

SOME PROPERTIES OF MULTIPLICATION MODULES

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Abstract. Let M be an R -module. The module M is called multiplication if for any submodule N of M we have $N = IM$, where I is an ideal of R . In this paper we state some basic properties of submodules of these modules. Also, we study the relationship between the submodules of a multiplication R -module M and ideals of ring R . Finally, by definition of semiprime submodule, we state some properties of radical submodules of multiplication modules.

Key words and Phrases: Multiplication module, prime submodule, primary submodule.

Abstrak. Diberikan ring R dan diketahui M adalah R -modul. Modul M disebut modul multiplikasi jika untuk setiap submodul N di M memenuhi sifat $N = IM$, untuk suatu ideal I di R . Dalam paper ini diberikan beberapa sifat dasar submodul-submodul dalam modul multiplikasi. Selain itu diberikan juga hubungan antara submodul di dalam modul multiplikasi M atas R dan ideal-ideal di dalam ring R . Dengan menggunakan definisi submodul semiprima, dihasilkan juga beberapa sifat submodul radikal dalam modul multiplikasi.

Kata kunci: Modul multiplikasi, submodul prima, submodul utama.

1. INTRODUCTION

In this paper all rings are commutative with identity and all modules over rings are unitary. Let K and N be submodules of an R -module M , we recall that $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R . Let N be a proper submodule of an R -module M , then N is called a prime submodule of M ,

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if for every $r \in R$, $x \in M$; $rx \in N$ we have $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p -prime. The set of all prime submodules of M is denoted by $Spec(M)$ and for a submodule N of M , $rad(N) = \bigcap_{L \in Spec(M), N \subseteq L} L$. If no prime submodule of M contains N , we write $rad(N) = M$. Also the set of all maximal submodules of M is denoted by $Max(M)$ and $RadM = \bigcap_{P \in Max(M)} P$. Also we recall that if I is an ideal of a ring R , then radical of I , i.e., $r(I)$ is defined as $\{r \in R \mid \exists k \in \mathbb{N}; r^k \in I\}$. Now, let \underline{a} be an ideal of a ring R and $\underline{a} = \bigcap_{i=1}^l q_i$, where $r(q_i) = p_i$ is a normal primary decomposition of \underline{a} , then $ass(\underline{a}) = \{p_1, \dots, p_l\}$.

An R -module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$. It can be shown that $N = (N : M)M$.

2. DEFINITIONS AND RESULTS

Lemma 2.1. *Let R be a non-trivial ring and let M be a multiplication R -module. Then $IM \neq M$ for any proper ideal I of R .*

PROOF. Let I be an arbitrary proper ideal of R , then there exists a maximal ideal \underline{m} of R such that $I \subseteq \underline{m}$. We remind that $R_{\underline{m}} = S^{-1}R$ and $M_{\underline{m}} = S^{-1}M$ are quotient ring of R and quotient module respectively where $S = R - \underline{m}$.

We show that $\underline{m}M \neq M$. By [1, Lemma 2(i)], $M_{\underline{m}}$ is a multiplication $R_{\underline{m}}$ -module and also by [3, Theorem 2.5], $(\underline{m}R_{\underline{m}})M_{\underline{m}}$ is the only maximal submodule of $M_{\underline{m}}$. Thus $(\underline{m}R_{\underline{m}})M_{\underline{m}} = (\underline{m}M)_{\underline{m}} \neq M_{\underline{m}}$ and hence $((\underline{m}M)_{\underline{m}} : M_{\underline{m}}) = \underline{m}R_{\underline{m}}$. Therefore $\underline{m}M \neq M$.

Now since $IM \subseteq \underline{m}M \neq M$, therefore $IM \neq M$ for every proper ideal I of R .

Lemma 2.2. *Let M be a multiplication R -module, then $pM \in Spec(M)$ and $(pM : M) = p$ for every $(0) \neq p \in Spec(R)$.*

PROOF. Let M be a multiplication R -module and let $(0) \neq p \in Spec(R)$ be arbitrary. By Lemma 2.1, $pM \neq M$. We show that $(pM : M) = p$.

Let $r \in (pM : M)$ be arbitrary then $rM \subseteq pM$ and hence we have:

$$(rM)_p \subseteq (pM)_p \implies \frac{r}{1}M_p \subseteq (pM)_p \implies \frac{r}{1} \in ((pM)_p : M_p).$$

But by [1, Lemma 2(i)], M_p is a multiplication R_p -module and by [3, Theorem 2.5], $(pM)_p$ is the only maximal submodule of M_p and so $((pM)_p : M_p) = pR_p$. Thus $r \in p$ and hence $(pM : M) = p$.

Now by [2, Corollary 2], pM is a primary submodule of M and also by [5, Proposition 1], $pM \in Spec(M)$. The proof is now completed.

Corollary 2.3. *Let R be an arbitrary ring and let M be a multiplication R -module. Then there exists a bijection between non-zero prime ideals of R and non-zero prime submodules of M .*

PROOF. We show that for every $N \in \text{Spec}(M)$, $N = pM$ where $p \in \text{Spec}(R)$. First, we show that, if M is a multiplication R -module and N is a submodule of M , then $N = (N : M)M$. Since M is a multiplication R -module, hence there exists an ideal I of R such that $N = IM$. From this, we have $I \subseteq (N : M)$, and $N = IM \subseteq (N : M)M \subseteq N$, hence $N = (N : M)M$. Second, let $N \in \text{Spec}(M)$. Since M is a multiplication R -module, $N = (N : M)M$ and by [5, Proposition 1], $(N : M) \in \text{Spec}(R)$.

Now we define $\psi : \text{Spec}(M) \rightarrow \text{Spec}(R)$ by $\psi(pM) = (pM : M)$ for any non-zero prime ideal p of R . Clearly ψ is well defined and by Lemma 2.2, ψ is a bijection.

Proposition 2.4. *Let M be a faithful multiplication R -module. Then every proper direct summand of M is prime. Hence M is indecomposable.*

PROOF. By [3, Lemma 4.1], M is torsion-free and by [5, Result 1], every direct summand of M is a prime submodule.

Now we show that M is indecomposable. If $M = M_1 \oplus M_2$ where $M_1, M_2 \neq 0$ then by the current form of the proposition, M_1 is a p -prime for some ideal p of R . Thus $M_1 = pM = pM_1 \oplus pM_2$. Hence $pM_2 = 0$. Since M is torsion-free and $M_2 \neq 0$, we have $p = (0)$ and hence $M_1 = 0$, a contradiction.

Proposition 2.5. *Let M be a multiplication R -module. Then for every submodule IM of M , if $IM \subseteq pM$ where $p \in \text{Spec}(R)$, then $I \subseteq p$.*

PROOF. Let $IM \subseteq pM$ for $I \triangleleft R$ and $p \in \text{Spec}(R)$. Since $I \subseteq (IM : M) \subseteq (pM : M)$, then by Lemma 2.2, $I \subseteq p$.

Corollary 2.6. *If M is a faithful multiplication R -module then M is finitely generated.*

PROOF. Since M is a multiplication R -module, hence by Lemma 2.1, $M \neq IM$ for every proper ideal I of R . Now, since M is a faithful multiplication R -module, then by [3, Theorem 3.1 part (i), (iv)], M is finitely generated.

Corollary 2.7. *Let M be a faithful multiplication R -module. Then for every ideal I of R , $(IM : M) = I$.*

PROOF. Let M is a faithful multiplication R -module, then by Corollary 2.6, M is a finitely generated R -module. Now, let $(IM : M) = q$ where I and q be ideals of R . Since $(IM : M) = q$, $qM \subseteq IM$ and by [3, Theorem 3.1 part (ii)], $q \subseteq I$. Now, since $I \subseteq (IM : M) = q$, therefore $I = (IM : M)$.

Lemma 2.8. *If M is a faithful multiplication R -module. Then there exists a bijection between ideals of R and submodules of M .*

PROOF. Since M is a multiplication R -module, hence for every submodule N of M there exists an ideal I of R such that $N = IM$ and by Corollary 2.7, $(N : M) =$

$(IM : M) = I$. Now we define $\psi : M \rightarrow R$ by $\psi(IM) = (IM : M)$ for any ideal I of R . Obviously ψ is well defined and also ψ is an epimorphism. Now let N_1, N_2 be submodules of M , then there exist $I_1, I_2 \leq R$ such that $N_1 = I_1M$ and $N_2 = I_2M$. If $\psi(N_1) = \psi(N_2)$ then $(I_1M : M) = (I_2M : M)$ and by Corollary 2.7, $I_1 = I_2$. Therefore ψ is a bijection.

Corollary 2.9. *Let M be a Noetherian multiplication R -module. Then R satisfies the ascending chain condition on prime ideals.*

PROOF. Let $p_1 \subseteq p_2 \subseteq p_3 \subseteq \dots$ be an ascending chain of prime ideals of R . Then $p_1M \subseteq p_2M \subseteq p_3M \subseteq \dots$. But, M is a Noetherian R -module, hence there exists submodule (by [3, Theorem 2.5 part (i)], specially a maximal submodule) N of M such that $p_1M \subseteq p_2M \subseteq p_3M \subseteq \dots \subseteq N$. But M is a multiplication R -module, hence by [3, Theorem 2.5 part (ii)], there exists a maximal ideal \underline{m} of R such that $N = \underline{m}M$. So we have $p_1M \subseteq p_2M \subseteq p_3M \subseteq \dots \subseteq \underline{m}M$ and hence $(p_1M : M) \subseteq (p_2M : M) \subseteq (p_3M : M) \subseteq \dots \subseteq (\underline{m}M : M)$. Now by Lemma 2.2, $p_1 \subseteq p_2 \subseteq p_3 \subseteq \dots \subseteq \underline{m}$. The proof is now completed.

Corollary 2.10. *Let R be an arbitrary ring and let M be a multiplication R -module. Then $\text{Ann}_R(M) \subseteq p$ for each $(0) \neq p \in \text{Spec}(R)$.*

PROOF. By the Lemma 2.2, $pM \in \text{Spec}(M)$ for every $(0) \neq p \in \text{Spec}(R)$. Therefore by [3, Corollary 2.11 part (i), (iii)], $\text{Ann}_R(M) \subseteq p$.

We recall that in the following lemma $J(R)$ and \underline{n}_R denote the Jacobson radical and nilradical of R , respectively.

Lemma 2.11. *Let R be a ring and M a multiplication R -module. Then $\bigcap_{\lambda \in \Lambda} (p_\lambda M) = (\bigcap_{\lambda \in \Lambda} p_\lambda)M$ for any non-empty collection of non-zero prime ideals p_λ ($\lambda \in \Lambda$) of R . Also if R is a ring which is not an integral domain then $\bigcap_{0 \neq P \in \text{Spec}(M)} P = \underline{n}_R M$ and $\text{Rad}M = J(R)M$.*

PROOF. Let M be a multiplication R -module and let p_λ ($\lambda \in \Lambda$) be any non-empty collection of non-zero prime ideals of R . By [3, Corollary 1.7], $\bigcap_{\lambda \in \Lambda} (p_\lambda M) = (\bigcap_{\lambda \in \Lambda} [p_\lambda + \text{Ann}_R(M)])M$. But by Corollary 2.10, $\bigcap_{\lambda \in \Lambda} (p_\lambda M) = (\bigcap_{\lambda \in \Lambda} p_\lambda)M$. By Lemma 2.2, $\bigcap_{0 \neq P \in \text{Spec}(M)} P = \bigcap_{(0) \neq p \in \text{Spec}(R)} (pM)$ and also by above we have $\bigcap_{(0) \neq p \in \text{Spec}(R)} (pM) = \underline{n}_R M$. So $\bigcap_{0 \neq P \in \text{Spec}(M)} P = \underline{n}_R M$. Also by Lemma 2.2, $\text{Rad}M = \bigcap_{\underline{m} \in \text{Max}(R)} (\underline{m}M)$ and by above $\bigcap_{\underline{m} \in \text{Max}(R)} \underline{m}M = J(R)M$. Hence $\text{Rad}M = J(R)M$.

Lemma 2.12. *Let R be a ring and M a multiplication R -module. Let IM be an arbitrary non-zero proper submodule of M for some ideal I of R . Then $\text{rad}(IM) = (\text{rad}I)M$ and $(\text{rad}(IM) : M) = \text{rad}I$, where $\text{rad}I = r(I)$.*

PROOF. It is easy to show that $\text{rad}(IM) = \bigcap_{p \in v(I)} (pM)$ (we recall that $v(I) = \{p \in \text{Spec}(R) \mid I \subseteq p\}$). By Lemma 2.11, $\text{rad}(IM) = (\text{rad}I)M$ and consequently $(\text{rad}(IM) : M) = \text{rad}I$.

We recall the following definition from [6].

Definition 2.13. *A proper submodule N of an R -module M is said to be semiprime in M , if for every ideal I of R and every submodule K of M , $I^2K \subseteq N$ implies that $IK \subseteq N$. Note that since the ring R is an R -module by itself, a proper ideal I of R is semiprime if for every ideals J and K of R , $J^2K \subseteq I$ implies that $JK \subseteq I$.*

Definition 2.14. *There exists another definition of semiprime submodules in [4] as follows:*

A proper submodule N of the R -module M is semiprime if whenever $r^k m \in N$ for some $r \in R$, $m \in M$ and positive integer k , then $rm \in N$.

By [7, Remark 2.6], we see that this definition is equivalent to Definition 2.13.

Definition 2.15. *Let M be an R -module and $N \leq M$. The envelope of the submodule N is denoted by $E_M(N)$ or simply by $E(N)$ and is defined as $E(N) = \{x \in M \mid \exists r \in R, a \in M; x = ra \text{ and } r^n a \in N \text{ for some positive integer } n\}$.*

The envelope of a submodule is not a submodule in general.

Let M be an R -module and $N \leq M$. If there exists a semiprime submodule of M which contains N , then the intersection of all semiprime submodules containing N is called *semi-radical* of N and is denoted by $S - rad_M N$, or simply $S - radN$. If there is no semiprime submodule containing N , then we define $S - radN = M$, in particular $S - radM = M$.

We say that M *satisfies the radical formula*, or M (s.t.r.f) if for every $N \leq M$, $radN = \langle E(N) \rangle$. Also we say that M *satisfies the semi-radical formula*, or M (s.t.s.r.f) if for every $N \leq M$, $S - radN = \langle E(N) \rangle$. Now let $x \in E(N)$ and P be a semiprime submodule of M containing N . Then $x = ra$ for some $r \in R$, $a \in M$ and for some positive integer n , $r^n a \in N$. But $r^n a \in P$ and since P is semiprime we have $ra \in P$. Hence $E(N) \subseteq P$. We see that $E(N) \subseteq \bigcap P$ (P is a semiprime submodule containing N). So $\langle E(N) \rangle \subseteq S - radN$. On the other hand, since every prime submodule of M is clearly semiprime, we have $S - radN \subseteq radN$. We conclude that $\langle E(N) \rangle \subseteq S - radN \subseteq radN$ and as a result if M (s.t.r.f) then it is also (s.t.s.r.f).

Lemma 2.16. *Let R be a ring and let M be a multiplication R -module. Then every proper submodule of M is a radical submodule, i.e., $radN = N$.*

PROOF. By [3, Theorem 2.12], $radN = rad(N : M)M$. But $rad(N : M)M \subseteq \langle E(N) \rangle \subseteq radN$, hence M (s.t.r.f) and so (s.t.s.r.f). Then $\langle E(N) \rangle = S - radN = radN$ for every proper submodule N of M . But by [6, Proposition 4.1], $S - radN = N$ and therefore $radN = N$.

Corollary 2.17. *Let R and M and IM be as in Lemma 2.12. Then $IM = (\text{rad}I)M$.*

PROOF. Let M be a multiplication R -module and IM be an arbitrary non-zero proper submodule of M for some ideal I of R . By Lemma 2.12, $\text{rad}(IM) = (\text{rad}I)M$ and by Lemma 2.16, $\text{rad}(IM) = IM$. Therefore $IM = (\text{rad}I)M$.

Theorem 2.18. *Let R be a ring and let M be a multiplication R -module. Then N is a primary submodule of M if and only if it is a prime submodule of M .*

PROOF. \Leftarrow . It is clear.

\Rightarrow . Let M be a multiplication R -module and let N be an arbitrary primary submodule of M . Then by [2, Corollary 2], there exists a primary ideal q ($\text{rad}q = p$) of R such that $N = qM$.

But by Lemma 2.16 and Corollary 2.17, $qM = (\text{rad}q)M = pM$. Therefore the proof is now completed.

Corollary 2.19. *Let R be a ring which satisfies ascending chain condition on semiprime ideals and let M be a multiplication R -module. Then M is a Noetherian R -module.*

PROOF. Let M be a multiplication R -module. Then M (s.t.r.f) and hence (s.t.s.r.f). Thus by [6, Proposition 4.1], every proper submodule of M is a semiprime submodule of M . Now, let $I_1M \subseteq I_2M \subseteq I_3M \subseteq \dots$ where I_i are ideals of R be ascending chain of submodules of M . Then $(I_1M : M) \subseteq (I_2M : M) \subseteq (I_3M : M) \subseteq \dots$. But by [6, Proposition 2.3(ii)], $(N : M)$ is a semiprime ideal of R for any semiprime submodule N of M , hence by assumption there exists $n \in \mathbb{N}$ such that $(I_nM : M) = (I_{n+k}M : M)$ for each $k \in \mathbb{N}$. But then $(I_nM : M)M = (I_{n+k}M : M)M$ and so $I_nM = I_{n+k}M$. Therefore M is a Noetherian R -module.

It should be noted that the above results (Lemma 2.16, Corollary 2.17, Theorem 2.18, Corollary 2.19) they are not necessarily true if $M = R$, the ring itself. Because according to [6, Theorem 4.4], R (s.t.s.r.f) if we have one of the following.

- (i) For every free R -module F , F (s.t.s.r.f).
- (ii) For every faithful R -module B , B (s.t.s.r.f).

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