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ON JOINTLY PRIME RADICALS OF (R,S)-MODULES

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Abstract. Let M be an (R, S)-module. In this paper a generalization of the m-system set of modules to (R, S)-modules is given. Then for an (R, S)-submodule N of M, we define $\sqrt[(R,S)]{N}$ as the set of $a \in M$ such that every m-system containing a meets N. It is shown that $\sqrt[(R,S)]{N}$ is the intersection of all jointly prime (R, S)-submodules of M containing N. We define jointly prime radicals of an (R, S)-module M as $rad_{(R,S)}(M) = \sqrt[(R,S)]{0}$. Then we present some properties of jointly prime radicals of an (R, S)-module.

Key words and Phrases: (R, S)-module, jointly prime (R, S)-submodule, m-system, prime radical.

Abstrak. Diberikan (R, S)-modul M. Dalam tulisan ini didefinisikan himpunan sistem-m pada suatu (R, S)-modul sebagai perumuman dari himpunan sistem-m suatu modul. Didefinisikan $\sqrt[(R,S)]{N}$ sebagai himpunan semua $a \in M$ yang memenuhi sifat setiap sistem-m yang memuat a irisannya dengan N tidak kosong, untuk suatu (R, S)-submodul N di M. Dapat ditunjukkan bahwa $\sqrt[(R,S)]{N}$ merupakan irisan dari semua (R, S)-submodul prima gabungan di M yang memuat N. Didefinisikan radikal prima gabungan dari (R, S)-modul M sebagai himpunan $rad_{(R,S)}(M) = \sqrt[(R,S)]{0}$. Kemudian, dalam tulisan ini disajikan beberapa sifat dari radikal prima gabungan suatu (R, S)-modul.

Kata kunci: (R, S)-modul, (R, S)-submodul prima gabungan, sistem-m, radikal prima.

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1. Introduction

All rings in this paper are arbitrary ring unless stated otherwise. Let R and S be arbitrary rings. Khumprapussorn et al. in [3] introduced (R, S)-modules as a generalization of (R, S)-bimodules. An (R, S)-module has an (R, S)-bimodule structure when both rings R and S have central idempotent elements.

In their paper, Khumprapussorn et al. also defined (R, S)-submodules of M as additive subgroups N of M such that $rns \in N$ for all $r \in R$, $n \in N$, and $s \in S$. Moreover, a proper (R, S)-submodule P of M is called a jointly prime (R, S)-submodule if for each left ideal I of R, right ideal J of S, and (R, S)-submodule N of M, $INJ \subseteq P$ implies $IMJ \subseteq P$ or $N \subseteq P$.

A jointly prime (R, S)-submodule P of M is called a minimal jointly prime (R, S)-submodule if it is minimal in the class of jointly prime (R, S)-submodules of M. Based on Goodearl and Warfield [2], we show that every jointly prime (R, S)-submodule of M contains a minimal jointly prime (R, S)-submodule.

Let T be a ring with unity. Lam [4] has defined that a nonempty set $J \subseteq T$ is said to be an m-system if for each pair $a, b \in J$, there exists $t \in T$ such that $atb \in J$. Furthermore, for an ideal I of T, the set $\sqrt{I} := \{a \in T \mid (\forall \text{ m-system } J \text{ of } T)\ a \in J \Rightarrow J \cap I \neq \emptyset\}$ equals to the intersection of all the prime ideals of T containing I. Based on this definition, Behboodi [1] has generalized the definition of m-system of unitary rings to modules. Let M be an unitary module over a ring T. A nonempty set $X \subseteq M \setminus \{0\}$ is called an m-system if for each (left) ideal I of T and for all submodules K, L of M, $(K+L)\cap X \neq \emptyset$ and $(K+IM)\cap X \neq \emptyset$ imply $(K+IL)\cap X \neq \emptyset$. It has been shown that the complement of a prime submodule is an m-system, and for any m-system X, a submodule disjoint from X and maximal with respect to this property is always a prime submodule. Moreover, for a submodule N of M, the set $\sqrt{N} := \{a \in M \mid (\forall \text{ m-system } X \text{ of } M)\ a \in X \Rightarrow X \cap N \neq \emptyset\}$ equals to the intersection of all prime submodules of M containing N.

In Section 2, we extend these facts to (R, S)-modules. In fact, we give a generalization of the notion of m-systems of modules to (R, S)-modules. Then for an (R, S)-submodule N of M, we define $\sqrt[(R,S)]{N} := \{a \in M \mid (\forall \text{ m-system } X \text{ of } M)\ a \in X \Rightarrow X \cap N \neq \emptyset\}$. And then we define jointly prime radicals of an (R, S)-module M as $rad_{(R,S)}(M) = \sqrt[(R,S)]{0}$. It is shown that $rad_{(R,S)}(M)$ is the intersection of all jointly prime (R, S)-submodules of M (note that, if M has no any jointly prime (R, S)-submodule, then $rad_{(R,S)}(M) := M$). In Section 3, we present some properties of jointly prime radicals of (R, S)-modules. These properties are as follows: every jointly prime radicals of (R, S)-submodules is contained in a jointly prime radical of its (R, S)-module; jointly prime radicals of (R, S)-modules M is either equal to M or the intersection of all minimal jointly prime (R, S)-submodules of M; and jointly prime radicals of quotient (R, S)-modules $M/rad_{(R,S)}(M)$ is zero.

2. **Jointly Prime Radicals of (R,S)-Modules**

Before we define m-systems of an (R, S)-module, we describe first the jointly prime (R, S)-submodule. As we have already stated earlier, a proper (R, S)-submodule P of M is called a jointly prime (R, S)-submodule if for each left ideal I of R, right ideal J of S, and (R, S)-submodule N of M, $INJ \subseteq P$ implies $IMJ \subseteq P$ or $N \subseteq P$. The following are some characterizations of jointly prime (R, S)-submodules given in [3].

Theorem 2.1. *Let M be an (R, S)-module satisfying $a \in RaS$ for all $a \in M$, and P a proper (R, S)-submodule of M. The following statements are equivalent:*

(1) P is a jointly prime (R, S)-submodule.
(2) For every right ideal I of R, $m \in M$, and left ideal J of S, $ImJ \subseteq P$ implies $IMJ \subseteq P$ or $m \in P$.
(3) For every right ideal I of R, (R, S)-submodule N of M, and left ideal J of S, $INJ \subseteq P$ implies $IMJ \subseteq P$ or $N \subseteq P$.
(4) For every left ideal I of R, $m \in M$, and right ideal J of S, $(IR)m(SJ) \subseteq P$ implies $IMJ \subseteq P$ or $m \in P$.
(5) For every $a \in R$, $m \in M$, and $b \in S$, $(aR)m(Sb) \subseteq P$ implies $aMb \subseteq P$ or $m \in P$.

If the (R, S)-module M satisfies $M = RMS$, the necessary and sufficient condition for a proper (R, S)-submodule P of M to be a jointly prime (R, S)-submodule is for all ideal I of R, ideal J of S, and (R, S)-submodule N of M, $INJ \subseteq P$ implies $IMJ \subseteq P$ or $N \subseteq P$.

Now, we define the notion of m-systems of (R, S)-modules.

Definition 2.2. *Let M be an (R, S)-module. A nonempty set $X \subseteq M \setminus \{0\}$ is called an m-system if for each left ideal I of R, right ideal J of S, and (R, S)-submodules K, L of M, $(K + L) \cap X \neq \emptyset$ and $(K + IMJ) \cap X \neq \emptyset$ imply $(K + ILJ) \cap X \neq \emptyset$.*

Based on Behboodi [1], we can show that the complement of a jointly prime (R, S)-submodule is an m-system.

Proposition 2.3. *Let P be a proper (R, S)-submodule of M. Then P is a jointly prime (R, S)-submodule of M if and only if $X = M \setminus P$ is an m-system.*

Proof. $(\Rightarrow)$. Suppose that P is a jointly prime (R, S)-submodule of M. Let I be a left ideal of R, J be a right ideal of S, and K, L be (R, S)-submodules of M such that $(K + L) \cap X \neq \emptyset$ and $(K + IMJ) \cap X \neq \emptyset$. If $(K + ILJ) \cap X = \emptyset$, then $K + ILJ \subseteq P$. Then, $ILJ \subseteq P$ and $K \subseteq P$. Since P is a jointly prime (R, S)-submodule of M, we have $L \subseteq P$ or $IMJ \subseteq P$. Thus $(K + L) \cap X = \emptyset$ or $(K + IMJ) \cap X = \emptyset$, a contradiction. Therefore, X is an m-system of M.
$(\Leftarrow)$. Suppose that X is an m-system of M. Let I be a left ideal of R, J be a right ideal of S, and L be an (R, S)-submodule of M such that $ILJ \subseteq P$. If $L \nsubseteq P$ and $IMJ \nsubseteq P$, then $L \cap X \neq \emptyset$ and $IMJ \cap X \neq \emptyset$. Since X is an m-system, $ILJ \cap X \neq \emptyset$ so that $ILJ \nsubseteq P$, a contradiction. Therefore, P is a jointly prime (R, S)-submodule of M. $\square$

Example 2.4. *Let $\mathbb{Z}$ be the ring of integers taken as an $(2\mathbb{Z}, 3\mathbb{Z})$-module. First, we show that $6\mathbb{Z}$ is a jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. Consider a left ideal $I = (2m)\mathbb{Z}$ of $2\mathbb{Z}$, a right ideal $J = (3n)\mathbb{Z}$ of $3\mathbb{Z}$, and an $(2\mathbb{Z}, 3\mathbb{Z})$-submodule $N = k\mathbb{Z}$ of $\mathbb{Z}$, for some $m, n, k \in \mathbb{N}$. It is true that $INJ = ((2m)\mathbb{Z})(k\mathbb{Z})((3n)\mathbb{Z}) = (6mkn)\mathbb{Z} \subseteq 6\mathbb{Z}$ and $N = k\mathbb{Z} =\nsubseteq 6\mathbb{Z}$. Then for each $m, n \in \mathbb{N}$, it is clear that $I\mathbb{Z}J = ((2m)\mathbb{Z})(\mathbb{Z})((3n)\mathbb{Z}) = (6mn)\mathbb{Z} \subseteq 6\mathbb{Z}$. Hence, $6\mathbb{Z}$ is a jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. Therefore, $\mathbb{Z} \setminus 6\mathbb{Z}$ is an m-system of $(2\mathbb{Z}, 3\mathbb{Z})$-module $\mathbb{Z}$.*

It is easy to prove that every maximal (R, S)-submodule of M is a jointly prime (R, S)-submodule. Furthermore, we prove a proposition that states that a maximal (R, S)-submodule P of M which is disjoint from an arbitrary m-system of M is a jointly prime (R, S)-submodule.

Proposition 2.5. *Let M be an (R, S)-module, X an m-system of M, and P a proper (R, S)-submodule of M maximal with respect to the property that $P \cap X = \emptyset$. Then, P is a jointly prime (R, S)-submodule of M.*

PROOF. Let I be a left ideal of R, J a right ideal of S, and N an (R, S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $IMJ \nsubseteq P$. Since P is maximal with respect to the property that $P \cap X = \emptyset$, we have $(P + N) \cap X \neq \emptyset$ and $(P + IMJ) \cap X \neq \emptyset$. Since X is an m-system of M, then $(P + INJ) \cap X \neq \emptyset$. Since $INJ \subseteq P$, it follows that $P \cap X \neq \emptyset$, a contradiction. Therefore, P must be a jointly prime (R, S)-submodule of M. $\square$

We recall the set introduced by Behboodi in [1],

$$\sqrt{N} := \{a \in M \mid (\forall \text{ m-system } X \text{ of } M)\ a \in X \Rightarrow X \cap N \neq \emptyset\}.$$

Now, we present a generalization of the notion of $\sqrt{N}$ for any (R, S)-submodules N of M and we denote it as $\sqrt[(R,S)]{N}$.

Definition 2.6. *Let M be an (R, S)-module. For an (R, S)-submodule N of M, if there is a jointly prime (R, S)-submodule containing N, then we define $\sqrt[(R,S)]{N} := \{a \in M \mid (\forall$ m-system X of $M)\ a \in X \Rightarrow X \cap N \neq \emptyset\}$. If there is no jointly prime (R, S)-submodules containing N, then we define $\sqrt[(R,S)]{N} := M$.*

Let M be an (R, S)-module. Then, the jointly prime spectrum of M is the set $Spec^{j_p}(M) := \{P \mid P \text{ is a jointly prime } (R, S)\text{-submodule of } M\}$. If N be an (R, S)-submodule of M, then we define $V^{j_p}(N) := \{P \in Spec^{j_p}(M) \mid N \subseteq P\}$. Next, we show that $\sqrt[(R,S)]{N}$ equals to the intersection of all jointly prime (R, S)-submodules of M.

Theorem 2.7. *Let M be an (R, S)-module and N be an (R, S)-submodule of M. Then either $\sqrt[(R,S)]{N} = M$ or $\sqrt[(R,S)]{N} = \bigcap\limits_{P \in V^{j_p}(N)} P$.*

PROOF. Suppose that $\sqrt[(R,S)]{N} \neq M$. It follows from Definition 2.6 that $V^{j_p}(N) \neq \emptyset$. We will show that $\sqrt[(R,S)]{N} = \bigcap\limits_{P \in V^{j_p}(N)} P$. Let $m \in \sqrt[(R,S)]{N}$ and $P \in V^{j_p}(N)$.

Consider the m-system $X := M \setminus P$ in M. Since $N \subseteq P$, we have $X \cap N = \emptyset$. Consequently, we get $m \notin X$ so that $m \in P$. Thus, we obtain $\sqrt[(R,S)]{N} \subseteq \bigcap_{P \in V^{j_p}(N)} P$.

Conversely, let $a \in \bigcap_{P \in V^{j_p}(N)} P$. If $a \notin \sqrt[(R,S)]{N}$, then there exists an m-system X such that $a \in X$ but $N \cap X = \emptyset$. Consider the following set:

$$\mathfrak{J} = \{J \mid N \subseteq J, J \text{ is an } (R,S)\text{-submodule of } M \text{ and } J \cap X = \emptyset\}.$$

By Zorn's Lemma, $\mathfrak{J}$ has a maximal element, which is an (R,S)-submodule $K \supseteq N$ maximal with respect to the property $K \cap X = \emptyset$. By Proposition 2.5, K is a jointly prime (R,S)-submodule of M, so $K \in V^{j_p}(N)$. Therefore, we have $a \in K$. Whereas $a \in X$, so we get $K \cap X \neq \emptyset$, a contradiction. Thus, $a \in \sqrt[(R,S)]{N}$ and it follows that $\bigcap_{P \in V^{j_p}(N)} P \subseteq \sqrt[(R,S)]{N}$. Hence, $\sqrt[(R,S)]{N} = \bigcap_{P \in V^{j_p}(N)} P$. $\square$

Example 2.8. *Let $\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-module and $8\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. We obtain the set $V^{j_p}(8\mathbb{Z}) = \{P \in Spec^{j_p}(\mathbb{Z}) \mid 8\mathbb{Z} \subseteq P\} = \{2\mathbb{Z}, 4\mathbb{Z}\}$. Therefore, $\sqrt[(2\mathbb{Z},2\mathbb{Z})]{8\mathbb{Z}} = \bigcap_{P \in V^{j_p}(8\mathbb{Z})} P = 4\mathbb{Z} \cap 2\mathbb{Z} = 4\mathbb{Z}$.*

Let I be an ideal of an unitary ring T. By Lam [4], $\sqrt{I}$ is equal to T or the intersection of all prime ideals of T containing I. From Khumprapussorn et al. [3], we know that the annihilator from M/N of the ring R, that is $(N : M)_R := \{r \in R \mid rMS \subseteq N\}$, is an ideal of R when the ring S satisfies $S^2 = S$. Therefore, when $S^2 = S$, $\sqrt{(N : M)_R}$ is equal to R or the intersection of all prime ideals of R containing $(N : M)_R$. Next, we present a connection between $\sqrt{(N : M)_R}MS$ and $\sqrt[(R,S)]{N}$.

Proposition 2.9. *Let M be an (R,S)-module and N be an (R,S)-submodule of M. If $S^2 = S$, then $\sqrt{(N : M)_R}MS \subseteq \sqrt[(R,S)]{N}$.*

Proof. Since $S^2 = S$, by [3] $(N : M)_R$ is an ideal of R. Also $\sqrt{(N : M)_R}$ is equal to R or equal to the intersection of all prime ideals of R that contain $(N : M)_R$. Suppose that $\sqrt[(R,S)]{N} = M$. Since $\sqrt{(N : M)_R} \subseteq R$, so

$$\sqrt{(N : M)_R}MS \subseteq RMS \subseteq M = \sqrt[(R,S)]{N}.$$

Suppose that $\sqrt[(R,S)]{N} \neq M$. Then $\sqrt[(R,S)]{N} = \bigcap_{P \in V^{j_p}(N)} P$. Let $P \in V^{j_p}(N)$, then P is a jointly prime (R,S)-submodule of M and $N \subseteq P$. Moreover, by Proposition 2.12 of [3], $(P : M)_R$ is a prime ideal of R. Furthermore, since $N \subseteq P$, it is clear that $(N : M)_R \subseteq (P : M)_R$. Since $(P : M)_R$ is a prime ideal of R and contains $(N : M)_R$, we obtain

$$\sqrt{(N : M)_R} \subseteq (P : M)_R.$$

Thus,

$$\sqrt{(N : M)_R}MS \subseteq (P : M)_RMS \subseteq P.$$

Therefore, this shows that $\sqrt{(N:M)_R}MS \subseteq \bigcap_{P\in V^{jp}(N)} P = \sqrt[(R,S)]{N}$. $\square$

The definition of jointly prime radicals of an (R,S)-module is given below.

Definition 2.10. *Let M be an (R,S)-module. If there is a jointly prime (R,S)-submodule of M, then we define jointly prime radicals of M as:*

$$rad_{(R,S)}(M) = \sqrt[(R,S)]{0} := \bigcap_{P\in Spec^{jp}(M)} P.$$

If there is no jointly prime (R,S)-submodule of M, then we define jointly prime radicals of M as $rad_{(R,S)}(M) := M$.

Example 2.11. *Let $\mathbb{Z}$ be an $(2\mathbb{Z},2\mathbb{Z})$-module. It is easy to show that $\{0\}$ is a jointly prime $(2\mathbb{Z},2\mathbb{Z})$-submodule of $\mathbb{Z}$. Since every jointly prime $(2\mathbb{Z},2\mathbb{Z})$-submodule of $\mathbb{Z}$ contains $\{0\}$, then jointly prime radical of $(2\mathbb{Z},2\mathbb{Z})$-module $\mathbb{Z}$ is $rad_{(2\mathbb{Z},2\mathbb{Z})}(\mathbb{Z}) = \{0\}$.*

3. Some Properties of Jointly Prime Radicals of (R,S)-Modules

In this section, we present some properties of jointly prime radicals of (R,S)-modules. Let N be an (R,S)-submodule of M. We show that the jointly prime radical of N is contained in the jointly prime radical of M.

Proposition 3.1. *Let N be an (R,S)-submodule of M. Then, $rad_{(R,S)}(N) \subseteq rad_{(R,S)}(M)$.*

PROOF. Let $P \in Spec^{jp}(M)$. If $N \subseteq P$ then $rad_{(R,S)}(N) \subseteq P$. If $N \nsubseteq P$ then it is easy to check that $N\cap P$ is a jointly prime (R,S)-submodule of N, and hence $rad_{(R,S)}(N) \subseteq N\cap P \subseteq P$. So, in any case we get $rad_{(R,S)}(N) \subseteq P$. Thus, it follows that $rad_{(R,S)}(N) \subseteq rad_{(R,S)}(M)$. $\square$

In module theory, we know that if T-module M is a direct sum of its submodules then the prime radicals of M is also a direct sum of prime radicals of its submodules. Evidently, this property is still maintained on (R,S)-modules M when M satisfies $a \in RaS$ for all $a \in M$.

Proposition 3.2. *Let M be an (R,S)-module and $\{N_i\}_{i\in I}$ be a collection of (R,S)-submodules of M. If M satisfies $a\in RaS$ for all $a\in M$ and $M = \bigoplus_{i\in I} N_i$ then we have $rad_{(R,S)}(M) = \bigoplus_{i\in I} rad_{(R,S)}(N_i)$.*

PROOF. Since each N_i is an (R,S)-submodule of M, we get $rad_{(R,S)}(N_i) \subseteq rad_{(R,S)}(M)$ for each $i \in I$. Thus, it follows that

$$\bigoplus_{i\in I} rad_{(R,S)}(N_i) \subseteq rad_{(R,S)}(M). \tag{1}$$

Now, let $m \in M$. Then, $m = \sum_{i\in I} m_i$ with $m_i \in N_i$ for each $i \in I$ and $m_i = 0$ except for finitely many indices $i \in I$. Suppose that $m \notin \bigoplus_{i\in I} rad_{(R,S)}(N_i)$. We will prove that $m \notin rad_{(R,S)}(M)$. Since $m \notin \bigoplus_{i\in I} rad_{(R,S)}(N_i)$, then there exists $k \in I$ such that $m_k \notin rad_{(R,S)}(N_k)$. Thus, there exists a jointly prime (R,S)-submodule N_k^* of N_k such that $m_k \notin N_k^*$. Consider $K = N_k^* \bigoplus \Big(\bigoplus_{i\neq k} N_i\Big)$. First, we prove that K is a jointly prime (R,S)-submodule of M. Let I be a right ideal of R, J be a left ideal of S, and $a \in M$ such that $IaJ \subseteq K$. Since M satisfies $a \in RaS$ for all $a \in M$, then based on Theorem 2.1 we will prove that $IMJ \subseteq K$ or $a \in K$. Since $a \in M$, $a = \sum_{i\in I} a_i$ where $a_i \in N_i$ for each $i \in I$ and $a_i = 0$ except for finitely many indices $i \in I$. Thus we get $IaJ = I\Big(\sum_{i\in I} a_i\Big)J = Ia_kJ + I\Big(\sum_{i\neq k} a_i\Big)J \subseteq K$, so that $Ia_kJ \subseteq N_k^*$. Since N_k^* is a jointly prime (R,S)-submodule of N_k, we have $IN_kJ \subseteq N_k^*$ or $a_k \in N_k^*$. Since $a_i \in N_i$ for each $i \in I$, $\sum_{i\neq k} a_i \in \bigoplus_{i\neq k} N_i$. Since for all $i \in I$, N_i is an (R,S)-submodule of M, $I\Big(\bigoplus_{i\neq k} N_i\Big)J \subseteq \bigoplus_{i\neq k} N_i$. Thus, it follows that $a = \sum_{i\in I} a_i \in K$ or $I\Big(\bigoplus_{i\in I} N_i\Big)J = IMJ \subseteq K$. Hence, K is a jointly prime (R,S)-submodule of M. Furthermore, because $m_k \notin N_k^*$ then $m \notin K$. Since K is a jointly prime (R,S)-submodule of M, $m \notin rad_{(R,S)}(M)$. Thus, it follows that

$$rad_{(R,S)}(M) \subseteq \bigoplus_{i\in I} rad_{(R,S)}(N_i). \tag{2}$$

From (1) and (2), we obtain $rad_{(R,S)}(M) = \bigoplus_{i\in I} rad_{(R,S)}(N_i)$. $\square$

It is easy to show that every jointly prime (R,S)-submodule of M contains a minimal jointly prime (R,S)-submodule of M. Based on this property, we get a relationship between jointly prime radicals of (R,S)-modules and minimal jointly prime (R,S)-submodules.

Proposition 3.3. *Let M be an (R,S)-module. The jointly prime radical of M is equal to M or the intersection of all minimal jointly prime (R,S)-submodules of M.*

Proof. Since every jointly prime (R,S)-submodule of M contains a minimal jointly prime (R,S)-submodule then for each $P \in Spec^{j_p}(M)$ there exists a minimal jointly prime (R,S)-submodule $P' \in Spec^{j_p}(M)$ such that $P' \subseteq P$. Furthermore, we can form the set:

$$\Im = \{P' \mid P' \text{ is a minimal jointly prime } (R,S)\text{-submodule}\}.$$

Suppose that $rad_{(R,S)}(M) \neq M$. We will prove that $rad_{(R,S)}(M) = \bigcap_{P'\in\Im} P'$. Since $\Im \subseteq Spec^{j_p}(M)$, we get $rad_{(R,S)}(M) \subseteq \bigcap_{P'\in\Im} P'$. On the other hand, for any

$P \in Spec^{j_p}(M)$ there is $P^* \in \Im$ with $P^* \subseteq P$. Thus $\bigcap_{P' \in \Im} P' \subseteq P^* \subseteq P$, which implies that $\bigcap_{P' \in \Im} P' \subseteq rad_{(R,S)}(M)$. Hence $rad_{(R,S)}(M) = \bigcap_{P' \in \Im} P'$. Therefore, this shows that $rad_{(R,S)}(M)$ is equal to the intersection of all minimal jointly prime (R, S)-submodules of M. $\square$

Now, we give an important lemma which will be used in the proof of the next property of jointly prime radicals of an (R, S)-module.

Lemma 3.4. *Let P_1 and P_2 be jointly prime (R, S)-submodules of M, and let $P_1/rad_{(R,S)}(M)$ and $P_2/rad_{(R,S)}(M)$ be (R, S)-submodules of $M/rad_{(R,S)}(M)$. Then,*

$$P_1/rad_{(R,S)}(M) \cap P_2/rad_{(R,S)}(M) = (P_1 \cap P_2)/rad_{(R,S)}(M).$$

Given an (R, S)-module M and (R, S)-submodules A, P of M with $A \subset P$. Then, it is easy to check that the necessary and sufficient condition for P to be a jointly prime (R, S)-submodule of M is P/A being a jointly prime (R, S)-submodule of M/A. By using this property, we can show that the jointly prime radical of the quotient (R, S)-module $M/rad_{(R,S)}(M)$ is zero.

Proposition 3.5. *Let M be an (R, S)-module. Then,*

$$rad_{(R,S)}\left(M/rad_{(R,S)}(M)\right) = \bar{0}.$$

PROOF. Suppose that M has no jointly prime (R, S)-submodules, then we get that quotient (R, S)-modules $M/rad_{(R,S)}(M)$ also has no jointly prime (R, S)-submodules. Thus, $rad_{(R,S)}(M) = M$ and then we obtain

$$rad_{(R,S)}\left(M/rad_{(R,S)}(M)\right) = rad_{(R,S)}\left(M/M\right) = rad_{(R,S)}(\bar{0}) = \bar{0}.$$

Suppose that M has a jointly prime (R, S)-submodule, then we obtain that quotient (R, S)-module $M/rad_{(R,S)}(M)$ also has a jointly prime (R, S)-submodule. From the definition,

$$rad_{(R,S)}\left(M/rad_{(R,S)}(M)\right) = \bigcap_{\bar{P} \in Spec^{j_p}\left(M/rad_{(R,S)}(M)\right)} \bar{P}.$$

Since Lemma 3.4 can be generalized for infinite number of P_i jointly prime (R, S)-submodules of M, then we get

$$\bigcap_{\bar{P} \in Spec^{j_p}\left(M/rad_{(R,S)}(M)\right)} \bar{P} = \left(\bigcap_{P \in Spec^{j_p}(M)} P\right)/rad_{(R,S)}(M).$$

So,

$$rad_{(R,S)}\left(M/_{rad_{(R,S)}(M)}\right) = rad_{(R,S)}(M)/_{rad_{(R,S)}(M)} = \bar{0}.$$

Hence, it's proved that $rad_T\left(M/_{rad_T(M)}\right) = \bar{0}$. $\square$

Given an (R, S)-module M and an ideal I of R such that $I \subseteq Ann_R(M)$. We can show that an (R, S)-module M is also an $(R/_I, S)$-module under the scalar multiplication operation that defined as follows:

$$\begin{array}{rcl} \_ \cdot \_ \cdot \_ : R/_I \times M \times S & \longrightarrow & M \\ (\bar{a}, m, s) & \longrightarrow & \bar{a} \cdot m \cdot s := ams \end{array}$$

for all $\bar{a} \in R/_I$, $m \in M$, and $s \in S$.

Moreover, it is easy to check that P is a jointly prime (R, S)-submodule of M if and only if P is a jointly prime $(R/_I, S)$-submodule of M.

Proposition 3.6. *Let M be an (R, S)-module and I be an ideal of R such that $I \subseteq Ann_R(M)$. Then, $rad_{(R,S)}(M) = rad_{(R/I,S)}(M)$.*

Proof. Let $a \in rad_{(R,S)}(M)$ and P be a jointly prime (R, S)-submodule of M. Then, $a \in P$. Since P is also a jointly prime $(R/_I, S)$-submodule of M, $a \in rad_{(R/I,S)}(M)$. Thus, we obtain

$$rad_{(R,S)}(M) \subseteq rad_{(R/I,S)}(M). \tag{3}$$

Furthermore, let $b \in rad_{(R/I,S)}(M)$ and N a jointly prime $(R/_I, S)$-submodule of M. Then, $b \in N$. Since N is also a jointly prime (R, S)-submodule of M, $b \in rad_{(R,S)}(M)$. Thus, we get

$$rad_{(R/I,S)}(M) \subseteq rad_{(R,S)}(M). \tag{4}$$

Based on (3) and (4), it's proved that $rad_{(R,S)}(M) = rad_{(R/I,S)}(M)$. $\square$

4. Concluding Remarks

Further work on the properties of jointly prime radicals of an (R, S)-module can be carried out. For example, the investigation of properties of jointly prime radicals can be done on any left multiplication (R, S)-module. The concept of left multiplication (R, S)-modules has been described by Khumprapussorn et al. [3].

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