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EDGE_TRANSITIVE DIHEDRAL COVERS OF THE HEAWOOD GRAPH

Mehdi Alaeiyan[1], Laleh Pourmokhtar[2]

[1]**Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran** 16844,
alaeiyan@iust.ac.ir
laleh_ pourmokhtar@iust.ac.ir

Abstract. A graph is called edge-transitive if its automorphism group acts transitively on its edge set and a regular cover of a connected graph is called *dihedral* if its transformation group is dihedral. In this paper, the authors classify all dihedral coverings of the Heawood graph whose fibre-preserving automorphism subgroups act edge-transitively.

Key words and Phrases: regular covering, edge-transitive graphs, the Heawood graph.

Abstrak. Suatu graf disebut transitif sisi jika grup automorfisma graf tersebut beraksi secara transitif pada himpunan sisinya. Suatu *regular cover* dari graf terhubung disebut dihedral jika grup transformasinya adalah dihedral. Dalam paper ini, penulis mengklasifikasikan semua *dihedral covering* dari graf Heawood yang subgrup automorfisma *fibre-preserving*-nya beraksi secara transitif sisi.

Kata kunci: *regular covering, graf transitif sisi, graf Heawood*

1. INTRODUCTION

Throughout this paper, we consider finite connected graphs without loops or multiple edges. For a graph X, each edge X gives rise to a pair of opposite arcs and we denote by $V(X)$, $E(X)$, $A(X)$ and $Aut(X)$ the vertex set, the edge set, the arc set and the full automorphism group of X, respectively. The neighbourhood of a vertex $v \in V(X)$, denoted by $N(v)$, is the set of vertices adjacent to v in X. Let a group G act on a set Ω, and let $\alpha \in \Omega$. We denote by G_α the stabilizer of α in G, that is the subgroup of G fixing α. The group G is said to be semiregular if

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$G_\alpha = 1$ for each $\alpha \in \Omega$, and regular if G is semiregular and transitive on Ω.
Let N be a subgroup of $Aut(X)$ such that N is intransitive on $V(X)$. The *quotient graph* X/N induced by N is defined as the graph for which the Σ set of N-orbits in $V(X)$ is the vertex set of X/N and $B, C \in \Sigma$ are adjacent if and only if there exists $u \in B$ and $v \in C$ such that $uv \in E(X)$.

A graph $\widetilde{X}$ is called a *covering* of a graph X with a projection $\rho : \widetilde{X} \to X$, if ρ is a surjection from $V(\widetilde{X})$ to $V(X)$ such that $\rho|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \rho^{-1}(v)$. The graph $\widetilde{X}$ is called the *covering graph* and X is the *base graph*. A covering $\widetilde{X}$ of X with a projection ρ is said to be regular (or K- covering) if there is a semiregular subgroup K of the automorphism group $Aut(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ρh of ρ and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic, elementary abelian or dihedral, then $\widetilde{X}$ is called a *cyclic*, *elementary abelian* or *dihedral* covering of X, and if $\widetilde{X}$ is connected, K becomes the *covering transformation group*. The *fibre* of an edge or a vertex is its preimage under ρ. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps each fibre on to itself. All of such fibre-preserving automorphisms form a group called *fibre-preserving group*.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$; in other words, it is a directed walk of length s which never includes a backtracking. A graph X is said to be *s-arc-transitive* if $Aut(X)$ acts transitively on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. An s-arc-transitive graph is said to be *s-transitive* if it is not $(s+1)$-arc-transitive. A symmetric graph X is said to be *s-regular* if for any two s-arcs in X, there is a unique automorphism of X mapping one to the other. In other words, the automorphism group $Aut(X)$ acts freely and transitively (i.e. regularly) on the set of s-arcs in X. A subgroup of the automorphism group of a graph X is said to be s-regular if it acts regularly on the set of s-arcs of X.

Regular coverings of a graph have received considerable attention. For example, for a graph X which is the complete graph K_4, the complete bipartite graph $K_{3,3}$, hypercube Q_3 or Petersen graph O_3, the s-regular cyclic or elementary abelian coverings of X, whose fibre-preserving groups are arc-transitive, classified for each $1 \leqslant s \leqslant 5$ in [5, 6, 7, 8, 10]. As an application of these classifications, all s-regular cubic graphs of order $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$ and $10p^2$ constructed for each $1 \leqslant s \leqslant 5$ and each prime p in [1, 5, 6, 8]. In [14], it was shown that all cubic graphs admitting a solvable edge-transitive group of automorphisms arise as regular covers of one of the following basic graphs: the complete graph K_4, the dipole Dip_3 with two vertices and three parallel edges, the complete bipartite graph $K_{3,3}$, the Pappus graph of order 18, and the Gray graph of order 54. In this paper all dihedral coverings of the Heawood graph, whose fibre- preserving automorphism subgroups act arc-transitively are determined.

2. PRELIMINARIES

We start with some notational conventions used throughout this paper. Let n be a positive integer. Denote by $\mathbb{Z}_n^*$ the multiplicative group consisting of numbers coprime to n. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For an abelian group H, the generalized dihedral group $Dih(H)$ is the semidirect product $H \rtimes \mathbb{Z}_2$, where the unique involution in $\mathbb{Z}_2$ maps each element of H to its inverse. In particular, $Dih(\mathbb{Z}_n)$ is the dihedral group D_{2n} of order $2n$. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. It is easy to see that $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.1. [12] *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $Aut(H)$ of H.*

Let X be a cubic graph and let $G \leqslant Aut(X)$ act transitively on the edges of X. Let N be a normal subgroup of G. The quotient graph X_N of X relative to N is defined as the graph with vertices the orbits of N in $V(X)$ and with two orbits adjacent if there is an edge in X between the vertices lying in those two orbits. Below we introduce two propositions, of which the first is a special case of [13, Theorem 9].

Proposition 2.2. *Let X be a cubic graph and let $G \leqslant Aut(X)$ be transitive on $E(X)$ and $V(X)$. Then G is an s-arc-regular subgroup of $Aut(X)$ for some integer s. If $N \lhd G$ has more than two orbits in $V(X)$, then N is semiregular on $V(X)$, X_N is a cubic symmetric graph with G/N as an s-arc-regular group of automorphisms, and X is an N-cover of X_N.*

Given a finite group G and an inverse closed subset $S \subseteq G - \{1\}$, the Cayley graph $Cay(G,S)$ on G relative to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. It is known that $Cay(G,S)$ is connected if and only if S generates G. Given $g \in G$, define the permutation $R(g)$ on G by $x \longmapsto xg$, $x \in G$. Then $R(G) = \{R(g) | g \in G\}$, called the right regular representation of G, is a permutation group isomorphic to G, which acts regularly on G. Thus the Cayley graph $Cay(G,S)$ is vertex-transitive. A Cayley graph $Cay(G,S)$ is said to be normal if $R(G)$ is normal in $Aut(Cay(G,S))$. It is easy to see that the group $Aut(G,S) = \{a \in Aut(G) | Sa = S\}$ is a sub group of $Aut(Cay(G,S))_1$, the stabilizer of the vertex 1 in $Aut(Cay(G,S))$. Godsil [11, Corollary 2.3] proved the following proposition (see also Xu [16, Proposition 1.5]).

Proposition 2.3. *$Cay(G,S)$ is normal if and only if $Aut(Cay(G,S))_1 = Aut(G,S)$.*

Let m and k be positive integers. Let $Dih(\mathbb{Z}_{mk} \times \mathbb{Z}_m) = \langle a, b, c \mid a^2 = b^{mk} = c^m = 1, aba = b^{-1}, aca = c^{-1}, bc = cb\rangle$. Assume that $\lambda = 0$ for $k = 1$ and $\lambda^2 + \lambda + 1 \equiv 0 \pmod{k}$ for $k > 1$. Define

$$DC(m, k, \lambda) = Cay(Dih(\mathbb{Z}_{mk} \times \mathbb{Z}_m), a, ab, ab^{-\lambda}c). \tag{1}$$

By [15, Theorem 1] or [3, 9], we have the following proposition.

Proposition 2.4. *Let $k > 1$ be an odd integer and m a positive integer. Then every connected cubic symmetric Cayley graph on $Dih(\mathbb{Z}_{mk} \times \mathbb{Z}_m)$ is isomorphic to some $DC(m, k, \lambda)$. Furthermore,*
(1) $DC(3, 1, 0)$ is the 3-arc-regular Pappus graph;
(2) $DC(1, 7, 2) \cong DC(1, 7, 4)$ is the 4-arc-regular Heawood graph;
(3) $DC(m, 1, 0)$ and $DC(m, 3, 1)$ $(m > 1)$ are 2-arc-regular and normal;
(4) If $k > 3$ and $(m, k) \neq (1, 7)$, then the graphs $DC(m, k, \lambda)$ are normal and 1-arc-regular, and for any two distinct values λ_1 and λ_2 satisfying the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_k$, $DC(m, k, \lambda_1) \cong DC(m, k, \lambda_2)$ if and only if $\lambda_1\lambda_2 \equiv 1 \pmod{k}$.

Proposition 2.5. *Let $n > 3$ be an integer. Then there exists a solution $\lambda \in \mathbb{Z}_n$ of the equation*

$$x^2 + x + 1 = 0 \tag{2}$$

if and only if $n = 3^t p^{k_1} 1 \dots p^{k_s}$, where $t \leqslant 1$, $s \geqslant 1$ and $p_i s$ are distinct primes such that $p_i \equiv 1 \pmod{3}$. Furthermore, if Equation (2) has a solution in $\mathbb{Z}_n$, then it has exactly 2^s solutions.

Let p be a prime congruent to 1 modulo 3. By Proposition 2.5, Equation (2) has exactly two solutions in $\mathbb{Z}_p$ which are just the two elements of $\mathbb{Z}_p^*$ of order 3. Combining this fact with Proposition 2.4, we know that $DC(1, p, \lambda)$ is independent of the choice of λ. Thus, we shall denote this graph by DC_{2p}.

3. DIHEDRAL COVERS OF THE HEAWOOD GRAPH

In [7], Feng and Kwak classified all dihedral covers of K_4, whose fiber preserving groups are edge-transitive. The main purpose of this section is to generalize this result to the Heawood graph. We first prove the following lemmas.

Lemma 3.1. *Let X be a connected cubic graph, and let $H \leqslant Aut(X)$ be abelian and act semiregularly on $V(X)$. If H has two orbits each of which contains no edges of X, then X is isomorphic to a Cayley graph on $Dih(H)$.*

Proof Let $\triangle = \{\triangle(h) \mid h \in H\}$ and $\acute{\triangle} = \{\acute{\triangle}(h) \mid h \in H\}$ be the two orbits of H in $V(X)$. One may assume that the actions of H on $\triangle$ and $\acute{\triangle}$ are just by right multiplication, that is, $\triangle(h)^g = \triangle(hg)$ and $\acute{\triangle}(h)^g = \acute{\triangle}(hg)$ for any $h, g \in H$. By assumption, there are no edges in $\triangle$ and $\acute{\triangle}$, implying that X is bipartite. Let the neighbors of $\triangle(1)$ be $\acute{\triangle}(h_1)$, $\acute{\triangle}(h_2)$ and $\acute{\triangle}(h_3)$, where $h_1, h_2, h_3 \in H$. Since H is abelian, for any $h \in H$, the neighbors of $\triangle(h)$ are $\acute{\triangle}(hh_1), \acute{\triangle}(hh_2)$ and $\acute{\triangle}(hh_3)$, and the neighbors of $\acute{\triangle}(h)$ are $\triangle(hh_1^{-1}), \triangle(hh_2^{-1})$ and $\triangle(hh_3^{-1})$. It is easy to see that the map α defined by $\triangle(h) \longmapsto \acute{\triangle}(h^{-1}), \acute{\triangle}(h) \longmapsto \triangle(h^{-1})$ for any $h \in H$, is an automorphism of X of order 2. For any $\acute{h}, h \in H$, one has $\triangle(\acute{h})^{\alpha h \alpha} = \triangle(\acute{h}h^{-1}) = \triangle(\acute{h})^{h^{-1}}$ and $\acute{\triangle}(\acute{h})^{\alpha h \alpha} = \acute{\triangle}(\acute{h}h^{-1}) = \acute{\triangle}(\acute{h})^{h^{-1}}$, implying that $h\alpha = h^{-1}$. It follows that $\langle H, \alpha \rangle \cong Dih(H)$ acts regularly on $V(X)$, and hence X is isomorphic to a Cayley graph on $Dih(H)$. □

Lemma 3.2. *Let $G \leqslant Aut(DC_{14})$ act edge-transitively on DC_{14}. Then G contains a subgroup acting regularly on the edges (not arcs) of DC_{14}.*

Proof We know that DC_{14} is the Heawood graph with automorphism group $PGL(2,7)$. Since G is edge-transitive on DC_{14}, $G \cong \mathbb{Z}_7, \mathbb{Z}_3, PSL(2,7)$ or $PGL(2,7)$. Thus, G has a subgroup $N \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ acting regularly on the edges of DC_{14}. □

By Propositions 2.4, 2.5 it is easy to see that the graph $DC(2, p, \lambda)$ is independent of the choice of λ. For the convenience of statement, we denote this graph by DC_{8p}.

The main purpose of this paper is to prove the following theorem.

Theorem 3.3. *Let X be the Heawood graph. Let $n > 1$ be an integer. Then $\tilde{X}$ is a connected edge-transitive D_{2n}-cover of X if and only if is isomorphic to DC_{56}.*

Proof First, we show the sufficiency. By Equation (1), $DC_{56} = Cay(G, \{a, ab, ab^{-\lambda}c\})$, where $G = \langle a, b, c \mid a^2 = b^{14} = c^2 = 1, aba = b^{-1}, ac = ca, bc = cb\rangle$ and $\lambda^2+\lambda+1 \equiv 0 \pmod 7$. From Proposition 2.4, it follows that $R(G) \lhd Aut(DC56)$. It is easy to see that $N = \langle R(b^p), R(c)\rangle \cong D_4$ is the maximal normal 2-subgroup of $R(G)$. So, N is characteristic in $R(G)$ and hence it is normal in $Aut(DC_{56})$. Clearly, N has more than two orbits in $V(DC_{56})$. By Proposition 2.2, the quotient graph $(DC_{56})_N$ of DC_{56} relative to N is a cubic symmetric graph of order 14, and DC_{56} is an N-cover of $(DC_{56})_N$. We know $(DC_{56})_N$ is a cubic symmetric graph of order 14 and by [2], $(DC_{56})_N \cong DC_{14}$. We note that, DC_{14} is the Heawood graph (the only cubic symmetric graph of order 14). Thus, DC_{56} is a D_4-cover of DC_{14}.

For the necessity, let $\tilde{X}$ be a connected edge-transitive D_{2n}-cover of the Heawood graph and $n > 1$ an integer. Let $K = D_{2n}$ and let F be the fibre-preserving group. Then $K \lhd F$. Since F is edge-transitive on $\tilde{X}$, F/K is an edge-transitive group of automorphisms of X.

Assume $n = 2$. Then $K \cong D_4$. By Lemma 3.2, F/K contains a subgroup $M/K (\cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ acting regularly on the edges of X. Let $C = C_M(K)$. Then $K \leqslant C$ and by Proposition 2.1, $M/CAut(K) \cong GL(2,3)$. Since $M/K \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, one has $7 \mid |C/K|$. Let $N/K \leqslant M/K$ such that $N/K \cong \mathbb{Z}_7$. Then N/K is the normal Sylow 7-subgroup of M/K. Since $7 \mid |C/K|$, it follows that $N/K \leqslant C/K$, and hence $N \cong \mathbb{Z}_{14} \times \mathbb{Z}_2$. Clearly, N acts semiregularly on $V(\tilde{X})$ with two orbits. Since M is edge-transitive on $\tilde{X}$, the normality of N in M implies that each orbit of N contains no edges of $\tilde{X}$. By Lemma 3.1, X is isomorphic to a Cayley graph on $Dih(\mathbb{Z}_{14} \times \mathbb{Z}_2)$, and by Proposition 2.4, $\tilde{X} \cong DC(2,7,\lambda) = DC_{56}$.

Assume $n > 2$. Recall that $K = D_{2n}$. Let N be the cyclic subgroup of K of order n. Since $n > 2$, N is characteristic in K. Then $N \lhd F$ because $K \lhd F$. By Proposition 2.2, the quotient graph X_N of $\tilde{X}$ relative to N is a connected cubic edge-transitive graph of order 28 with F/N as an edge-transitive group of automorphisms. By [4], every connected cubic edge-transitive graph of order 28 is also arc-transitive. Then, X_N is the 3-arc-regular Coxeter graph of order 28, which is non-bipartite by [2]. It follows that F/N is also arc-transitive one X_N. Since $Aut(\tilde{X_N}) \cong PGL(2,7)$, one has $F/N \cong PSL(2,7)$ or $PGL(2,7)$. However, since $K \lhd F$, $K/N \cong \mathbb{Z}_2$ is a normal subgroup of F/N, a contradiction.

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