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STRONGLY CLEAN ELEMENTS IN A CERTAIN BLOCK MATRIX RING

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Abstract. In this paper, we investigate the concepts of strongly clean, strongly π-regular and strongly J_n-clean related to the ring $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$. Moreover, we give several equivalent conditions such that an element $\alpha \in End(A)$ is strongly clean.

Key words and Phrases: Strongly clean, strongly π-regular, strongly J_n-clean, endomorphism.

Abstrak. Dalam makalah ini, konsep dari *strongly clean, strongly π-regular* dan *strongly J_n-clean* yang berhubungan dengan ring $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$ diselidiki. Selanjutnya, beberapa kondisi ekivalen diberikan sedemikian sehingga sebuat element $\alpha \in End(A)$ adalah *strongly clean*

Kata kunci:Strongly clean, strongly π-regular, strongly J_n-clean, endomorphism.

1. Introduction

An element in a ring R is strongly clean provided that it is the sum of an idempotent and a unit that commute with each other. This notion was firstly introduced by Nicholson in [**?**]. Let R, S be two rings, and let M be an $R-S$-bimodule. This means that M is a left R-module and a right S-module such that $(rm)s = r(ms)$ for all $r \in R$, $m \in M$, and $s \in S$. Given such a bimodule M, we can form

$$A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

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and define a multiplication on A by using formal matrix multiplication:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms' \\ 0 & ss' \end{pmatrix}.$$

In [?], a characterization of strongly J_n-clean rings by virtue of strongly π-regularity is given. The main purpose of this note is to study about strongly clean, strongly π-regular and strongly J_n-clean of the ring A. We give several equivalent conditions under which such element $\alpha \in End(A)$ is strongly clean.

2. Strongly clean, strongly π-regular and strongly J_n-clean

Let us define the following subsets of a ring R.

$$\mathcal{U}(R) = \{r \in R \mid r \text{ is a unit of } R\},$$
$$\mathcal{ID}(R) = \{r \in R \mid r \text{ is an idempotent of } R\}.$$

We begin with

Proposition 2.1. Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are commutative rings and ${}_RM_S$ is a bimodule. Then, A is a strongly clean ring if and only if R and S are strongly clean rings.

Proof. Suppose that $r \in R$, $s \in S$ and $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in A$. Then, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix}$ such that $r_1 \in \mathcal{U}(R)$, $s_1 \in \mathcal{U}(S)$, $r_2 \in \mathcal{ID}(R)$, $s_2 \in \mathcal{ID}(S)$, $m_1, m_2 \in M$ and $r_2m_2 + m_2s_2 = m_2$. So, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & m_1 + m_2 \\ 0 & s_1 + s_2 \end{pmatrix}$. Hence, $r = r_1 + r_2$ and $s = s_1 + s_2$.

For the converse, suppose that R and S are strongly clean rings and $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in A$. Then, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & m \\ 0 & s_1 + s_2 \end{pmatrix}$ such that $r_1 \in \mathcal{U}(R)$, $s_1 \in \mathcal{U}(S)$, $r_2 \in \mathcal{ID}(R)$ and $s_2 \in \mathcal{ID}(S)$. So, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix}$, where $m_1 + m_2 = m$ and $r_2m_2 + m_2s_2 = m_2$. Hence, $\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \in \mathcal{U}(A)$ and $\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} \in \mathcal{ID}(A)$. □

Definition 2.2. *An element in a ring R is called strongly π-regular if for every $a \in R$, the chain $aR \supseteq a^2R \supseteq \ldots$ terminates (or equivalently, the chain $Ra \supseteq Ra^2 \supseteq \ldots$ terminates, see* [**?**, **?**].

As defined by artinian ring, if $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is artinian, then the chain

$$A \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \supseteq A \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}^2 \supseteq \ldots$$

terminates. So, A is strongly π-regular.

Remark 1. [?] Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are rings and ${}_RM_S$ is a bimodule. Then, A is artinian if and only if R and S are artinian.

Proposition 2.3. Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are artinian rings and M as a $R-S$-module is artinian. Then, A is strongly π-regular.

Proof. The verification is straightforward. □

Definition 2.4. *We say that $x \in R$ is strongly J_n-clean provided that there exists an idempotent $e \in R$ such that $x - e \in \mathcal{U}(R)$, $ex = xe$ and $(ex)^n \in J(R)$, where $J(R)$ is the Jacobson radical of R.*

Proposition 2.5. Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are commutative local rings and ${}_RM_S$ is a bimodule. Then, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in A$ is strongly J_n-clean if there exists an idempotent $\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \in A$ such that $r_1m + m_1s = rm_1 + ms_1$ and $(r_1r)^n = (s_1s)^n = 0$.

Proof. By $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ and $J(A) = \begin{pmatrix} J(R) & M \\ 0 & J(S) \end{pmatrix} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$, the proof is clear. □

3. Endomorphisms and strongly clean

In this section, we study the endomorphisms of the ring A and also necessary and sufficient conditions under which an endomorphism of the ring A is strongly clean.

Lemma 3.1. *Let $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$, $\alpha \in End(A)$ and let $\pi^2 = \pi \in End(A)$. Then,*

(1) *$\pi(A)$ is α-invariant if and only if $\pi\alpha = \pi\alpha\pi$.*
(2) *Both $\pi(A)$ and $(1 - \pi)(A)$ are α-invariant if and only if $\pi\alpha = \alpha\pi$.*

Proof. The proof is a routine verification. □

Proposition 3.2. Let $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$, $\alpha \in End(A)$ and let

$$A = \underbrace{\begin{pmatrix} R & {}_RP_S \\ 0 & S \end{pmatrix}}_{A'} \oplus \underbrace{\begin{pmatrix} R & {}_RQ_S \\ 0 & S \end{pmatrix}}_{A''},$$

where A' and A'' are both α-invariant. The following are equivalent.

(1) $\alpha \mid_{A'}$ is an isomorphism.
(2) $ker(\alpha) \subseteq A''$ and $A' \subseteq \alpha(A)$.

Proof. (1) $\Rightarrow$ (2). If $F \in ker(\alpha)$ write $F = G + H$. Then, $0 = \alpha(F) = \alpha(G+H) = \alpha(G) + \alpha(H) \in A' \oplus A''$, so $0 = \alpha(G) = \alpha \mid_{A'} (G)$. Thus, we have $G = 0$ by (1), so $F = H \in A''$. Hence, $ker(\alpha) \subseteq A''$. Finally, $A' = \alpha \mid_{A'} (A') = \alpha(A') \subseteq \alpha(A)$.

(2) $\Rightarrow$ (1). $\alpha \mid_{A'}$ is one-to-one because $ker(\alpha \mid_{A'}) = ker(\alpha) \cap A' \subseteq A'' \cap A' = 0$. Let $\pi^2 = \pi \in End(A)$ satisfy $ker(\pi) = A''$ and $\pi(A) = A'$. Then, $\alpha\pi = \pi\alpha$ by **??**, so $A' = \pi(A') \subseteq \pi\alpha(A) = \alpha\pi(A) = \alpha(A')$ using (2). Hence, $\alpha \mid_{A'}$ is onto. □

Remark 2. In Proposition **??**, it is important the existence of a ring R with the property $R \cong R \oplus R$. Here we mention a well known fact. If V is an infinite dimensional vector space over a field F, then $R = End(V_F)$ is a ring with the property that $R \cong R \oplus R$.

Theorem 3.3. *Let* $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$ *and* $E = End(A)$. *The following are equivalent for* $\alpha \in E$.

(1) *α is strongly clean in E.*
(2) *There exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi$ is a unit in $\pi E\pi$ and $(1-\alpha)(1-\pi)$ is a unit in $(1-\pi)E(1-\pi)$.*
(3) $A = \underbrace{\begin{pmatrix} R & {}_RP_S \\ 0 & S \end{pmatrix}}_{A'} \oplus \underbrace{\begin{pmatrix} R & {}_RQ_S \\ 0 & S \end{pmatrix}}_{A''}$, *where A' and A'' are α-invariant, and $\alpha \mid_{A'}$ and $(1-\alpha) \mid_{A''}$ are isomorphisms.*
(4) *$A = A' \oplus A''$, where A' and A'' are α-invariant, $ker(\alpha) \subseteq A'' \subseteq (1-\alpha)(A)$ and $ker(1-\alpha) \subseteq A' \subseteq \alpha(A)$.*
(5) *$A = A_1 \oplus A_2 \oplus ... \oplus A_n$ for some $n \geq 1$ where A_i is α-invariant and $\alpha \mid_{A_i}$ is strongly clean in $End(A_i)$ for each i.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha = (1-\pi) + \sigma$ where $\pi\sigma = \sigma\pi$, $\pi^2 = \pi$, and $\sigma \in \mathcal{U}(E)$. Note that and α, π, and σ all commute. Now $\alpha - \sigma = 1 - \pi$ so $\alpha\pi = \sigma\pi$. Since $\sigma^{-1}\alpha \in \pi E\pi$ this gives $(\alpha\pi)(\sigma^{-1}\pi) = (\sigma\pi)(\sigma^{-1}\pi) = \pi$. Similarly, $(\sigma^{-1}\pi)(\alpha\pi) = \pi$ so $\alpha\pi$ is a unit in $\pi E\pi$. Finally, observe that $1 - \alpha = \pi + (-\sigma)$ is strongly clean too, so an analog of the above argument shows that $(1-\alpha)(1-\pi)$ is a unit in $(1-\pi)E(1-\pi)$.

(2) $\Rightarrow$ (3). Given π as in (2) let $A' = \pi(A)$ and $A'' = (1-\pi)(A)$. Then, $A = A' \oplus A''$ and $\alpha(A') = \alpha\pi(A) = \pi\alpha(A) \subseteq \pi(A) = A'$, so A' is α-invariant. Similarly, $\alpha(A'')$. If $(\alpha\pi)^{-1} = \pi\gamma\pi$ in $\pi E\pi$, let $\gamma_0 = (\pi\gamma\pi) \mid_{A'}$. Then, $\gamma_0 \in$

$End(A')$ so, if $F \in A'$ we have $(\gamma_0(\alpha \mid_{A'}))(F) = (\gamma_0\alpha)(\pi(F)) = \gamma_0(\alpha\pi)(F) = (\gamma_0\pi\alpha)(F) = \pi(F) = F$. Thus $\gamma_0\alpha \mid_{A'} = 1_{A'}$. Similarly $\alpha \mid_{A'} \gamma_0 = 1_{A'}$, because $\alpha \mid_{A'} \gamma_0(F) = (\alpha \mid_{A'} (\pi\gamma\pi))(F) = (\alpha\pi\gamma\pi)(F) = \pi(F) = F$. Thus $\alpha \mid_{A'}$ is a unit in $End(A')$. A similar argument shows that $(1-\alpha) \mid_{A''}$ is a unit in $End(A'')$.

(3) $\Rightarrow$ (4). This follows from Proposition **??**.

(4) $\Rightarrow$ (5). Suppose that $A = A' \oplus A''$ as in (4). Then, $\alpha \mid_{A'}$ is strongly clean in $End(A')$ because it is a unit by Proposition **??**. Similarly A' and A'' are $(1-\alpha)$-invariant so $(1-\alpha) \mid_{A''}$ is a unit in $End(A'')$, again by Proposition **??**. This gives (5) with $n = 2$, $A_1 = A'$ and $A_2 = A''$.

(5) $\Rightarrow$ (1). Given the situation in (5), extend maps $\lambda_i \in End(A_i)$ to $\widehat{\lambda_i} \in End(A)$ by defining

$$\widehat{\lambda_i}(F_1 + F_2 + ... + F_n) = \lambda_i(F_i).$$

Then, $\widehat{\lambda_i}\widehat{\lambda_j} = 0$ if $i \neq j$ while $\widehat{\lambda_i}\widehat{\mu_i} = \widehat{\lambda_i\mu_i}$ for all $\mu_i \in End(A_i)$. Now let $\alpha \mid_{A_i} = \pi_i + \sigma_i \in End(A_i)$ where $\pi_i^2 = \pi_i$, $\sigma_i \in \mathcal{U}(End(A_i))$ and $\pi_i\sigma_i = \sigma_i\pi_i$. If $\pi = \Sigma_i\widehat{\pi_i}$ and $\sigma = \Sigma_i\widehat{\sigma_i}$, then $\pi^2 = \Sigma_i\widehat{\pi_i}^2 = \pi$, $\pi\sigma = \Sigma_i\widehat{\pi_i}\widehat{\sigma_i} = \Sigma_i\widehat{\sigma_i}\widehat{\pi_i} = \sigma\pi$, and $\sigma \in \mathcal{U}(End(A))$ because $\sigma^{-1} = \Sigma_i\widehat{\sigma_i^{-1}}$. Since $\alpha = \Sigma_i\widehat{\alpha \mid_{A_i}} = \Sigma_i\widehat{(\pi + \sigma_i)} = \pi + \sigma$, the proof of (1) is complete. □

A ring is called uniquely clean [**?**] if every element is uniquely the sum of an idempotent and a unit.

Definition 3.4. *An element a of a ring R is called uniquely strongly clean (or USC for short) if a has a unique strongly clean expression in R as stated above. The ring R is called uniquely strongly clean (or USC for short) if every element of R is uniquely strongly clean.*

Proposition 3.5. Let $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$ and $\alpha \in End(A)$. The following are equivalent:

1. α is USC in $End(A)$.
2. There exists a unique decomposition $A = A' \oplus A''$ where A' and A'' are α-invariant, and $\alpha \mid_{A'}$ and $(1-\alpha) \mid_{A''}$ are isomorphisms.

Conclusion. A ring is said to be clean if every element of A can be written as a sum of an idempotent and a unit. Till now, many authors considered clean rings and obtained many results in this respect. In this work, we investigated the ring $A = \begin{pmatrix} R & {}_RM_S \\ 0 & S \end{pmatrix}$. In particular, we studied the endomorphisms of the ring A and gave necessary and sufficient conditions under which an endomorphism of A is strongly clean.

References

[1] Azumaya, G., "Strongly π-regular rings", *J. Fac. Sci. Hokkiado U.*, **13** (1954), 34-39.

[2] Chen, J., Wang, Z. and Zhou, Y., "Rings in which elements are uniquely the sum of an idempotent and a unit that commute", *J. Pure Appl. Algebra*, **213** (2009), 215-223.
[3] Chen, H., "Some classes of strongly clean rings", *Bull. Iran. Math. Soc.*, **30:6** (2013), 1099-1115.
[4] Nicholson, W.K., "Strongly clean rings and Fitting's lemma", *Comm. Algebra*, **27:8** (1999), 3583-3592.
[5] Lam, T.Y., *A First Course in Noncommutative Rings*, Springer-Verlag, 1991.
[6] Rajeswari, K.N. and Aziz, R., "A note on clean matrices in $M_2(\mathbb{Z})$", *Int. J. Algebra*, **3:(5-8)** (2009), 241-248.