$$

with the best possible parameter $c = \ln 2/(1+\ln 2) = 0.40938\cdots$.

It is the aim of this paper to answer the question: for $\alpha \in (0,1)$, what are the greatest value $p = p(\alpha)$ and least value $q = q(\alpha)$, such that the double inequality

$$L_p(a,b) \leq A^{\alpha}(a,b)H^{1-\alpha}(a,b) \leq L_q(a,b)$$

holds for all $a,\ b > 0$?

2. Main Results

Theorem 2.1. *For $\alpha \in (0,1)$ and all $a, b > 0$ we have*

(1) $L_{6\alpha-5}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-1/\alpha}(a,b)$ *for* $\alpha = 1/3$ *or* $\alpha = 1/2$*;*

(2) $L_{-1/\alpha}(a,b) \le A^{\alpha}(a,b)H^{1-\alpha}(a,b) \le L_{6\alpha-5}(a,b)$ *for* $\alpha \in (0,1/3)\bigcup(1/2,1)$ *and* $L_{-1/\alpha}(a,b) \ge A^{\alpha}(a,b)H^{1-\alpha}(a,b) \ge L_{6\alpha-5}(a,b)$ *for* $\alpha \in (1/3,1/2)$*, with equality if and only if* $a = b$*, and the parameters* $-1/\alpha$ *and* $6\alpha - 5$ *in either case are best possible.*

PROOF. (1) If $\alpha = 1/3$ or $\alpha = 1/2$, and $a = b$, then we clearly see that $L_{6\alpha-5}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-1/\alpha}(a,b) = a$.

If $\alpha = 1/3$ and $a \neq b$, then we have

$$L_{6\alpha-5}(a,b) = L_{-1/\alpha}(a,b) = L_{-3}(a,b)$$

$$= \frac{2^{1/3}(ab)^{2/3}}{(a+b)^{1/3}} = A^{1/3}(a,b)H^{2/3}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b).$$

If $\alpha = 1/2$ and $a \neq b$, then we get

$$L_{6\alpha-5}(a,b) = L_{-1/\alpha}(a,b) = L_{-2}(a,b)$$

$$= (ab)^{1/2} = A^{1/2}(a,b)H^{1/2}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b).$$

(2) If $\alpha \in (0,1)$ and $a = b$, then we clearly see that $L_{-1/\alpha}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{6\alpha-5}(a,b)$. Without loss of generality, we assume that $t = a/b > 1$ in the following discussion.

If $\alpha = 2/3$, then one has

$$\begin{aligned} & \ln L_{6\alpha-5}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\ = \; & \ln L_{-1}(a,b) - \ln[A^{2/3}(a,b)H^{1/3}(a,b)] \\ = \; & \ln(\frac{t-1}{\ln t}) - \ln[2^{-1/3}(1+t)^{1/3}t^{1/3}]. \end{aligned} \tag{1}$$

Let $f_1(t) = \ln(\frac{t-1}{\ln t}) - \ln[2^{-1/3}(1+t)^{1/3}t^{1/3}]$, then simple computations lead to

$$\lim_{t\to1^+} f_1(t) = 0, \tag{2}$$

$$f_1'(t) = \frac{g_1(t)}{3t(t-1)(t+1)\ln t}, \tag{3}$$

where $g_1(t) = (t^2 + 4t + 1)\ln t - 3(t^2 - 1)$.

$$g_1(1) = 0, \tag{4}$$

$$g_1'(t) = \frac{h_1(t)}{t}, \tag{5}$$

where $h_1(t) = 2t(t+2)\ln t - 5t^2 + 4t + 1$.

$$\begin{aligned} g_1'(1) &= h_1(1) = 0, && (6)\\ h_1'(t) &= 4(t+1)\ln t - 8(t-1), &&\\ h_1'(1) &= 0, && (7)\\ h_1''(t) &= \frac{4}{t}v_1(t), && (8) \end{aligned}$$

where $v_1(t) = t\ln t - t + 1$.

$$h_1''(1) = v_1(1) = 0 \tag{9}$$

and

$$v_1'(t) = \ln t > 0 \tag{10}$$

for $t > 1$.

From (1)-(10) we know that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha = 2/3$ and $a \neq b$.

If $\alpha = 5/6$, then

$$\begin{aligned} & \ln L_{6\alpha-5}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\ = {} & \ln L_0(a,b) - \ln[A^{5/6}(a,b)H^{1/6}(a,b)] \\ = {} & \ln[\frac{1}{e}t^{\frac{t}{t-1}}] - \ln[2^{-2/3}t^{1/6}(1+t)^{2/3}]. \qquad (11) \end{aligned}$$

Let $f_2(t) = \ln[\frac{1}{e}t^{\frac{t}{t-1}}] - \ln[2^{-2/3}t^{1/6}(1+t)^{2/3}]$, then elementary calculations yield

$$\begin{aligned} \lim_{t\to 1^+} f_2(t) &= 0, && (12)\\ f_2'(t) &= \frac{g_2(t)}{6t(t-1)^2(t+1)}, && (13)\\ g_2(t) &= (t^3 + 9t^2 - 9t - 1) - 6t(t+1)\ln t, &&\\ g_2(1) &= 0, && (14)\\ g_2'(t) &= 3t^2 + 12t - 6(2t+1)\ln t - 15, &&\\ g_2'(1) &= 0, && (15)\\ g_2''(t) &= \frac{6}{t}h_2(t), && (16)\\ h_2(t) &= t^2 - 2t\ln t - 1, &&\\ g_2''(1) &= h_2(1) = 0, && (17)\\ h_2'(t) &= 2(t - \ln t - 1), &&\\ h_2'(1) &= 0 && (18) \end{aligned}$$

and

$$h_2''(t) = 2(1 - \frac{1}{t}) > 0 \tag{19}$$

for $t > 1$.

From (11)-(19) we know that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha = 5/6$ and $a \neq b$.

If $\alpha \in (0,1)\backslash\{2/3, 5/6\}$, then

$$\begin{aligned} & \ln L_{6\alpha-5}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\ = & \frac{1}{6\alpha-5}\ln\frac{t^{6\alpha-4}-1}{(6\alpha-4)(t-1)} - \ln[2^{1-2\alpha}t^{1-\alpha}(t+1)^{2\alpha-1}]. \end{aligned} \tag{20}$$

Let $f_3(t) = \frac{1}{6\alpha-5}\ln\frac{t^{6\alpha-4}-1}{(6\alpha-4)(t-1)} - \ln[2^{1-2\alpha}t^{1-\alpha}(t+1)^{2\alpha-1}]$, then simple computations lead to

$$\lim_{t\to 1^+} f_3(t) = 0, \tag{21}$$

$$f_3'(t) = \frac{g_3(t)}{t(t^2-1)(t^{6\alpha-4}-1)}, \tag{22}$$

where $g_3(t) = (1-\alpha)t^{6\alpha-2} + \frac{4(1-\alpha)(1-3\alpha)}{6\alpha-5}t^{6\alpha-3} - \frac{(1-2\alpha)(1-3\alpha)}{6\alpha-5}t^{6\alpha-4} + \frac{(1-2\alpha)(1-3\alpha)}{6\alpha-5}t^2 - \frac{4(1-\alpha)(1-3\alpha)}{6\alpha-5}t - (1-\alpha)$.

$$g_3(1) = 0, \tag{23}$$

$$\begin{aligned} g_3'(t) = & 2(1-\alpha)(3\alpha-1)t^{6\alpha-3} + \frac{12(1-\alpha)(1-3\alpha)(2\alpha-1)}{6\alpha-5}t^{6\alpha-4} \\ & - \frac{2(1-2\alpha)(1-3\alpha)(3\alpha-2)}{6\alpha-5}t^{6\alpha-5} + \frac{2(1-2\alpha)(1-3\alpha)}{6\alpha-5}t \\ & - \frac{4(1-\alpha)(1-3\alpha)}{6\alpha-5}, \end{aligned}$$

$$g_3'(1) = 0, \tag{24}$$

$$\begin{aligned} g_3''(t) = & 6(1-\alpha)(1-2\alpha)(1-3\alpha)t^{6\alpha-4} \\ & + \frac{24(1-\alpha)(1-2\alpha)(1-3\alpha)(2-3\alpha)}{6\alpha-5}t^{6\alpha-5} \\ & - 2(1-2\alpha)(1-3\alpha)(3\alpha-2)t^{6\alpha-6} + \frac{2(1-2\alpha)(1-3\alpha)}{6\alpha-5}, \end{aligned}$$

$$g_3''(1) = 0 \tag{25}$$

and

$$g_3'''(t) = -12(1-\alpha)(1-2\alpha)(1-3\alpha)(2-3\alpha)t^{6\alpha-7}(t-1)^2. \tag{26}$$

If $\alpha \in (0, 1/3)\bigcup(1/2, 2/3)$, then

$$t^{6\alpha-4} - 1 < 0 \tag{27}$$

and (26) implies that

$$g_3'''(t) < 0 \tag{28}$$

for $t > 1$.

From (20)-(21) and (27)-(28) we know that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (0,1/3)\bigcup(1/2,2/3)$ and $a \neq b$.

If $\alpha \in (2/3,5/6)\bigcup(5/6,1)$, then

$$t^{6\alpha-4} - 1 > 0 \tag{29}$$

and (26) implies that

$$g_3'''(t) > 0 \tag{30}$$

for $t > 1$.

From (20)-(25) and (29)-(30) we conclude that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (2/3,5/6)\bigcup(5/6,1)$ and $a \neq b$.

If $\alpha \in (1/3,1/2)$, then (27) and (30) again hold. From (20)-(25) and (27) together with (30) we know that $L_{6\alpha-5}(a,b) < A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (1/3,1/2)$ and $a \neq b$.

Next we compare the values of $L_{-1/\alpha}(a,b)$ with $A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (0,1)$ and $a \neq b$. It is not difficult to verify that

$$\begin{aligned}
& \ln L_{-1/\alpha}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\
= \; & \alpha \ln \frac{(1-1/\alpha)(t-1)}{t^{1-1/\alpha}-1} - \ln[2^{1-2\alpha}t^{1-\alpha}(1+t)^{2\alpha-1}].
\end{aligned} \tag{31}$$

Let $f_4(t) = \alpha \ln \frac{(1-1/\alpha)(t-1)}{t^{1-1/\alpha}-1} - \ln[2^{1-2\alpha}t^{1-\alpha}(1+t)^{2\alpha-1}]$, then elementary calculations yield

$$\lim_{t\to 1^+} f_4(t) = 0, \tag{32}$$

$$f_4'(t) = \frac{g_4(t)}{(t^2-1)(t^{1/\alpha-1}-1)}, \tag{33}$$

where $g_4(t) = (3\alpha-1)t^{1/\alpha-1} + (1-\alpha)t^{1/\alpha-2} + (\alpha-1)t + (1-3\alpha)$.

$$g_4(1) = 0, \tag{34}$$

$$g_4'(t) = \frac{(1-\alpha)(3\alpha-1)}{\alpha}t^{1/\alpha-2} + \frac{(1-\alpha)(1-2\alpha)}{\alpha}t^{1/\alpha-3} + (\alpha-1),$$

$$g_4'(1) = 0, \tag{35}$$

$$g_4''(t) = \frac{(1-\alpha)(1-2\alpha)(3\alpha-1)}{\alpha^2}t^{1/\alpha-4}(t-1). \tag{36}$$

If $\alpha \in (0,1/3)\bigcup(1/2,1)$, then(36) implies

$$g_4''(t) < 0 \tag{37}$$

for $t > 1$.

From (31)-(35) and (37) we know that $L_{-1/\alpha}(a,b) < A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (0,1/3)\bigcup(1/2,1)$ and $a \neq b$.

If $\alpha \in (1/3,1/2)$, then (36) implies

$$g_4''(t) > 0 \tag{38}$$

for $t > 1$. Therefore, $L_{-1\alpha}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (1/3, 1/2)$ and $a \neq b$ follows from (31)-(35) and (38).

Finally, we prove that the parameters $-1/\alpha$ and $6\alpha - 5$ in either case are best possible.

Firstly, we show that the parameter $6\alpha - 5$ in either case is best possible. We divide the proof into seven cases.

Case 1. $\alpha = 2/3$. For any $\epsilon > 0$ and $x > 0$ one has

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{1+\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{1+\epsilon} \\ = & [A^{2/3}(1,1+x)H^{1/3}(1,1+x)]^{1+\epsilon} - [L_{-1-\epsilon}(1,1+x)]^{1+\epsilon} \\ = & \frac{f_1(x)}{(1+x)^{\epsilon} - 1}, \end{aligned} \tag{39}$$

where $f_1(x) = [(1+x)^{\epsilon} - 1](1+x)^{(1+\epsilon)/3}(1+x/2)^{(1+\epsilon)/3} - \epsilon x(1+x)^{\epsilon}$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_1(x) = \frac{\epsilon^2(1+\epsilon)}{24}x^3 + o(x^3). \tag{40}$$

Equations (39) and (40) imply that for $\epsilon > 0$ there exists $\delta_1 = \delta_1(\epsilon) > 0$, such that $L_{6\alpha-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)H^{(1-\alpha)}(1,1+x)$ for $\alpha = 2/3$ and $x \in (0, \delta_1)$.

Case 2. $\alpha = 5/6$. For any $\epsilon \in (0,1)$ and $x > 0$ we have

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{\epsilon} \\ = & [A^{5/6}(1,1+x)H^{1/6}(1,1+x)]^{\epsilon} - [L_{-\epsilon}(1,1+x)]^{\epsilon} \\ = & \frac{f_2(x)}{(1+x)^{1-\epsilon} - 1}, \end{aligned} \tag{41}$$

where $f_2(x) = [(1+x)^{1-\epsilon} - 1](1+x)^{\epsilon/6}(1+x/2)^{2\epsilon/3} - (1-\epsilon)x$.

Letting $x \to 0$ and making using of Taylor expansion we get

$$f_2(x) = \frac{\epsilon(1-\epsilon)}{24}x^3 + o(x^3). \tag{42}$$

Equations (41) and (42) show that for $\epsilon \in (0,1)$ there exists $\delta_2 = \delta_2(\epsilon) > 0$, such that $L_{6\alpha-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)H^{(1-\alpha)}(1,1+x)$ for $\alpha = 5/6$ and $x \in (0, \delta_2)$.

Case 3. $\alpha \in (0, 1/3)$. For any $\epsilon > 0$ and $x > 0$ one has

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5+\epsilon-6\alpha} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{5+\epsilon-6\alpha} \\ = & \frac{f_3(x)}{[(1+x)^{4+\epsilon-6\alpha} - 1](1+\frac{x}{2})^{(1-2\alpha)(5+\epsilon-6\alpha)}}, \end{aligned} \tag{43}$$

where $f_3(x) = [(1+x)^{4+\epsilon-6\alpha} - 1](1+x)^{(1-\alpha)(5+\epsilon-6\alpha)} - (4+\epsilon-6\alpha)x(1+x)^{4+\epsilon-6\alpha}(1+x/2)^{(1-2\alpha)(5+\epsilon-6\alpha)}$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_3(x) = \frac{\epsilon(4+\epsilon-6\alpha)(5+\epsilon-6\alpha)}{24}x^3 + o(x^3). \tag{44}$$

Equations (43) and (44) imply that for any $\alpha \in (0, 1/3)$ and $\epsilon > 0$ there exists $\delta_3 = \delta_3(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_3)$.

Case 4. $\alpha \in (1/3, 1/2)$. For any $\epsilon \in (0, 4-6\alpha)$ and $x > 0$ we get

$$\begin{aligned} & [L_{6\alpha-5+\epsilon}(1,1+x)]^{5-6\alpha-\epsilon} - [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5-6\alpha-\epsilon} \\ = \quad & \frac{f_4(x)}{[(1+x)^{4-6\alpha-\epsilon}-1](1+\frac{x}{2})^{(1-2\alpha)(5-6\alpha-\epsilon)}}, \end{aligned} \tag{45}$$

where $f_4(x) = (4-6\alpha-\epsilon)x(1+x)^{4-6\alpha-\epsilon}(1+x/2)^{(1-2\alpha)(5-6\alpha-\epsilon)} - [(1+x)^{4-6\alpha-\epsilon} - 1](1+x)^{(1-\alpha)(5-6\alpha-\epsilon)}$

Letting $x \to 0$ and making using Taylor expansion one has

$$f_4(x) = \frac{\epsilon(4+\epsilon-6\alpha)(5+\epsilon-6\alpha)}{24}x^3 + o(x^3). \tag{46}$$

Equations (45) and (46) imply that for any $\alpha \in (1/3, 1/2)$ and $\epsilon \in (0, 4-6\alpha)$ there exists $\delta_4 = \delta_4(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5+\epsilon}(1, 1+x) > A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_4)$.

Case 5. $\alpha \in (1/2, 2/3)$. For any $\epsilon > 0$ and $x > 0$ we have

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5-6\alpha+\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{5-6\alpha+\epsilon} \\ = \quad & \frac{f_5(x)}{(1+x)^{4-6\alpha+\epsilon}-1}, \end{aligned} \tag{47}$$

where $f_5(x) = [(1+x)^{4-6\alpha+\epsilon} - 1](1+x/2)^{(2\alpha-1)(5-6\alpha+\epsilon)}(1+x)^{(1-\alpha)(5-6\alpha+\epsilon)} - (4-6\alpha+\epsilon)x(1+x)^{4-6\alpha+\epsilon}$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_5(x) = \frac{\epsilon(4+\epsilon-6\alpha)^2(5+\epsilon-6\alpha)}{24}x^3 + o(x^3). \tag{48}$$

Equations (47) and (48) imply that for any $\alpha \in (1/2, 2/3)$ and $\epsilon > 0$ there exists $\delta_5 = \delta_5(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_5)$.

Case 6. $\alpha \in (2/3, 5/6)$. For any $\epsilon \in (0, 6\alpha-4)$ and $x > 0$ one has

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5-6\alpha+\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{5-6\alpha+\epsilon} \\ = \quad & \frac{f_6(x)}{(1+x)^{6\alpha-4-\epsilon}-1}, \end{aligned} \tag{49}$$

where $f_6(x) = [(1+x)^{6\alpha-4-\epsilon} - 1](1+x/2)^{(2\alpha-1)(5-6\alpha+\epsilon)}(1+x)^{(1-\alpha)(5-6\alpha+\epsilon)} - (6\alpha-4-\epsilon)x$.

Letting $x \to 0$ and making using Taylor expansion we obtain

$$f_6(x) = \frac{\epsilon(5+\epsilon-6\alpha)(6\alpha-4-\epsilon)}{24}x^3 + o(x^3). \tag{50}$$

Equations (49) and (50) imply that for any $\alpha \in (2/3, 5/6)$ and $\epsilon \in (0, 6\alpha-4)$ there exists $\delta_6 = \delta_6(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_6)$.

Case 7. $\alpha \in (5/6, 1)$. For any $\epsilon \in (0, 6\alpha-5)$ and $x > 0$ we have

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{6\alpha-5-\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{6\alpha-5-\epsilon} \\ = & \frac{f_7(x)}{(6\alpha-4-\epsilon)x}, \end{aligned} \tag{51}$$

where $f_7(x) = (6\alpha-4-\epsilon)x(1+x/2)^{(2\alpha-1)(6\alpha-5-\epsilon)}(1+x)^{(1-\alpha)(6\alpha-5-\epsilon)} - [(1+x)^{6\alpha-4-\epsilon} - 1]$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_7(x) = \frac{\epsilon(6\alpha-5-\epsilon)(6\alpha-4-\epsilon)}{24}x^3 + o(x^3). \tag{52}$$

Equations (51) and (52) imply that for any $\alpha \in (5/6, 1)$ and $\epsilon \in (0, 6\alpha-5)$ there exists $\delta_7 = \delta_7(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_7)$.

Secondly, we prove that the parameter $-1/\alpha$ in either case is the best possible. The proof is divided into two cases.

Case A. $\alpha \in (0, 1/3)\bigcup(1/2, 1)$. For any $\epsilon \in (0, 1/\alpha-1)$ and $t > 0$, we have

$$\begin{aligned} & L_{1/\alpha+\epsilon}(1,t) - A^{\alpha}(1,t)H^{1-\alpha}(1,t) \\ = & t^{\frac{\alpha}{1-\epsilon\alpha}}\{[\frac{(\frac{1}{\alpha}-1-\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1-\epsilon)}}]^{\frac{\alpha}{1-\epsilon\alpha}} - 2^{1-2\alpha}t^{-\frac{\epsilon\alpha^2}{1-\epsilon\alpha}}(1+\frac{1}{t})^{2\alpha-1}\} \end{aligned} \tag{53}$$

and

$$\begin{aligned} & \lim_{t\to+\infty}\{[\frac{(\frac{1}{\alpha}-1-\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1-\epsilon)}}]^{\frac{\alpha}{1-\epsilon\alpha}} - 2^{1-2\alpha}t^{-\frac{\epsilon\alpha^2}{1-\epsilon\alpha}}(1+\frac{1}{t})^{2\alpha-1}\} \\ = & (\frac{1}{\alpha}-1-\epsilon)^{\frac{\alpha}{1-\epsilon\alpha}} > 0. \end{aligned} \tag{54}$$

Equation (53) and inequality (54) imply that for any $\alpha \in (0, 1/3)\bigcup(1/2, 1)$ and $\epsilon \in (0, 1/\alpha-1)$ there exists $T_1 = T_1(\epsilon, \alpha) > 1$, such that $L_{-1/\alpha+\epsilon}(1, t) > A^{\alpha}(1, t)H^{1-\alpha}(1, t)$ for $t \in (T_1, \infty)$.

Case B. $\alpha \in (1/3, 1/2)$. For any $\epsilon > 0$ and $t > 0$, we have

$$\begin{aligned} & A^{\alpha}(1,t)H^{1-\alpha}(1,t) - L_{-1/\alpha-\epsilon}(1,t) \\ = & t^{\alpha}\{2^{1-2\alpha}(1+\frac{1}{t})^{2\alpha-1} - t^{-\frac{\epsilon\alpha^2}{1+\epsilon\alpha}}[\frac{(\frac{1}{\alpha}-1+\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1+\epsilon)}}]^{\frac{\alpha}{1+\epsilon\alpha}}\} \end{aligned} \tag{55}$$

and

$$\begin{aligned}
&\lim_{t\to+\infty}\{2^{1-2\alpha}(1+\frac{1}{t})^{2\alpha-1}-t^{-\frac{\epsilon\alpha^2}{1+\epsilon\alpha}}[\frac{(\frac{1}{\alpha}-1+\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1+\epsilon)}}]^{\frac{\alpha}{1+\epsilon\alpha}}\}\\
= \quad & 2^{1-\alpha}>0. \qquad (56)
\end{aligned}$$

From (55) and (56) we clearly see that for any $\alpha \in (1/3, 1/2)$ and $\epsilon > 0$ there exists $T_2 = T_2(\epsilon, \alpha) > 1$, such that $L_{-1/\alpha-\epsilon}(1,t) < A^{\alpha}(1,t)H^{1-\alpha}(1,t)$ for $t \in (T_2, \infty)$.

Acknowledgement The authors wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions. This research was supported by the Natural Science Foundation of China under Grants 11071069 and 11171307, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

References

[1] Abuhany, A. A. K., Salem, S. R. and Salman, I. M., "On Steffensen's integral inequality with applications", *J. Rajasthan Acad. Phys. Sci.* **5** (2006), 1-12.

[2] Alzer, H., "Ungleichungen für $(e/a)^a(b/e)^b$", *Elem. Math.* **40** (1985), 120-123.

[3] Alzer, H., "Ungleichungen für Mittelwerte", *Arch. Math.* **47**(1986), 422-426.

[4] Alzer, H. and Qiu, S.-L., "Inequalities for means in two variables", *Arch. Math.* **80** (2003), 201-215.

[5] Bullen, P. S., Mitrinović, D. S. and Vasić, P. M., *Means and Their Inequalities*, D. Reidel Pubishing Co., 1988.

[6] Burk, F., "The geometric, logarithmic and arithmic mean inequality", *Amer. Math. Monthly* **94** (1987), 527-528.

[7] Carlson, B. C., "The logarithmic mean", *Amer. Math. Monthly* **79** (1972), 615-618.

[8] Chen, Ch.-P., "The monotonicity of the ratio between generalized logarithmic means", *J. Math. Anal. Appl.* **345** (2008), 86-89.

[9] Chen, Ch.-P. and Qi, F., "Monotonicity properties for generalized logarithmic means", *Aust. J. Math. Anal. Appl.* **1** (2004), Article 2, 1-4.

[10] Kahlig, P. and Matkowski, J., "Functional equations involving the logarithmic mean", *Z. Angew. Math. Mech.* **76** (1996), 385–390.

[11] Leach, E. B. and Sholander, M. C., "Extended mean values II", *J. Math. Anal. Appl.* **92** (1983), 207-223.

[12] Lin, T. P., "The power mean and the logarithmic mean", *Amer. Math. Monthly* **81** (1974), 879-883.

[13] Mond, B., Pearce, C. E. M. and Pečarić, J., "The logarithmic mean is a mean", *Math. Commun.* **2** (1997), 35-39.

[14] Pearce, C. E. M. and Pečarić, J., "Some theorems of Jensen type for generalized logarithmic means", *Rev. Roumaine Math. Pures Appl.* **40** (1995), 789-795.

[15] Pittenger, A. O., "Inequalities between arithmetic and logarithmic means", *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **678-715** (1980), 15-18.

[16] Pittenger, A. O., "The symmetric, logarithmic and power means", *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **678-715** (1980), 19-23.

[17] Pittenger, A. O., "The logarithmic mean in n variables", *Amer. Math. Monthly* **92** (1985), 99-104.

[18] Pólya, G. and Szegö, G., *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.

[19] Qi, F., Chen Sh.-X. and Chen, Ch.-P., "Monotonicity of ratio between the generalized logarithmic means", *Math. Inequal. Appl.* **10** (2007), 559-564.

[20] Qi, F., Li, X.-A. and Chen, Sh.-X., "Refinements, extensions and generalizations of the second Kershaw's double inequality", *Math. Inequal. Appl.* **11** (2008), 457-465.

[21] Sándor, J., "A note on some inequalities for means", *Arch. Math.* **56** (1991), 471-473.

[22] Shi, H.-N., "Schur-convex functions related to Hadamard-type inequalities", *J. Math. Inequal.* **1** (2007), 127-136.

[23] Stolarsky, K. B., "The power and generalized logarithmic means", *Amer. Math. Monthly* **87** (1980), 545-548.