

A CLASSIFICATION OF SEMISYMMETRIC CUBIC GRAPHS OF ORDER $28p^2$

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Abstract. A graph is said to be semisymmetric if its full automorphism group acts transitively on its edge set but not on its vertex set. In this paper, we prove that there is only one semisymmetric cubic graph of order $28p^2$, where p is a prime.

Key words: Semisymmetric, covering graphs, semiregular subgroup.

Abstrak. Suatu graf dikatakan semisimetris jika grup automorfisma penuhnya bekerja secara transitif pada himpunan sisinya tapi tidak pada himpunan titiknya. Pada paper ini, kami membuktikan bahwa terdapat hanya satu graf kubik semisimetris dengan orde $28p^2$, dimana p adalah sebuah bilangan prima.

Kata kunci: Semisimetris, graf selimut, subgroup semireguler.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notations not defined here we refer to [16, 20]. Given a graph X , we let $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ (or A) be the *vertex set*, the *edge set*, the *arc set* and the *full automorphism group* of X , respectively. For $u, v \in V(X)$, we denote by $\{u, v\}$ the edge incident to u and v in X . If a subgroup G of $\text{Aut}(X)$ acts transitively on $V(X)$ and $E(X)$, we say that X is G -*vertex-transitive* and G -*edge-transitive*, respectively. In the special case when $G = \text{Aut}(X)$ we say that X is vertex-transitive and edge-transitive, respectively. It can be shown that a G -edge-transitive but not G -vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of

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$G \leq \text{Aut}(X)$. Moreover, if X is regular then these two parts have the same cardinality. A regular G -edge-transitive but not G -vertex-transitive graph will be referred to as a G -semisymmetric graph. In particular, if $G = \text{Aut}(X)$ the graph is said to be semisymmetric. We denote by K_4 , $F28$ and $S112$, the symmetric cubic complete graph of order 4, the symmetric cubic graph of order 28 and the semisymmetric cubic graph of order 112, which is Z_2^3 -covering graph of the symmetric cubic graph $F14$ (the Heawood graph) of order 14, respectively (for more details see [3, 4]). Let X be a graph and let N be a subgroup of $\text{Aut}(X)$. For $u \in V(X)$ denote by $N_X(u)$ the set of vertices adjacent to u in X . The quotient graph X/N or X_N induced by N is defined as the graph such that the set Σ of N -orbits in $V(X)$ is the vertex set of X/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A graph \tilde{X} is called a covering of a graph X with projection $\varphi : \tilde{X} \rightarrow X$ if there exists a surjection $\varphi : V(\tilde{X}) \rightarrow V(X)$ such that $\varphi|_{N_{\tilde{X}}(\tilde{v})} : N_{\tilde{X}}(\tilde{v}) \rightarrow N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \varphi^{-1}(v)$. A covering graph \tilde{X} of X with a projection φ is said to be regular (or K -covering) if there is a semiregular subgroup K of the automorphism group $\text{Aut}(\tilde{X})$ such that graph X is isomorphic to the quotient graph \tilde{X}/K , say by h , and the quotient map $\tilde{X} \rightarrow \tilde{X}/K$ is the composition φh of φ and h . The fibre of an edge or a vertex is its preimage under φ . The group of automorphisms of which maps every fibre to itself is called the covering transformation subgroup of $\text{Aut}(\tilde{X})$.

Let X be a graph and let K be a finite group. By a^{-1} we mean the reverse arc to an arc a . A voltage assignment (or, K -voltage assignment) of X is a function $\phi : A(X) \rightarrow K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called voltages, and K is the voltage group. The graph $X \times_{\phi} K$ derived from a voltage assignment $\phi : A(X) \rightarrow K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_{\phi} K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = \{u, v\}$. Clearly, the derived graph $X \times_{\phi} K$ is a covering of X with the first coordinate projection $\varphi : X \times_{\phi} K \rightarrow X$, which is called the natural projection. By defining $(u, g')^g = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_{\phi} K)$, K becomes a subgroup of $\text{Aut}(X \times_{\phi} K)$ which acts semiregularly on $V(X \times_{\phi} K)$. Therefore, $X \times_{\phi} K$ can be viewed as a K -covering. Conversely, each regular covering \tilde{X} of X with a covering transformation group K can be derived from a K -voltage assignment.

Covering techniques have long been known as a powerful tool in topology and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. Some general methods of elementary abelian coverings were developed in [6, 13, 14], which are useful tools for investigation of semisymmetric and symmetric graphs. The class of semisymmetric graphs was introduced by Folkman [8]. He constructed several infinite families of such graphs and posed eight open problems. Afterwards, Bouwer [2], Titov [18], Klin [11], Iofinova and Ivanov A.A [9], Ivanov A.V [10], Du and Xu [7] and others did much work on semisymmetric graphs. They gave new constructions of such graphs by combinatorial and group-theoretical methods. By now, the answers to most of Folkman's open problems are

known. By using the covering technique, cubic semisymmetric graphs of order $6p^2$, $8p^2$ and $2p^3$ were classified in [12, 1, 15], respectively.

The next proposition is a special case of [19, proposition 2.5].

Proposition 2.1. *Let X be a G -semisymmetric cubic graph with bipartition sets $U(X)$ and $W(X)$, where $G \leq A := \text{Aut}(X)$. Moreover, suppose that N is a normal subgroup of G . Then,*

(1) *If N is intransitive on bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is a regular N -covering of G/N -semisymmetric graph X_N .*

(2) *If 3 does not divide $|A/N|$, then N is semisymmetric on X .*

Proposition 2.2. [15, Proposition 2.4]. *The vertex stabilizers of a connected G -semisymmetric cubic graph X have order $2^r \cdot 3$, where $0 \leq r \leq 7$. Moreover, if u and v are two adjacent vertices, then the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v .*

Proposition 2.3. [16, pp.236]. *Let G be a finite group and let p be a prime. If G has an abelian Sylow p -subgroup, then p does not divide $|G' \cap Z(G)|$.*

Proposition 2.4. [20, Proposition 4.4]. *Every transitive abelian group G on a set Ω is regular and the centralizer of G in the symmetric group on Ω is G .*

2. Main Result

Theorem 2.1. *Let p be a prime. Then, the graph $S112$ is the only semisymmetric cubic graph of order $28p^2$.*

Proof. Let X be a cubic semisymmetric graph of order $28p^2$. If $p < 7$, then by [3] there is only one cubic semisymmetric graph $S112$ of order $28p^2$, in which $p = 2$. Hence, we can assume that $p \geq 7$. Set $A := \text{Aut}(X)$. By Proposition 2.2, $|A_v| = 2^r \cdot 3$, where $0 \leq r \leq 7$ and hence $|A| = 2^{r+1} \cdot 3 \cdot 7 \cdot p^2$. Let $Q = O_p(A)$ be the maximal normal p -subgroup of A . We first show that $Q \neq 1$ and $|Q| \neq p$ as follows.

Let N be a minimal normal subgroup of A . Thus, $N \cong T \times T \times \cdots \times T = T^k$, where T is a simple group. Let N be unsolvable. By [5], T is isomorphic to $PSL(2, 7)$ or $PSL(2, 13)$ of orders $2^3 \cdot 3 \cdot 7$ and $2^2 \cdot 3 \cdot 7 \cdot 13$, respectively. Note that $3^2 \nmid |N|$, forcing $k = 1$. Then, 3 does not divide $|A/N|$ and hence by Proposition 2.1, N is semisymmetric on X . Consequently, $14p^2 \mid |N|$, a contradiction. Therefore, N is solvable and so elementary abelian. It follows that N acts intransitively on bipartition sets of X and by Proposition 2.1, it is semiregular on each partition. Hence, $|N| \mid 14p^2$.

Suppose first that $Q = 1$. It implies two cases, $N \cong Z_2$ and $N \cong Z_7$ and we get a contradiction in each case as follows.

case (I): $N \cong Z_2$.

By Proposition 2.1, X_N is a cubic A/N -semisymmetric graph of order $14p^2$. Let T/N be a minimal normal subgroup of A/N . If T/N is unsolvable then by a similar argument as above T/N is isomorphic to $PSL(2, 7)$ or $PSL(2, 13)$ and so $|T| = 2^4 \cdot 3 \cdot 7$ or $|T| = 2^3 \cdot 3 \cdot 7 \cdot 13$, respectively. However, 3 does not divide $|A/T|$ and by Proposition 2.1, T is semisymmetric on X , a contradiction because $14p^2 \nmid |T|$ where $p \geq 7$. Hence, T/N is solvable and so elementary abelian. Therefore, T/N acts intransitively on bipartition sets of X_N and by Proposition 2.1, it is semiregular on each partition, which force $|T/N| \mid 7p^2$.

If $|T/N| = p^i$ ($i = 1, 2$), then $|T| = 2p^i$ and so the Sylow p -subgroup of T is characteristic and consequently normal in A . It contradicts our assumption that $Q = 1$. Hence, $|T/N| = 7$ and so $|T| = 14$. Thus, T acts intransitively on bipartition sets of X and by Proposition 2.1, X_T is a cubic A/T -semisymmetric graph of order $2p^2$. Let K/T be a minimal normal subgroup of A/T . One can see that K/T is solvable and so elementary abelian. If K/T acts transitively on one partition of X_T then by Proposition 2.4, $|K/T| = p^2$ and hence $|K| = 14p^2$. Thus K has a normal subgroup of order $7p^2$, say H . Since $p \geq 7$, the Sylow p -subgroup of H is characteristic in K and consequently normal in A , a contrary to the fact that $Q = 1$. Therefore, K/T acts intransitively on bipartition sets of X_T and by Proposition 2.1, $|K/T| = p^i$ ($i = 1, 2$). Again, one can show that A has a normal p -subgroup, a contradiction.

case (II): $N \cong Z_7$.

Consequently, by Proposition 2.1, X_N is a cubic A/N -semisymmetric graph of order $4p^2$. Let L/N be a minimal normal subgroup of A/N . By a similar argument as case (I), L/N is solvable and so elementary abelian. If L/N acts transitively on one partition of X_N , then by Proposition 2.4 $|L/N| = 2p^2$, a contradiction. Therefore, L/N acts intransitively on bipartition sets of X_N and by Proposition 2.1, it is semiregular on each partition, which force $|L/N| \mid 2p^2$. If $|L/N| = 2$ then $|L| = 14$, a contradiction (see case (I)). Hence, $|L/N| = p^i$ ($i = 1, 2$) and so $|L| = 7p^i$. Again, A has a normal p -subgroup, a contradiction.

We now suppose that $|Q| = p$ and we show it is impossible. Set $C := C_A(Q)$ the centralizer of Q in A . Let P be a Sylow p -subgroup of A . Clearly, $Q < P$ and also $P \leq C$ because P is abelian. Thus, $p^2 \mid |C|$. If $p^2 \mid |C'|$ (C' is the derived subgroup of C) then $Q \leq C'$ and hence $p \mid |C' \cap Q|$, a contradiction by Proposition 2.3, because $Q \leq Z(C)$. Consequently, $p^2 \nmid |C'|$ and so C' acts intransitively on bipartition sets of X . Then by Proposition 2.1, it is semiregular and hence $|C'| \mid 14p^2$. Let K/C' be a Sylow p -subgroup of C/C' . Since C/C' is abelian, K/C' is characteristic and hence normal in A/C' , implying that $K \triangleleft A$. Note that $p^2 \mid |K|$ and $|K| \mid 14p^2$. If $|K| = tp^2 < 14p^2$, where $t \mid 14$ then the Sylow p -subgroup of K is characteristic in C and also in A , because $K \triangleleft C$ and $C \triangleleft A$. If $|K| = 14p^2$, then the Sylow p -subgroup of A is normal as case (I), which contradicts our assumption that $|Q| = p$.

Therefore it is clear when $p > 7$, $|Q| = p^2$. Then, by Proposition 2.1 the semisymmetric graph X is a regular Q -covering of A/Q -semisymmetric graph X_Q and so X_Q is an edge-transitive bipartite cubic graph of order 28. But by [3,4] the only edge-transitive cubic graph of order 28 is the symmetric graph F_{28} , which is

not bipartite, a contradiction. Suppose that $p = 7$. Thus, $|Q| = 7^2$ or $|Q| = 7^3$. If $|Q| = 7^2$, we get the same contradiction as above. Let $|Q| = 7^3$. By Proposition 2.1, the semisymmetric graph X is a regular Q -covering of A/Q -semisymmetric graph X_Q and so X_Q is an edge-transitive bipartite cubic graph of order 4. But by [3,4] the only edge-transitive cubic graph of order 4 is the symmetric graph K_4 , which is not bipartite, a contradiction. Hence, the result now follows. \square

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