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INSTABILITY OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS OF FIFTH ORDER

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Abstract. This paper establishes certain sufficient conditions to guarantee the non-existence of periodic solutions for a class of nonlinear vector differential equations of fifth order. With this work, we extend and improve two related results in the literature from scalar cases to vectorial cases. An example is given to illustrate the theoretical analysis made in this paper.

Key words and Phrases: Instability, Lyapunov function, nonlinear, vector differential equation, fifth order.

Abstrak. Makalah ini membahas kondisi cukup untuk menjamin non-eksistensi dari solusi periodek untuk suatu kelas vektor nonlinear dari persamaan differensial orde lima. Dalam pembahasan makalah ini, dua hasil terkait yang ada dalam literatur telah dikembankan dan diperluas dari kasus skalar menjadi kasus vektor. Sebuah contoh diberikan untuk mengilustrasikan analisis teoritis yang dibahas dalam makalah ini.

Kata kunci: .

1. Introduction

It is well known that in applied sciences, some practical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, etc. are associated with certain differential equations of higher order. Here, we would not like to give the details of them. By this time, perhaps, the most effective basic tool in the

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literature to investigate the qualitative behaviors of certain differential equations whose orders are more than two is the Lyapunov's direct method. Hence, Lyapunov functions have been successfully used and are still being used to discuss stability, instability, existence and non-existence of periodic solutions, etc. of differential equations whose orders are more than two.

In (2012), Tejumola [10] investigated the non-existence of periodic solutions to the following nonlinear scalar differential equations of fifth order

$$x^{(5)} + \phi_1(\dddot{x})x^{(4)} + \phi_2(\ddot{x})\dddot{x} + \phi_3(\dot{x})\ddot{x} + \phi_4(\dot{x}) + \phi_5(x) = 0 \tag{1}$$

and

$$x^{(5)} + b_1 x^{(4)} + \psi_2(\ddot{x})\dddot{x} + \psi_3(\dot{x})\ddot{x} + \psi_4(\dot{x}) + \psi_5(x) = 0. \tag{2}$$

The author established certain sufficient conditions which guarantee that Eq.(1) and Eq.(2) have no non-trivial periodic solutions of whatever period with the aid of the Lyapunov's direct method.

In this direction, in recent years, Ezeilo [1]-[3], Li and Duan [6], Li and Yu [7], Sadek [8], Sun and Hou [9], Tejumola [10], Tunc [11]-[13], [15], Tunc and Erdogan [16], Tunc and Karta [17], Tunc and Şevli [18], etc., continued to discuss the existence, non-existence of periodic solutions and instability of solutions to certain nonlinear scalar and vector differential equations of fifth order by the Lyapunov's second method. These researchers obtained many new and considerable results concerning to the mentioned topics. It should be noted that throughout these mentioned papers, the Lyapunov's direct method has been used as a basic tool to investigate the main results thereof.

In this paper, we focus on the work of Tejumola [10]. Namely, instead of Eq.(1) and Eq.(2), we consider their following vectorial forms:

$$X^{(5)} + \Phi_1(\dddot{X})X^{(4)} + \Phi_2(\ddot{X})\dddot{X} + \Phi_3(\dot{X})\ddot{X} + \Phi_4(\dot{X}) + \Phi_5(X) = 0 \tag{3}$$

and

$$X^{(5)} + AX^{(4)} + \Psi_2(\ddot{X})\dddot{X} + \Psi_3(\dot{X})\ddot{X} + \Psi_4(\dot{X}) + \Psi_5(X) = 0, \tag{4}$$

respectively, where $X \in \Re^n$; A is a constant $n \times n$-symmetric matrix; Φ_1, Φ_2, Φ_3, Ψ_2 and Ψ_3 are $n \times n$-symmetric continuous matrix functions; $\Phi_4 : \Re^n \to \Re^n$, $\Phi_5 : \Re^n \to \Re^n, \Psi_4 : \Re^n \to \Re^n, \Psi_5 : \Re^n \to \Re^n$ with $\Phi_4(0) = \Phi_5(0) = \Psi_4(0) = \Psi_5(0) = 0$ are continuous functions and so constructed such that the uniqueness theorem is valid.

Instead of Eq.(3) and Eq.(4), we consider their equivalent differential systems:

$$\dot{X} = Y,\ \dot{Y} = Z,\ \dot{Z} = W,\ \dot{W} = U,$$

$$\dot{U} = -\Phi_1(W)U - \Phi_2(Z)W - \Phi_3(Y)Z - \Phi_4(Y) - \Phi_5(X), \tag{5}$$

and

$$\dot{X} = Y,\ \dot{Y} = Z,\ \dot{Z} = W,\ \dot{W} = U,$$

$$\dot{U} = -AU - \Psi_2(Z)W - \Psi_3(Y)Z - \Psi_4(Y) - \Psi_5(X), \tag{6}$$

respectively.

For the sake of the brevity, we assume that the symbol $J_{\Phi_1}(W),\ J_{\Phi_2}(Z),\ J_{\Phi_4}(Y),\ J_{\Phi_5}(X),\ J_{\Psi_2}(Z),\ J_{\Psi_3}(Y),\ J_{\Psi_4}(Y)$ and $J_{\Psi_5}(X)$ denote the Jacobian matrices corresponding to $\Phi_1,\ \Phi_2,\ \Phi_4,\ \Phi_5, \Psi_2, \Psi_3, \Psi_4$ and Ψ_5, respectively. In addition, it is assumed, as basic throughout the paper, that these Jacobian matrices exist and are continuous and symmetric.

We establish two new theorems on the non-existence of periodic solutions of Eq.(3) and Eq.(4). This paper is inspired by the results established in the aforementioned papers, Tunc [14] and in the literature. Our aim is to generalize and improve the results of Tejumola [10, Theorem 3, 5]. This paper has also a contribution to the subject in the literature, and it may be useful for researchers who work on the qualitative behaviors of solutions. The equation considered and the assumptions to be established here are different from those in aforementioned papers and in the literature.

The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$ and $\lambda_i(A),\ (A = (a_{ij})),\ (i, j = 1, 2, \ldots, n)$ are the eigenvalues of the $n \times n$-symmetric matrix A and the matrix $A = (a_{ij})$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \Re^n$ and X^T denotes the transpose of X.

Consider the linear constant coefficient differential equation of fifth order:

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + a_3 \ddot{x} + a_4 \dot{x} + a_5 x = 0, \tag{7}$$

where $a_1,\ a_2, \ldots, a_5$ are some real constants. It can be seen from Tejumola [10] that if either of the conditions

(i) $a_1 \neq 0,\ sgn\ a_1 = sgn\ a_5,\ a_3\ sgn\ a_1 < 0$

and

(ii) $a_2 < 0,\ a_4 > 0$

holds, then Eq.(7) has no non-trivial periodic solutions of any period. It should also be noted that these odd and even subscripts features run through the generalized criteria obtained for the non-linear equations studied here.

2. Main Results

The following lemma plays a key role in proving our main results.

Lemma 2.1. *(Horn and Johnson* [4]*) Let A be a real $n \times n$-symmetric matrix and*

$$a' \geq \lambda_i(A) \geq a,\ \ (i = 1, 2, \ldots, n)$$

where a' and a are some positive constants. Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle\,.$$

Our first main result is the following theorem.

Theorem 2.2. *In addition to the basic assumptions imposed on* $\Phi_1,\ \Phi_2,\ \Phi_3,\ \Phi_4$ *and* Φ_5 *that appearing in Eq.(3), we assume that there are positive constants* $a_1,\ a_4$ *and* a_5 *and* $a_3(<0)$ *such that the following assumptions hold:*

$$\lambda_i(\Phi_1(W)) \geq a_1,\ \lambda_i(\Phi_3(Y)) \leq a_3,\ \lambda_i(J_{\Phi_4}(Y)) \geq a_4,$$

$$\Phi_5\,(0) = 0,\ \Phi_5(X) \neq 0,\ \textit{when } X \neq 0, \lambda_i(J_{\Phi_5}(X)) \geq a_5.$$

Then, Eq.(3) has no non-trivial periodic solution of whatever period.

Remark 1. There is no restriction on matix function Φ_2 except Φ_2 is an $n \times n$-symmetric continuous matrix function.

Remark 2. To complete the proof of Theorem 1, subject to the assumptions of Theorem 1, we have to show that there exists a Lyapunov function $V = V(X, Y, Z, W, U)$, which satisfies the following Krasovskiis [5] criteria:

- (K_1)**:** In every neighborhood of $(0, 0, 0, 0, 0)$ there exists a point $(\xi, \eta, \zeta, \mu, \rho)$ such that $V(\xi, \eta, \zeta, \mu, \rho) > 0$;
- (K_2)**:** the time derivative $\dot{V}$ along solution paths of the system (5) is positive semi-definite;
- (K_3)**:** the only solution $(X, Y, Z, W, U) = (X(t), Y(t), Z(t), W(t), U(t))$ of system (5) which satisfies $\dot{V} = 0, (t \geq 0)$, is the trivial solution $(0, 0, 0, 0, 0)$.

These properties guarantee that Eq.(3) has no non-trivial periodic solution of any period.

It should be noted that a similar discussion can be made for our second main result, Theorem 2.

PROOF. To prove Theorem 1, we define a Lyapunov function $V = V(X, Y, Z, W, U)$:

$$\begin{aligned} V \;&=\; \int_0^1 \langle \Phi_1(\sigma W)W, Z\rangle\, d\sigma + \int_0^1 \langle \sigma\Phi_2(\sigma Z)Z, Z\rangle\, d\sigma - \frac{1}{2}\langle W, W\rangle \\ &\quad + \int_0^1 \langle \Phi_4(\sigma Y), Y\rangle\, d\sigma + \langle \Phi_5(X), Y\rangle + \langle U, Z\rangle\,. \end{aligned} \tag{8}$$

First, it is easy to see from (8) that

$$V(0, 0, 0, 0, 0) = 0.$$

In view of the estimates $\frac{\partial}{\partial\sigma}\Phi_4(\sigma Y) = J_{\Phi_4}(\sigma Y)Y$ and $\Phi_4(0) = 0$, it follows, on integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$, that

$$\Phi_4(Y) = \int_0^1 J_{\Phi_4}(\sigma_1 Y)Y\, d\sigma_1.$$

Hence, we have

$$\begin{aligned} \int_0^1 \langle \Phi_4(\sigma Y), Y\rangle\, d\sigma \;&=\; \int_0^1\int_0^1 \langle \sigma_1 J_{\Phi_4}(\sigma_1\sigma_2 Y)Y, Y\rangle\, d\sigma_2 d\sigma_1 \\ &\geq\; \int_0^1\int_0^1 \langle \sigma_1 a_4 Y, Y\rangle\, d\sigma_2 d\sigma_1 \geq \frac{1}{2}a_4\,\langle Y, Y\rangle \end{aligned}$$

by the assumption $\lambda_i(J_{\Phi_4}(Y)) \geq a_4$.

Using this estimate, one can easily see that

$$V(0,\varepsilon,0,0,0) \geq \frac{1}{2}a_4 \langle \varepsilon,\varepsilon\rangle = \frac{1}{2}a_4 \|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \neq 0,\ \varepsilon \in \Re^n$, which verifies the property (K_1) of Krasovskii [5].

Let $(X,Y,Z,W,U) = (X(t),Y(t),Z(t),W(t),U(t))$ be an arbitrary solution of system (5). Differentiating the Lyapunov function V with respect to the time t along this solution, we get

$$\begin{aligned}
\frac{d}{dt}V \quad = \quad & \frac{d}{dt}\int_0^1 \langle \Phi_1(\sigma W)W, Z\rangle\, d\sigma + \langle J_{\Phi_5}(X)Y, Y\rangle + \frac{d}{dt}\int_0^1 \langle \sigma\Phi_2(\sigma Z)Z, Z\rangle\, d\sigma \\
& - \langle \Phi_3(Y)Z, Z\rangle + \frac{d}{dt}\int_0^1 \langle \Phi_4(\sigma Y), Y\rangle\, d\sigma - \langle \Phi_1(W)U, Z\rangle \\
& - \langle \Phi_2(Z)W, Z\rangle - \langle \Phi_4(Y), Z\rangle\,.
\end{aligned} \tag{9}$$

It can be checked that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \langle \Phi_1(\sigma W)W, Z\rangle\, d\sigma \quad = \quad & \int_0^1 \langle \Phi_1(\sigma W)W, W\rangle\, d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma} \langle \Phi_1(\sigma W)U, Z\rangle\, d\sigma \\
& + \int_0^1 \langle \Phi_1(\sigma W)U, Z\rangle\, d\sigma \\
= \quad & \sigma \langle \Phi_1(\sigma W)U, Z\rangle \big|_0^1 + \int_0^1 \langle \Phi_1(\sigma W)W, W\rangle\, d\sigma \\
= \quad & \langle \Phi_1(W)U, Z\rangle + \int_0^1 \langle \Phi_1(\sigma W)W, W\rangle\, d\sigma,
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \langle \sigma\Phi_2(\sigma Z)Z, Z\rangle\, d\sigma \quad = \quad & \int_0^1 \sigma\frac{\partial}{\partial\sigma} \langle \Phi_2(\sigma Z)W, Z\rangle\, d\sigma + \int_0^1 \langle \sigma\Phi_2(\sigma Z)W, Z\rangle\, d\sigma \\
= \quad & \sigma^2 \langle \Phi_2(\sigma Z)W, Z\rangle \big|_0^1 = \langle \Phi_2(\sigma Z)W, Z\rangle
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \langle \Phi_4(\sigma Y), Y\rangle\, d\sigma \quad = \quad & \int_0^1 \sigma \langle J_{\Phi 4}(\sigma Y)Z, Y\rangle\, d\sigma + \int_0^1 \langle \Phi_2(\sigma Y), Z\rangle\, d\sigma \\
= \quad & \int_0^1 \sigma\frac{\partial}{\partial\sigma} \langle \Phi_4(\sigma Y), Z\rangle\, d\sigma + \int_0^1 \langle \Phi_4(\sigma Y), Z\rangle\, d\sigma \\
= \quad & \sigma \langle \Phi_4(\sigma Y), Z\rangle \big|_0^1 = \langle \Phi_4(Y), Z\rangle\,.
\end{aligned}$$

Combining the last three estimates into (9) and in viewing of Lemma and the assumptions of Theorem 1, $\lambda_i(\Phi_1(W)) \geq a_1 > 0,\ \lambda_i(\Phi_3(Y)) \leq a_3 < 0,\ \lambda_i(J_{\Phi_5}(X)) \geq$

$a_5 > 0$, we obtain

$$\dot{V} = \langle J_{\Phi_5}(X)Y, Y\rangle - \langle \Phi_3(Y)Z, Z\rangle + \int_0^1 \langle \Phi_1(\sigma W)W, W\rangle \, d\sigma \geq 0,$$

which verifies the property (K_2) of Krasovskii [5].

Thus, the assumptions of Theorem 1 imply that $\dot{V}(t) \geq 0$ for all $t \geq 0$, that is, $\dot{V}$ is positive semi-definite. Finally, $\dot{V} = 0, (t \geq 0)$, necessarily implies that $Y = 0$ for all $t \geq 0$, and $Z = \dot{Y} = 0,\ W = \ddot{Y} = 0,\ \ \dot{W} = \dddot{Y} = 0$ for all $t \geq 0$ so that

$$X = \xi,\ (\xi \neq 0 \text{ is a constant vector}),\ Y = Z = W = U = 0.$$

From the last estimate and system (5), we have $\Phi_5(\xi) = 0$ which necessarily implies that $\xi = 0$ since $\Phi_5(0) = 0$. Then

$$X = Y = Z = W = U = 0 \text{ for all } t \geq 0,$$

which verifies the property (K_3) of Krasovskii [5]. Therefore, the Lyapunov function has the entire criteria of Krasovskii [5] if the assumptions of Theorem 1 hold. Thus, the basic properties of the Lyapunov function which were shown above, prove that system (5) has no non-trivial periodic solutions of whatever period. Since system (5) is equivalent to Eq.(3), this completes the proof of Theorem 1.

Example. As a special case of system (5), we choose $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and Φ_5 as the following:

$$\Phi_1(W) = \begin{bmatrix} 2 + (1 + w^2)^{-1} & 1 \\ 1 & 2 + (1 + w^2)^{-1} \end{bmatrix},$$

$$\Phi_2(Z) = \begin{bmatrix} -2 - z^2 & 1 \\ 1 & -2 - z^2 \end{bmatrix},$$

$$\Phi_3(Z) = \begin{bmatrix} -4 - y^2 & 1 \\ 1 & -4 - y^2 \end{bmatrix},$$

$$\Phi_4(Z) = \begin{bmatrix} 2y + \arctan y \\ 2y + \arctan y \end{bmatrix},\ \Phi_4(0) = 0,$$

$$\Phi_5(Z) = \begin{bmatrix} 3x + \arctan x \\ 3x + \arctan x \end{bmatrix},\ \Phi_5(0) = 0.$$

It follows from Φ_4 and Φ_5 that

$$J_{\Phi_4}(Y) = \begin{bmatrix} 2 + (1 + y^2)^{-1} & 0 \\ 0 & 2 + (1 + y^2)^{-1} \end{bmatrix}$$

and

$$\lambda_i(J_{\Phi_5}(X)) = \begin{bmatrix} 3 + (1 + x^2)^{-1} & 0 \\ 0 & 3 + (1 + x^2)^{-1} \end{bmatrix}.$$

Then, respectively, we get

$$\lambda_1(\Phi_1(W)) = 1 + \frac{1}{1 + w^2},\ \lambda_2(\Phi_1(W)) = 3 + \frac{1}{1 + w^2},$$

$$\lambda_i(\Phi_1(W)) \geq 1 = a_1,$$

$$\lambda_1(\Phi_2(Z)) = -1 - z^2,\ \lambda_2(\Phi_2(Z)) = -3 - z^2,$$

$$\lambda_i(\Phi_2(Z)) < 0,$$

$$\lambda_1(\Phi_3(Y)) = -3 - y^2,\ \lambda_2(\Phi_3(Y)) = -5 - y^2,$$

$$\lambda_i(\Phi_3(Y)) \leq -3 = a_3,$$

$$\lambda_1(J_{\Phi_4}(Y)) = 2 + \frac{1}{1+y^2},\ \lambda_2(J_{\Phi_4}(Y)) = 2 + \frac{1}{1+y^2},$$

$$\lambda_i(J_{\Phi_4}(Y)) \geq 2 = a_4,$$

$$\lambda_1(J_{\Phi_5}(X)) = 3 + \frac{1}{1+x^2},\ \lambda_2(J_{\Phi_5}(X)) = 3 + \frac{1}{1+x^2},$$

$$\lambda_i(J_{\Phi_5}(X)) \geq 3 = a_5.$$

Thus, it is shown that all the assumptions of Theorem1 hold.

Our second main result is the following theorem.

Theorem 2.3. *In addition to the basic assumptions imposed on $A,\ \Psi_2,\ \Psi_3,\ \Psi_4$ and Ψ_5 that appearing in Eq.(4), we assume that there are constants $b_1(>0), b_2(<0),\ b_3(<0), b_4(>0)$ and $b_5(>0)$ such that the following conditions hold:*

$$\lambda_i(A) \geq b_1,\ \lambda_i(\Psi_2(Z)) \leq b_2,\quad \lambda_i(\Psi_3(Y)) \leq b_3,$$

$$\lambda_i(J_{\Psi_4}(Y)) \geq b_4,\ \Psi_5(0) = 0,\ \Psi_5(X) \neq 0 \quad \textit{when } X \neq 0,\quad \lambda_i(\Psi_5(Y)) \geq b_5.$$

Then, Eq.(4) has no non-trivial periodic solution of whatever period.

Proof. To prove Theorem 2, we define a Lyapunov function $V_1 = V_1\,(X, Y, Z, W, U)$:

$$\begin{aligned} V_1 \quad = \quad & -\int_0^1 \langle \Psi_2(\sigma Z)Z, Y\rangle \, d\sigma - \langle U, Y\rangle - \int_0^1 \langle \sigma \Psi_3(\sigma Y)Y, Y\rangle \, d\sigma \\ & -\int_0^1 \langle \Psi_5(\sigma X), X\rangle \, d\sigma - \langle AY, W\rangle + \frac{1}{2}\langle AZ, Z\rangle + \langle Z, W\rangle\,. \end{aligned} \tag{10}$$

It is easy to see from (10) that

$$V\,(0,0,0,0,0) = 0.$$

and

$$\begin{aligned} V(0,0,\varepsilon,\varepsilon,0) &\geq \frac{1}{2}\langle A\varepsilon,\varepsilon\rangle + \langle\varepsilon,\varepsilon\rangle \\ &\geq \frac{1}{2}(b_1+1)\|\varepsilon\|^2 > 0, \end{aligned}$$

for all arbitrary $\varepsilon \neq 0,\ \varepsilon \in \Re^n$ by the assumption $\lambda_i(A) \geq b_1 > 0$.

Finally, let $(X,Y,Z,W,U) = (X(t),Y(t),Z(t),W(t),U(t))$ be an arbitrary solution of system (6). Differentiating the Lyapunov function V_1 with respect to the time t along this solution, we obtain

$$\begin{aligned} \dot{V}_1 &= -\frac{d}{dt}\int_0^1 \langle\Psi_2(\sigma Z)Z,Y\rangle\, d\sigma - \frac{d}{dt}\int_0^1 \langle\sigma\Psi_3(\sigma Y)Y,Y\rangle\, d\sigma \\ &\quad -\frac{d}{dt}\int_0^1 \langle\Psi_5(\sigma X),X\rangle\, d\sigma + \langle\Psi_2(Z)W,Y\rangle \\ &\quad +\langle\Psi_3(Y)Z,Y\rangle + \langle\Psi_4(Y),Y\rangle + \langle\Psi_5(X),Y\rangle + \langle W,W\rangle . \end{aligned} \tag{11}$$

It can be checked that

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle\Psi_2(\sigma Z)Z,Y\rangle\, d\sigma &= \int_0^1 \langle\Psi_2(\sigma Z)Z,Z\rangle\, d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle\Psi_2(\sigma Z)W,Y\rangle\, d\sigma \\ &\quad + \int_0^1 \langle\Psi_2(\sigma Z)W,Y\rangle\, d\sigma \\ &= \sigma\langle\Psi_2(\sigma Z)W,Y\rangle\big|_0^1 + \int_0^1 \langle\Psi_2(\sigma Z)Z,Z\rangle\, d\sigma \\ &= \langle\Psi_2(Z)W,Y\rangle + \int_0^1 \langle\Psi_2(\sigma Z)Z,Z\rangle\, d\sigma, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle\Psi_3(\sigma Y)Y,Y\rangle\, d\sigma &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle\sigma\Psi_3(\sigma Y)Z,Y\rangle\, d\sigma + \int_0^1 \langle\sigma\Psi_2(\sigma Y)Z,Y\rangle\, d\sigma \\ &= \sigma^2\langle\Psi_3(\sigma Y)Z,Y\rangle\big|_0^1 = \langle\Psi_3(Y)Z,Y\rangle , \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle\Psi_5(\sigma X),X\rangle\, d\sigma &= \int_0^1 \sigma\langle J_{\Psi_5}(\sigma X)Y,X\rangle\, d\sigma + \int_0^1 \langle\Psi_5(\sigma X),Y\rangle\, d\sigma \\ &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle\Psi_5(\sigma X),Y\rangle\, d\sigma + \int_0^1 \langle\Psi_5(\sigma X),Y\rangle\, d\sigma \\ &= \sigma\langle\Psi_5(\sigma X),Y\rangle\big|_0^1 = \langle\Psi_5(X),Y\rangle . \end{aligned}$$

Substituting the last three estimates into (11), we have

$$\dot{V}_1 = \langle\Psi_4(Y),Y\rangle - \int_0^1 \langle\Psi_2(\sigma Z)Z,Z\rangle\, d\sigma + \langle W,W\rangle .$$

On the other hand, it is clear that

$$\Psi_4(Y) = \int_0^1 J_{\Psi_4}(\sigma_1 Y) Y d\sigma_1$$

so that

$$\langle \Psi_4(Y), Y \rangle = \left\langle \int_0^1 J_{\Psi_4}(\sigma_1 Y) Y d\sigma_1, Y \right\rangle \geq \frac{1}{2} b_4 \langle Y, Y \rangle$$

by $\lambda_i \left(J_{\Phi_4} (Y) \right) \geq b_4 > 0$.

Then,

$$\dot{V}_1 \geq \frac{1}{2} b_4 \langle Y, Y \rangle - \int_0^1 \langle \Psi_2(\sigma Z) Z, Z \rangle \, d\sigma + \langle W, W \rangle \geq 0$$

by the assumptions of Theorem 2. The rest of the proof is similar to the proof of Theorem 1. Therefore, we omit the details of the poof.

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