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# On Spectrum and Energy of Non-Commuting Graph for Group $U_{6n}$

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**Abstract.** Let $G$ be a group and $Z(G)$ be the center of $G$. In this paper, we discuss a specific type of graph known as the non-commuting graph, denoted by $\Gamma_G$, whose vertex set contains all group elements excluding central elements, $G\backslash Z(G)$. This graph has to satisfy a condition in which $v_p, v_q \in G\backslash Z(G)$ where $v_p \neq v_q$, are adjacent whenever $v_p v_q \neq v_q v_p$. This paper presents the spectrum and energy of the non-commuting graph for $U_{6n}$, $\Gamma_{U_{6n}}$ associated with the adjacency, degree sum, and degree sum adjacency matrices and their energy relationship.

*Key words and Phrases*: spectrum, energy of a graph, non-commuting graph, $U_{6n}$ group.

## 1. INTRODUCTION

Neumann [1] first proposed the idea of the non-commuting graph in 1976. However, in 2006, Abdollahi et al. [2] began a thorough investigation of the characteristics of the non-commuting graph. The non-commuting graph on a group $G$ is represented by $\Gamma_G$. The vertex set of $\Gamma_G$ is $G\backslash Z(G)$, with $Z(G)$ being the center of $G$. An edge connects two distinct vertices $v_p, v_q \in G\backslash Z(G)$ when $v_p v_q \neq v_q v_p$. Note that $\Gamma_G$ is a finite and simple directed graph with The number of vertices is $|G\backslash Z(G)|$.

Graphs have been extensively studied for their spectral properties, including their characteristic polynomials. The molecular orbital energies of the $\pi$ electrons in conjugated hydrocarbons have been demonstrated to correlate well with the

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eigenvalues of the spectral graph theory. Gutman first discovered the energy of a finite graph. This concept involves the ordinary graph spectrum of the adjacency matrix [3]. Ramane et al. [4] presented the polynomial of the Seidel Laplacian matrix of graphs as well as the Seidel signless Laplacian matrix. Furthermore, Shi et al. [5] identified the energy of fuzzy picture graphs, consistent with the signless Laplacian energy [6]. The equienergetic properties of the graphs corresponding to the Seidel matrix are discussed in [7].

In addition to the adjacency matrix ($A$), as defined by Ramane et al. [8],the degree sum ($DS$) matrix for a graph is introduced, along with the $DS$-energy. In the following, Jog and Kotambari [9] showed the $DS$-energy of six types of simple graphs. In addition to discussing $DS$-energy, Hosamani and Ramane [10] also established lower bounds for $DS$-energy in simple graphs. Recently, Shinde et al. [11], who defined the degree sum adjacency matrix ($DSA$) of a graph, have given a different approach to $DS$-matrix and formulated the characteristic polynomial. These graph matrices are the property used in this paper. In addition, the reader can also see the graph matrix extension in [12].

Over the years, algebraic graph theory has made significant progress in the area of non-commuting graphs. It can be seen in [13, 14, 15], which provides a detailed discussion of the spectral and energy problems of non-commuting graphs for dihedral groups involving several degree-based matrices. Rather et al. [16] also presented the properties of the Sombor spectrum of threshold graphs. Zheng and Zhou [17] investigated the spectral properties of closeness Laplacian and closeness signless Laplacian. Furthermore, a spectral radius for $\Gamma_G$ is defined as the highest of absolute eigenvalues corresponding to the graph matrices [18].

This paper presents a complete formula of the adjacency, degree sum, and degree sum adjacency energies of $\Gamma_G$ in which $G$ is the group $U_{6n}$, or simply written as $\Gamma_{U_{6n}}$. We also analyze some spectral properties of $\Gamma_{U_{6n}}$ corresponding to the adjacency, degree sum, and degree sum adjacency matrices.

## 2. PRELIMINARIES

In this part, we have an overview of the concepts that we need in this article.

**Definition 2.1.** [19] *The $n \times n$ adjacency matrix of $\Gamma_G$ is $A(\Gamma_G) = [a_{ij}]$ in which $(i,j)$-th entry is*

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

**Definition 2.2.** [8] *The $n \times n$ degree sum matrix of $\Gamma_G$ is $DS(\Gamma_G) = [ds_{ij}]$ in which $(i,j)$-th entry is*

$$ds_{ij} = \begin{cases} d_{v_i} + d_{v_j}, & \text{if } v_i \neq v_j \\ 0, & \text{if } v_i = v_j. \end{cases}$$

**Definition 2.3.** [11] *The $n \times n$ degree sum adjacency matrix of $\Gamma_G$ is $DSA(\Gamma_G) = [dsa_{ij}]$ in which $(i,j)$-th entry is*

$$dsa_{ij} = \begin{cases} d_{v_i} + d_{v_j}, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The $DS-$spectrum of $\Gamma_G$ can be written as:

$$\sigma_{DS}(\Gamma_G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ k_1 & k_2 & \dots & k_n \end{pmatrix},$$

where the eigenvalues of $DS(\Gamma_G)$ are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ (not necessarily distinct) and their multiplicities are $k_1, k_2, \ldots, k_n$. Afterward, the $DS-$energy can be defined as follows:

$$E_{DS}(\Gamma_G) = \Sigma_{i=1}^{n} |\lambda_i|,$$

and the $DS-$spectral radius is defined as

$$\rho_{DS}(\Gamma_G) = max\{|\lambda| : \lambda \in \sigma_{DS}(\Gamma_G)\},$$

The above notations also apply to matrices $A$ and $DSA$. Now by Definition 2.1, 2.2, and 2.3, we can construct the $A$, $DS$, and $DSA-$matrix of $\Gamma_G$ to derive its characteristic polynomial. The following lemma helps ease the process of formulating the characteristic formula of $\Gamma_G$.

**Lemma 2.4.** [20] *If a square matrix $J_n$ where all the entries of $J_n$ are equal to one and $e, f, g$ and $h$ are real numbers, consequently the determinant of an $(n_1 + n_2) \times (n_1 + n_2)$ matrix*

$$\begin{vmatrix} (\lambda + e)I_{n_1} - eJ_{n_1} & -gJ_{n_1 \times n_2} \\ -hJ_{n_2 \times n_1} & (\lambda + f)I_{n_2} - fJ_{n_2} \end{vmatrix}$$

*are able to be expressed more simply*

$$(\lambda + e)^{n_1 - 1}(\lambda + f)^{n_2 - 1}\left((\lambda - (n_1 - 1)\,a)\,(\lambda - (n_2 - 1)\,b) - n_1 n_2 gh\right),$$

*where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.*

Furthermore, we can construct $A$, $DS$, and $DSA$ matrices and compute the eigenvalues from the solution of the characteristic polynomial in determinant form. The following theorem is the guideline to simplify the determinant formula of a matrix.

**Theorem 2.5.** [21] *If $M = \begin{bmatrix} T & U \\ V & W \end{bmatrix}$ is a square matrix that can be divided into four submatrices, $T, U, V, W$, where $T$ is non-singular, then the determinant of $M$ is*

$$|M| = \begin{vmatrix} T & U \\ 0 & W - VT^{-1}U \end{vmatrix} = |T|\left|W - VT^{-1}U\right|.$$

Throughout this paper, the discussion focuses on the $DS-$matrix of $\Gamma_G$ for the group $U_{6n}$. James and Liebeck [22] introduced the group $U_{6n}$ in their famous book. The group of order $6n$, for $n \geq 1$ is denoted as $U_{6n}$ and defined by

$$U_{6n} = \left\langle a, b : a^{2n} = b^3 = e, a^{-1}ba = b^{-1} \right\rangle.$$

We define the sets $G_1 = \{a^{2r+1} : 0 \leq r \leq n-1\}$, $G_2 = \{a^{2r+1}b : 0 \leq r \leq n-1\}$, $G_3 = \{a^{2r+1}b^2 : 0 \leq r \leq n-1\}$, $G_4 = \{a^{2r}b : 0 \leq r \leq n-1\}$, and $G_5 = \{a^{2r}b^2 : 0 \leq r \leq n-1\}$. It is clear that $|G_1| = |G_2| = |G_3| = |G_4| = |G_5| = n$. Clearly, the set of vertex of $\Gamma_{U_{6n}}$ is $G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5 = U_{6n} \backslash Z(U_{6n})$, and the number of vertices is $5n$.

**Lemma 2.6.** [23] *For the group $U_{6n}$ and $0 \leq r \leq n-1$, we have the following:*

*(1) The center of $U_{6n}$ is $Z(U_{6n}) = \langle a^2 \rangle$.*
*(2) The centralizer of element $a^{2r+1}$ is $C_{U_{6n}}\left(a^{2r+1}\right) = \langle e \rangle$.*
*(3) The centralizer of element $a^{2r+1}b$ is*
*$C_{U_{6n}}\left(a^{2r+1}b\right) = \langle e^2 \rangle \cdot \langle \{a^{2s+1}b : 0 \leq s \leq n-1\} \rangle$.*
*(4) The centralizer of element $a^{2r+1}b^2$ is*
*$C_{U_{6n}}\left(a^{2r+1}b^2\right) = \langle e^2 \rangle \cdot \langle \{a^{2s+1}b^2 : 0 \leq s \leq n-1\} \rangle$.*
*(5) The centralizer of element $a^{2r}b$ is*
*$C_{U_{6n}}\left(a^{2r}b\right) = \langle e^2 \rangle \cdot \langle \{a^{2s}b, a^{2s}b^2 : 0 \leq s \leq n-1\} \rangle$.*

**Corollary 2.7.** [24] *For $n \geq 1$ and $\Gamma_{U_{6n}}$, then for $0 \leq r \leq n-1$,*

*(1) $d_{a^{2r+1}} = 4n$,*
*(2) $d_{a^{2r+1}b} = 4n$,*
*(3) $d_{a^{2r+1}b^2} = 4n$,*
*(4) $d_{a^{2r}b} = 3n$,*
*(5) $d_{a^{2r}b^2} = 3n$.*

Therefore, the discussion of this paper is devoted to the non-commuting graph for $U_{6n}$, denoted by $\Gamma_{U_{6n}}$. Moreover, one interesting graph property is hyperenergetic when the energy of a graph with $n$ vertices is greater than that of a complete graph on $n$ vertices, $K_n$. Since $U_{6n}$ has $5n$ vertices, we have the following definition.

**Definition 2.8.** [19] *An $5n-$vertex graph $\Gamma_{U_{6n}}$ is hyperenergetic if the energy is more than $2(5n-1)$.*

## 3. MAIN RESULTS

We determine spectral properties for $\Gamma_{U_{6n}}$ with respect to the $A$, $DS$ and $DSA$ matrices in this section.

### 3.1. Adjacency Energy.

We initiate by presenting the adjacency matrix of $\Gamma_{U_{6n}}$ that will be employed in establishing its characteristic polynomial.

**Theorem 3.1.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the characteristic polynomial of $A(\Gamma_{U_{6n}})$ is*

$$P_{A(\Gamma_{U_{6n}})}(\lambda) = \lambda^{5n-4}(\lambda + n^2)^2(\lambda^2 + (2-4n)\lambda - 6n^4).$$

*Proof.* Based on the number of vertices of $\Gamma_{U_{6n}}$, then $A(\Gamma_{U_{6n}})$ is a matrix of size $5n \times 5n$. By Lemma 2.6, then the $(i,j)-$entries of $A(\Gamma_{U_{6n}})$ are:

(1) $a_{ij} = 0$, for $3n+1 \leq i,j \leq 5n$, $1 \leq i,j \leq n$, $n+1 \leq i,j \leq 2n$, and $2n+1 \leq i,j \leq 3n$;
(2) $a_{ij} = 1$, otherwise.

Then $A(\Gamma_{U_{6n}})$ is

$$\begin{array}{c|ccccccccccccccc}
 & a & \cdots & a^{2n-1} & ab & \cdots & a^{2n-1}b & ab^2 & \cdots & a^{2n-1}b^2 & b & \cdots & a^{2(n-1)}b & b^2 & \cdots & a^{2(n-1)}b^2 \\
\hline
a & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1} & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
ab & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
ab^2 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b^2 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
b & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2(n-1)}b & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
b^2 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & & \\
a^{2(n-1)}b^2 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}.$$

Now we can express $A(\Gamma_{U_{6n}})$ to the block matrices form:

$$A(\Gamma_{U_{6n}}) = \begin{pmatrix} 0_n & J_n & J_n & J_n & J_n \\ J_n & 0_n & J_n & J_n & J_n \\ J_n & J_n & 0_n & J_n & J_n \\ J_n & J_n & J_n & 0_n & 0_n \\ J_n & J_n & J_n & 0_n & 0_n \end{pmatrix}.$$

The characteristic formula of $A(\Gamma_{U_{6n}})$ is as given below:

$$P_{A(\Gamma_{U_{6n}})}(\lambda) = \begin{vmatrix} \lambda I_n & -J_n & -J_n & -J_n & -J_n \\ -J_n & \lambda I_n & -J_n & -J_n & -J_n \\ -J_n & -J_n & \lambda I_n & -J_n & -J_n \\ -J_n & -J_n & -J_n & \lambda I_n & 0_n \\ -J_n & -J_n & -J_n & 0_n & \lambda I_n \end{vmatrix}. \tag{1}$$

In order to get the formula of $P_{A(\Gamma_{U_{6n}})}(\lambda)$, row and column operations need to be performed. Let $R_i$ and $C_i$ be the $i-$th row and column of $P_{A(\Gamma_{U_{6n}})}(\lambda)$, respectively. We apply the following steps:

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{3n}$, for $i = 1, 2, \ldots, 2n$.
(2) $R_{j-i} \longrightarrow R_{j-i} - R_{3n-i}$, for $i = 1, 2, \ldots, n$ and $j = n, 2n$.
(3) $R_{3n-i} \longrightarrow R_{3n-i} - R_{3n}$, for $i = 1, 2, \ldots, n-1$.
(4) $C_{2n+i} \longrightarrow C_{2n+i} + C_{j+i}$, for $i = 1, 2, \ldots, n$ and $j = 0, n$.
(5) $R_{j+i} \longrightarrow R_{j+i} - R_{n+j}$, for $i = 1, 2, \ldots, n-1$ and $j = 0, n$.
(6) $C_{3n+1} \longrightarrow C_{3n+1} + C_{3n+2} + C_{3n+3} + \ldots + C_{4n} + C_{4n+1} + \ldots + C_{5n}$.
(7) $C_j \longrightarrow C_j + C_{j-1} + C_{j-2} + \ldots + C_{j-(n-1)}$, $j = n, 2n, 3n$.
(8) $R_{3n} \longrightarrow R_{3n} + \frac{1}{\lambda} R_{3n+2} + \frac{1}{\lambda} R_{3n+3} + \ldots + \frac{1}{\lambda} R_{4n} + \frac{1}{\lambda} R_{4n+1} + \ldots + \frac{1}{\lambda} R_{5n}$,

then Equation (1) can be written as the following determinant

$$\begin{vmatrix} \lambda I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ J_{1\times(n-1)} & \lambda + n & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} & 0 & 0 & 0_{1\times(n-1)} & 0_{1\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & \lambda I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 0_{1\times(n-1)} & 0 & J_{1\times(n-1)} & \lambda + n & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & \lambda I_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ -J_{1\times(n-1)} & -n & -J_{1\times(n-1)} & -n & -2J_{1\times(n-1)} & \lambda - 2n & -2n & 0_{1\times(n-1)} & 0_{1\times n} \\ -J_{1\times(n-1)} & -n & -J_{1\times(n-1)} & -n & -3J_{1\times(n-1)} & -3n & \lambda & 0_{1\times(n-1)} & 0_{1\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & \lambda I_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & \lambda I_{n\times n} \end{vmatrix}. \tag{2}$$

Hence, according to Theorem 2.5, we can state Equation (1) as follows:

$$P_{A(\Gamma_{U_{6n}})}(\lambda) = \lambda^{5n-4}(\lambda + n)^2(\lambda^2 - 2n\lambda + 6n^2).$$

□

The subsequent theorems provide insight into the spectrum and energy of $\Gamma_{U_{6n}}$ associated with $A(\Gamma_{U_{6n}})$.

**Theorem 3.2.** *The $A-$spectral radius of $\Gamma_{U_{6n}}$ is*

$$\rho_A(\Gamma_{U_{6n}}) = n + n\sqrt{7}.$$

*Proof.* As a result of Theorem 3.2 when we find the roots of $P_{A(\Gamma_{U_{6n}})}(\lambda) = 0$, we arrive at the four eigenvalues. There are three of these, with multiplicity 2, $\lambda_1 = -n$, $\lambda_2 = 0$ with multiplicity $5n-4$, and the other two are $\lambda_{3,4} = n \pm n\sqrt{7}$ with multiplicity 1 respectively. These eigenvalues are the adjacency spectrum of $\Gamma_{U_{6n}}$,

$$\sigma_A(\Gamma_{U_{6n}}) = \begin{pmatrix} n + n\sqrt{7} & 0 & -n & n - n\sqrt{7} \\ 1 & 5n-4 & 2 & 1 \end{pmatrix}.$$

By taking the largest of $|\lambda_i|$, $i = 1, 2, 3, 4$ then the adjacency spectral radius of $\Gamma_{U_{6n}}$ is $n + n\sqrt{7}$. □

**Theorem 3.3.** *The $A-$energy of $\Gamma_{U_{6n}}$ is*

$$E_A(\Gamma_{U_{6n}}) = 2n + 2n\sqrt{7}.$$

*Proof.* On computing the absolute eigenvalues in the proof of Theorem 3.2, then the adjacency energy of $\Gamma_{U_{6n}}$ is

$$\begin{aligned} E_A(\Gamma_{U_{6n}}) =& \left| n \pm n\sqrt{7} \right| + (5n-4)|0| + (2)\,|-n| \\ =& 2n + 2n\sqrt{7}. \end{aligned}$$

□

### 3.2. **Degree Sum Energy.**

We begin with the formulation of the characteristic polynomial of $\Gamma_{U_{6n}}$.

**Theorem 3.4.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the characteristic polynomial of $DS(\Gamma_{U_{6n}})$ is*

$$P_{DS(\Gamma_{U_{6n}})}(\lambda) = (\lambda{+}8n)^{3n-1}(\lambda{+}6n)^{2n-1}\left(\left(\lambda - 8n\left(3n-1\right)\right)\left(\lambda - 6n\left(2n-1\right)\right) - 294n^4\right).$$

*Proof.* Since the set of vertex of $\Gamma_{U_{6n}}$ is $U_{6n}\backslash Z(U_{6n})$, then we have $5n$ vertices in $DS(\Gamma_{U_{6n}})$ which are the members of $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$. By using the information of the degree of every vertex in Corollary 2.7, we can construct $DS(\Gamma_{U_{6n}})$ of the size $5n \times 5n$ whose entries are:

(1) $ds_{ij} = 4n + 4n = 8n$, for $i \neq j$ and $1 \leq i, j \leq 3n$;
(2) $ds_{ij} = 4n + 3n = 7n$, for $1 \leq i \leq 3n$ and $3n + 1 \leq j \leq 5n$;
(3) $ds_{ij} = 3n + 4n = 7n$, for $3n + 1 \leq i \leq 5n$, and $1 \leq j \leq 3n$;
(4) $ds_{ij} = 3n + 3n = 6n$, for $3n + 1 \leq i, j \leq 5n$;
(5) $ds_{ij} = 0$, for $i = j$.

Then $DS(\Gamma_{U_{6n}})$ is

$$\begin{array}{cc}
 & \begin{matrix} a & \cdots & a^{2n-1} & ab & \cdots & a^{2n-1}b & ab^2 & \cdots & a^{2n-1}b^2 & b & \cdots & a^{2(n-1)}b & b^2 & \cdots & a^{2(n-1)}b^2 \end{matrix} \\
\begin{matrix} a \\ \vdots \\ a^{2n-1} \\ ab \\ \vdots \\ a^{2n-1}b \\ ab^2 \\ \vdots \\ a^{2n-1}b^2 \\ b \\ \vdots \\ a^{2(n-1)}b \\ b^2 \\ \vdots \\ a^{2(n-1)}b^2 \end{matrix} &
\begin{pmatrix}
0 & \cdots & 8n & 8n & \cdots & 8n & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
8n & \cdots & 0 & 8n & \cdots & 8n & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\
8n & \cdots & 8n & 0 & \cdots & 8n & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
8n & \cdots & 8n & 8n & \cdots & 0 & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\
8n & \cdots & 8n & 8n & \cdots & 8n & 0 & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
8n & \cdots & 8n & 8n & \cdots & 8n & 8n & \cdots & 0 & 7n & \cdots & 7n & 7n & \cdots & 7n \\
7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 0 & \cdots & 6n & 6n & \cdots & 6n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 6n & \cdots & 0 & 6n & \cdots & 6n \\
7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 6n & \cdots & 6n & 0 & \cdots & 6n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 6n & \cdots & 6n & 6n & \cdots & 0
\end{pmatrix}
\end{array}$$

In other words, the following is the way in which $DS(\Gamma_{U_{6n}})$ can be divided into four submatrices:

$$DS(\Gamma_{U_{6n}}) = \begin{bmatrix} 8n(J-I)_{3n} & 7nJ_{3n\times 2n} \\ 7nJ_{2n\times 3n} & 6n(J-I)_{2n} \end{bmatrix}.$$

The characteristic polynomial of $DS(\Gamma_{U_{6n}})$ is

$$P_{DS(\Gamma_{U_{6n}})}(\lambda) = \begin{vmatrix} \left(\lambda+8n\right)I_{3n} - 8nJ_{3n} & -7nJ_{3n\times 2n} \\ -7nJ_{2n\times 3n} & \left(\lambda+6n\right)I_{2n} - 6nJ_{2n} \end{vmatrix}. \tag{3}$$

By Lemma 2.4 with $a = 8n$, $b = 6b$, $c = d = 7n$, $n_1 = 3n$ and $n_2 = 2n$, we can express Equation 3 as

$$P_{DS(\Gamma_{U_{6n}})}(\lambda) = (\lambda{+}8n)^{3n-1}(\lambda{+}6n)^{2n-1}\left(\left(\lambda - 8n\left(3n-1\right)\right)\left(\lambda - 6n\left(2n-1\right)\right) - 294n^4\right).$$

□

The following Theorem gives the $DS-$spectrum and $DS-$spectral radius of $\Gamma_{U_{6n}}$.

**Theorem 3.5.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the $DS-$spectral radius of $\Gamma_{U_{6n}}$ is*

$$\rho_{DS}(\Gamma_{U_{6n}}) = n(18n-7) + n\sqrt{330n^2 - 492n + 97}.$$

*Proof.* The result according to Theorem 3.4 is that the formula of $P_{DS(\Gamma_{U_{6n}})}$ gives four eigenvalues. They are $\lambda_1 = -8n$ of multiplicity $3n-1$, $\lambda_2 = -6n$ of multiplicity $2n-1$, and the quadratic formula give us $\lambda_{3,4} = n(18n-7) \pm n\sqrt{330n^2 - 492n + 97}$. Thus, the list of these eigenvalues is the spectrum of $\Gamma_{U_{6n}}$ associated with $DS-$matrix,

$$\begin{pmatrix} n(18n-7) + n\sqrt{330n^2 - 492n + 97} & -6n & -8n & n(18n-7) - n\sqrt{330n^2 - 492n + 97} \\ 1 & 2n-1 & 3n-1 & 1 \end{pmatrix}.$$

By taking the maximum of absolute eigenvalues of $\sigma_{DS}(\Gamma_{U_{6n}})$, we then obtain the required result. □

The following theorem gives the $DS-$energy of $\Gamma_{U_{6n}}$.

**Theorem 3.6.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the $DS-$energy of $\Gamma_{U_{6n}}$ is*

$$E_{DS}(\Gamma_{U_{6n}}) = 2n(18n-7) + 2n\sqrt{330n^2 - 492n + 97}.$$

*Proof.* By Theorem 3.5 and calculating the absolute eigenvalues of $\sigma_{DS}(\Gamma_{U_{6n}})$, the $DS$-energy of $\Gamma_{U_{6n}}$ is the given by

$$\begin{aligned} E_{DS}(\Gamma_{U_{6n}}) =& (3n-1)|-8n| + (2n-1)|-6n| + \left| n(18n-7) \pm n\sqrt{330n^2 - 492n + 97} \right| \\ =& 2n(18n-7) + 2n\sqrt{330n^2 - 492n + 97}. \end{aligned}$$

□

### 3.3. **Degree Sum Adjacency Energy.**

The primary outcome of this subsection is the presentation of the degree sum adjacency energy of $\Gamma_{U_{6n}}$.

**Theorem 3.7.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the characteristic polynomial of $DSA(\Gamma_{U_{6n}})$ is*

$$P_{DSA(\Gamma_{U_{6n}})}(\lambda) = \lambda^{5n-4}(\lambda + 8n^2)^2(\lambda^2 + (16 - 32n)\lambda - 294n^4).$$

*Proof.* We know that $|U_{6n} \backslash Z(U_{6n})| = 5n$ which implies $DSA(\Gamma_{U_{6n}})$ has $5n$ vertices, and they are the members of $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$. By Lemma 2.6 and Corollary 2.7, we can provide an $5n \times 5n$ matrix $DSA(\Gamma_{U_{6n}})$ whose entries are:

(1) $dsa_{ij} = 4n + 4n = 8n$, for $1 \leq i \leq n$ and $n+1 \leq j \leq 3n$;
(2) $dsa_{ij} = 4n + 4n = 8n$, for $n+1 \leq i \leq 3n$ and $1 \leq i \leq n$;
(3) $dsa_{ij} = 4n + 4n = 8n$, for $n+1 \leq i \leq 2n$ and $2n+1 \leq i \leq 3n$;
(4) $dsa_{ij} = 4n + 4n = 8n$, for $2n+1 \leq i \leq 3n$ and $n+1 \leq i \leq 2n$;

(5) $dsa_{ij} = 4n + 3n = 7n$, for $1 \leq i \leq 3n$ and $3n + 1 \leq j \leq 5n$;
(6) $dsa_{ij} = 3n + 4n = 7n$, for $3n + 1 \leq i \leq 5n$, and $1 \leq j \leq 3n$;
(7) $dsa_{ij} = 0$, for $3n + 1 \leq i, j \leq 5n$, $1 \leq i, j \leq n$, $n + 1 \leq i, j \leq 2n$, and $2n + 1 \leq i, j \leq 3n$.

Now $DSA(\Gamma_{U_{6n}})$ can be performed as follows:

$$\begin{array}{c@{\;}c} & \begin{array}{ccccccccccccccc} a & \cdots & a^{2n-1} & ab & \cdots & a^{2n-1}b & ab^2 & \cdots & a^{2n-1}b^2 & b & \cdots & a^{2(n-1)}b & b^2 & \cdots & a^{2(n-1)}b^2 \end{array} \\ \begin{array}{c} a \\ \vdots \\ a^{2n-1} \\ ab \\ \vdots \\ a^{2n-1}b \\ ab^2 \\ \vdots \\ a^{2n-1}b^2 \\ b \\ \vdots \\ a^{2(n-1)}b \\ b^2 \\ \vdots \\ a^{2(n-1)}b^2 \end{array} & \begin{pmatrix} 0 & \cdots & 0 & 8n & \cdots & 8n & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 8n & \cdots & 8n & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\ 8n & \cdots & 8n & 0 & \cdots & 0 & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 8n & \cdots & 8n & 0 & \cdots & 0 & 8n & \cdots & 8n & 7n & \cdots & 7n & 7n & \cdots & 7n \\ 8n & \cdots & 8n & 8n & \cdots & 8n & 0 & \cdots & 0 & 7n & \cdots & 7n & 7n & \cdots & 7n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 8n & \cdots & 8n & 8n & \cdots & 8n & 0 & \cdots & 0 & 7n & \cdots & 7n & 7n & \cdots & 7n \\ 7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 7n & \cdots & 7n & 7n & \cdots & 7n & 7n & \cdots & 7n & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}.$$

In other words, the matrix $DSA(\Gamma_{U_{6n}})$ can be partitioned into 25 blocks as given below:

$$DSA(\Gamma_{U_{6n}}) = \begin{bmatrix} 0_n & 8nJ_n & 8nJ_n & 7nJ_n & 7nJ_n \\ 8nJ_n & 0_n & 8nJ_n & 7nJ_n & 7nJ_n \\ 8nJ_n & 8nJ_n & 0_n & 7nJ_n & 7nJ_n \\ 7nJ_n & 7nJ_n & 7nJ_n & 0_n & 0_n \\ 7nJ_n & 7nJ_n & 7nJ_n & 0_n & 0_n \end{bmatrix}.$$

The characteristic formula of $DSA(\Gamma_{U_{6n}})$ is

$$P_{DSA(\Gamma_{U_{6n}})}(\lambda) = \begin{vmatrix} \lambda I_n & -8nJ_n & -8nJ_n & -7nJ_n & -7nJ_n \\ -8nJ_n & \lambda I_n & -8nJ_n & -7nJ_n & -7nJ_n \\ -8nJ_n & -8nJ_n & \lambda I_n & -7nJ_n & -7nJ_n \\ -7nJ_n & -7nJ_n & -7nJ_n & \lambda I_n & 0_n \\ -7nJ_n & -7nJ_n & -7nJ_n & 0_n & \lambda I_n \end{vmatrix}. \tag{4}$$

By applying the similar row and column operations as the proofing part of Theorem 3.1 with the following steps:

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{3n}$, for $i = 1, 2, \ldots, 2n$.
(2) $R_{j-i} \longrightarrow R_{j-i} - R_{3n-i}$, for $i = 1, 2, \ldots, n$ and $j = n, 2n$.
(3) $R_{3n-i} \longrightarrow R_{3n-i} - R_{3n}$, for $i = 1, 2, \ldots, n - 1$.
(4) $C_{2n+i} \longrightarrow C_{2n+i} + C_{j+i}$, for $i = 1, 2, \ldots, n$ and $j = 0, n$.
(5) $C_{3n+1} \longrightarrow C_{3n+1} + C_{3n+2} + C_{3n+3} + \ldots + C_{4n} + C_{4n+1} + \ldots + C_{5n}$.
(6) $R_{j+i} \longrightarrow R_{j+i} - R_{n+j}$, for $i = 1, 2, \ldots, n - 1$ and $j = 0, n$.
(7) $C_j \longrightarrow C_j + C_{j-1} + C_{j-2} + \ldots + C_{j-(n-1)}$, $j = n, 2n, 3n$.

(8) $R_{3n} \longrightarrow R_{3n} + \frac{7n}{\lambda} R_{3n+2} + \frac{7n}{\lambda} R_{3n+3} + \ldots + \frac{7n}{\lambda} R_{4n} + \frac{7n}{\lambda} R_{4n+1} + \ldots + \frac{7n}{\lambda} R_{5n}$,

then Equation (4) can be written as the following determinant

$$\begin{vmatrix} \lambda I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 8nJ_{1\times(n-1)} & \lambda + 8n^2 & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} & 0 & 0 & 0_{1\times(n-1)} & 0_{1\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & \lambda I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 0_{1\times(n-1)} & 0 & 8nJ_{1\times(n-1)} & \lambda + 8n^2 & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & \lambda I_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ -8nJ_{1\times(n-1)} & -8n^2 & -8nJ_{1\times(n-1)} & -8n^2 & -16nJ_{1\times(n-1)} & \lambda - 16n^2 & -14n^2 & 0_{1\times(n-1)} & 0_{1\times n} \\ -7nJ_{1\times(n-1)} & -7n^2 & -7nJ_{1\times(n-1)} & -7n^2 & -21nJ_{1\times(n-1)} & -21n^2 & \lambda & 0_{1\times(n-1)} & 0_{1\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & \lambda I_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & \lambda I_{n\times n} \end{vmatrix}. \tag{5}$$

By Theorem 2.5, Equation (5) can be stated as

$$P_{DSA(\Gamma_{U_{6n}})}(\lambda) = \lambda^{5n-4}(\lambda + 8n^2)^2(\lambda^2 + 16n^2\lambda - 294n^4).$$

$\square$

By applying Theorem 3.7, we can analyze the spectrum of $\Gamma_{U_{6n}}$ with devoted to $DSA(\Gamma_{U_{6n}})$ to calculate the spectral radius and energy of $\Gamma_{U_{6n}}$.

**Theorem 3.8.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the $DSA-$spectral radius of $\Gamma_{U_{6n}}$ is*

$$\rho_{DSA}(\Gamma_{U_{6n}}) = (8 + \sqrt{358})n^2.$$

*Proof.* The three eigenvalues of $\Gamma_{U_{6n}}$ are determined by the roots of equation $P_{DSA(\Gamma_{U_{6n}})}(\lambda) = 0$, this is shown in Theorem 3.7. The eigenvalues are $\lambda_{1,2} = 8n^2 \pm n^2\sqrt{358}$ of multiplicity 1 respectively, $\lambda_3 = 0$ of multiplicity $5n - 4$, and $\lambda_4 = -8n^2$ with multiplicity 2. Therefore, we get the spectrum of $\Gamma_{U_{6n}}$ associated with $DSA-$matrix,

$$\begin{pmatrix} 8n^2 + n^2\sqrt{358} & 0 & -8n^2 & 8n^2 - n^2\sqrt{358} \\ 1 & 5n-4 & 2 & 1 \end{pmatrix}.$$

It is clear that the largest absolute $\lambda_i$ for $i = 1, 2, 3, 4$ is $8n^2 + n^2\sqrt{358}$. $\square$

**Theorem 3.9.** *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for $U_{6n}$, then the $DSA-$energy of $\Gamma_{U_{6n}}$ is*

$$E_{DSA}(\Gamma_{U_{6n}}) = 2(8 + \sqrt{358})n^2.$$

*Proof.* Using the result of Theorem 3.8, then the $DSA$-energy of $\Gamma_{U_{6n}}$ as follows:

$$\begin{aligned} E_{DSA}(\Gamma_{U_{6n}}) =& \left|8n^2 \pm n^2\sqrt{358}\right| + (5n-4)|0| + (2)\left|-8n^2\right| \\ =& 16n^2 + 2n^2\sqrt{358} \\ =& 2(8 + \sqrt{358})n^2. \end{aligned}$$

$\square$

3.4. **Example.**

The major contribution of this research lies in its exploration of the spectrum and energy of $\Gamma_{U_{6n}}$ corresponding to adjacency, degree sum, and degree sum adjacency matrices. As an illustration of this, we provide an example when $n = 1$ as given below:

**Example 3.10.** *Let* $U_6 = \{e, a, b, ab, b^2, ab^2\}$ *and* $Z(U_6) = \{e\}$*, where* $C_{U_6}(a) = \{e\}$*,* $C_{U_6}(b) = \{e, b, b^2\} = C_{U_6}(b^2)$*,* $C_{U_6}(ab) = \{e, ab\}$*,* $C_{U_6}(ab^2) = \{e, ab^2\}$*. For* $G = U_6 \backslash Z(U_6)$*, by using the centralizer of every element in* $U_6$*, then* $\Gamma_{U_6}$ *is as in Figure 1. It can be seen that* $d_a = d_{ab} = d_{ab^2} = 4$*, and* $d_b = d_{b^2} = 3$*. Consequently,*

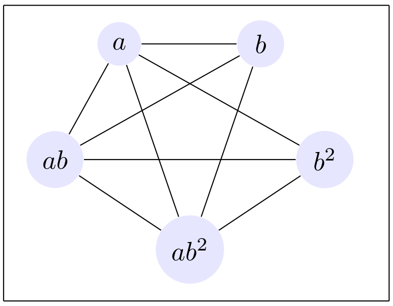


FIGURE 1. Non-commuting graph on $U_6$

*we can construct the adjacency, degree sum, and degree sum adjacency matrices of order* $5 \times 5$ *for* $\Gamma_{U_6}$ *as follows:*

$$A(\Gamma_{U_6}) = \begin{array}{c} \\ a \\ ab \\ ab^2 \\ b \\ b^2 \end{array} \begin{array}{c} \begin{array}{ccccc} a & ab & ab^2 & b & b^2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$DS(\Gamma_{U_6}) = \begin{array}{c} \\ a \\ ab \\ ab^2 \\ b \\ b^2 \end{array} \begin{array}{c} \begin{array}{ccccc} a & ab & ab^2 & b & b^2 \end{array} \\ \begin{pmatrix} 0 & 8 & 8 & 7 & 7 \\ 8 & 0 & 8 & 7 & 7 \\ 8 & 8 & 0 & 7 & 7 \\ 7 & 7 & 7 & 0 & 6 \\ 7 & 7 & 7 & 6 & 0 \end{pmatrix} \end{array}$$

$$DSA(\Gamma_{U_6}) = \begin{array}{c} \\ a \\ ab \\ ab^2 \\ b \\ b^2 \end{array} \begin{array}{c} \begin{array}{ccccc} a & ab & ab^2 & b & b^2 \end{array} \\ \begin{pmatrix} 0 & 8 & 8 & 7 & 7 \\ 8 & 0 & 8 & 7 & 7 \\ 8 & 8 & 0 & 7 & 7 \\ 7 & 7 & 7 & 0 & 0 \\ 7 & 7 & 7 & 0 & 0 \end{pmatrix} \end{array}$$

*Therefore, the characteristic polynomial of $A(\Gamma_{U_6})$, $DS(\Gamma_{U_6})$ and $DSA(\Gamma_{U_6})$ are as follows:*

$$\begin{aligned} P_{A(\Gamma_{U_6})}(\lambda) =& \lambda(\lambda+1)^2(\lambda^2-2\lambda-6). \\ P_{DS(\Gamma_{U_6})}(\lambda) =& (\lambda+6)(\lambda+8)^2(\lambda^2-(22+2\sqrt{319})\lambda+440). \\ P_{DSA(\Gamma_{U_6})}(\lambda) =& \lambda(\lambda+8)^2(\lambda^2-16\lambda-294) \end{aligned}$$

*By using Maple, we have confirmed that the spectrum and spectral radius of respective matrices are given by*

$$\begin{aligned} \sigma_A(\Gamma_{U_6}) &= \begin{pmatrix} 1+\sqrt{7} & 0 & -1 & 1-\sqrt{7} \\ 1 & 1 & 2 & 1 \end{pmatrix} \text{ and } \rho_A(\Gamma_{U_6}) = 1+\sqrt{7}, \\ \sigma_{DS}(\Gamma_{U_6}) &= \begin{pmatrix} 11+\sqrt{319} & -6 & -8 & 11-\sqrt{319} \\ 1 & 1 & 2 & 1 \end{pmatrix} \text{ and } \rho_{DS}(\Gamma_{U_6}) = 11+\sqrt{319}, \\ \sigma_{DSA}(\Gamma_{U_6}) &= \begin{pmatrix} 8+\sqrt{358} & 0 & -8 & 8-\sqrt{358} \\ 1 & 1 & 2 & 1 \end{pmatrix} \text{ and } \rho_{DSA}(\Gamma_{U_6}) = 8+\sqrt{358}. \end{aligned}$$

*Therefore, the $DS-$energy of $\Gamma_{U_6}$ is as follows:*

$$\begin{aligned} E_A(\Gamma_{U_6}) =& (1)|0|+(2)|-1|+|1\pm\sqrt{7}| = 2+2\sqrt{7} = 2\cdot\rho_A(\Gamma_{U_6}), \\ E_{DS}(\Gamma_{U_6}) =& (1)|-6|+(2)|-8|+|11\pm\sqrt{319}| = 22+2\sqrt{319} = 2\cdot\rho_{DS}(\Gamma_{U_6}), \\ E_{DSA}(\Gamma_{U_6}) =& (1)|0|+(2)|-8|+|8\pm\sqrt{358}| = 16+2\sqrt{358} = 2\cdot\rho_{DSA}(\Gamma_{U_6}). \end{aligned}$$

### 3.5. **Further Discussions.**

Moreover, from Theorem 3.3, 3.6, 3.9, and Definition 2.8, we derive the statement as given below:

**Corollary 3.11.** *The graph $\Gamma_{U_{6n}}$ is a hyperenergetic associated with the adjacency, degree sum, and degree sum adjacency energies.*

**Corollary 3.12.** *In $\Gamma_{U_{6n}}$, $E_A(\Gamma_{U_{6n}})$, $E_{DS}(\Gamma_{U_{6n}})$, and $E_{DSA}(\Gamma_{U_{6n}})$ are never an odd integer.*

The result in Corollary 3.12 aligns with the well-known facts from [25] and [26]. Moreover, the energies obtained in this paper are always twice as large as their spectral radius.

**Corollary 3.13.** *For the non-commuting graph $\Gamma_{U_{6n}}$, then*

*(1)* $E_A(\Gamma_{U_{6n}}) = 2\cdot\rho_A(\Gamma_{U_{6n}})$.

*(2)* $E_{DS}(\Gamma_{U_{6n}}) = 2 \cdot \rho_{DS}(\Gamma_{U_{6n}})$.
*(3)* $E_{DSA}(\Gamma_{U_{6n}}) = 2 \cdot \rho_{DSA}(\Gamma_{U_{6n}})$.

In addition, by comparing the results from Theorem 3.3 and 3.6, we may quickly conclude the following:

**Corollary 3.14.** *In* $\Gamma_{U_{6n}}$, $E_{DS}(\Gamma_{U_{6n}}) > E_{DSA}(\Gamma_{U_{6n}}) > E_A(\Gamma_{U_{6n}})$.

## 4. CONCLUDING REMARKS

The energy formula of $\Gamma_{U_{6n}}$ based on the adjacency, degree sum, and degree sum adjacency matrices has been presented. It is found that the energy formulas from the results of this study are hyperenergetic, associated with degree sum and degree sum adjacency matrices. Moreover, the three energies of $\Gamma_{U_{6n}}$ are never an odd integer. All results are aligned with previous literature which states that the energy of a graph is never an odd integer.

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## REFERENCES

[1] B. H. Neumann, "Problem of paul erdos on groups," *Journal of the Australian Mathematical Society*, vol. 21, no. 4, pp. 467–472, 1976. `https://doi.org/10.1017/S1446788700019303`.
[2] A. Abdollahi, S. Akbari, and H. R. Maimani, "Non-commuting graph of a group," *Journal of Algebra*, vol. 298, no. 2, pp. 468–492, 2006. `https://doi.org/10.1016/j.jalgebra.2006.02.015`.
[3] I. Gutman, "The energy of graph," *Ber. Math. Statist. Sekt. Forschungszenturm Graz.*, vol. 103, pp. 1–22, 1978.
[4] H. S. Ramane, K. Ashoka, and D. Patil, "On the seidel laplacian and seidel signless laplacian polynomials of graphs," *Kyungpook Mathematical Journal*, vol. 61, pp. 155–168, 2021. `https://doi.org/10.5666/KMJ.2021.61.1.155`.
[5] X. Shi, S. Kosari, A. A. Talebi, S. H. Sadati, and H. Rashmanlou, "Investigation of the main energies of picture fuzzy graph and its applications," *International Journal of Computational Intelligence Systems*, vol. 15, no. 1, p. 31, 2022. `https://link.springer.com/article/10.1007/s44196-022-00086-5`.
[6] M. U. Romdhini, F. Al-Sharqi, A. Nawawi, A. Al-Quran, and H. Rashmanlou, "Signless laplacian energy of interval-valued fuzzy graph and its applications," *Sains Malaysiana*, vol. 52, no. 7, pp. 2127–2138, 2023. `https://doi.org/10.17576/jsm-2023-5207-18`.
[7] S. K. Vaidya and K. M. Popat, "Some new results on seidel equienergetic graphs," *Kyungpook Mathematical Journal*, vol. 59, pp. 335–340, 2019. `https://doi.org/10.5666/KMJ.2019.59.2.335`.

[8] H. S. Ramane, D. S. Revankar, and J. B. Patil, "Bounds for the degree sum eigenvalues and degree sum energy of a graph," *International Journal of Pure and Applied Mathematics*, vol. 6, no. 2, pp. 161–167, 2013. https://www.ripublication.com/ijpams/ijpamsv6n2_08.pdf.

[9] S. R. Jog and R. Kotambari, "Degree sum energy of some graphs," *Annals of Pure and Applied Mathematics*, vol. 11, no. 1, pp. 17–27, 2016. https://www.researchmathsci.org/apamart/apam-v11n1-3.pdf.

[10] S. M. Hosamani and H. S. Ramane, "On degree sum energy of a graph," *European Journal of Pure Applied Mathematics*, vol. 9, no. 3, pp. 340–345, 2016.

[11] S. S. Shinde, H. S. Ramane, S. B. Gudimani, and N. Swamy, "Degree sum adjacency polynomial of standard graphs and graph operations," *Electronic Journal of Graph Theory and Applications*, vol. 10, no. 1, pp. 15–32, 2022. http://dx.doi.org/10.5614/ejgta.2022.10.1.2.

[12] F. Al-Sharqi, M. U. Romdhini, and A. Al-Quran, "Group decision-making based on aggregation operator and score function of q-neutrosophic soft matrix," *Journal of Intelligent and Fuzzy Systems*, vol. 45, pp. 1–17, 2023. https://doi.org/10.3233/JIFS-224552.

[13] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, A. Al-Quran, , and S. R. Kamali, "Wiener-hosoya energy of non-commuting graph for dihedral groups," *Asia Pacific Journal of Mathematics*, vol. 11, no. 9, pp. 1–9, 2024. https://doi.org/10.28924/APJM/11-9.

[14] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, and A. Al-Quran, "Closeness energy of non-commuting graph for dihedral groups," *European Journal of Pure and Applied Mathematics*, vol. 17, no. 1, pp. 212–221, 2024. https://doi.org/10.29020/nybg.ejpam.v17i1.5020.

[15] M. U. Romdhini, A. Nawawi, and B. N. Syechah, "Seidel laplacian and seidel signless laplacian energies of commuting graph for dihedral groups," *Malaysian Journal of Fundamental and Applied Sciences*, vol. 20, no. 3, p. 701–713, 2024. https://doi.org/10.11113/mjfas.v20n3.3465.

[16] M. Rather, B A Imran, Z. Raza, and F. B. Farooq, "Spectral properties of sombor matrix of threshold graphs," *RAIRO-Operations Research*, vol. 58, no. 4, pp. 2845–2864, 2024. https://doi.org/10.1051/ro/2024111.

[17] L. Zheng and B. Zhou, "Spectra of closeness laplacian and closeness signless laplacian of graphs," *RAIRO-Operations Research*, vol. 56, no. 5, pp. 3525–3543, 2022. https://doi.org/10.1051/ro/2022161.

[18] R. A. Horn and C. A. Johnson, *Matrix analysis*. Cambridge University Press, 1985.

[19] X. Li, Y. Shi, and I. Gutman, *Graph Energy*. Springer, 2012. https://link.springer.com/book/10.1007/978-1-4614-4220-2.

[20] H. S. Ramane and S. S. Shinde, "Degree exponent polynomial of graphs obtained by some graph operations," *Electronic Notes in Discrete Mathematics*, vol. 63, pp. 161–168, 2017. https://doi.org/10.1016/j.endm.2017.11.010.

[21] F. R. Gantmacher, *The Theory of Matrices*. Chelsea Publishing Company, 1959.

[22] G. James and M. Liebeck, *Representations and characters of groups*. Cambridge University Press, 2001. https://doi.org/10.1017/CBO9780511814532.

[23] M. Mirzargar and A. R. Ashrafi, "Some distance-based topological indices of a non-commuting graph," *Hacettepe Journal of Mathematics and Statistics*, vol. 41, no. 4, pp. 515–526, 2012. https://dergipark.org.tr/en/pub/hujms/issue/7753/101338.

[24] S. Khasraw, C. H. Jaf, N. H. Sarmin, and I. Gambo, "On the non-commuting graph of the group $u_{6n}$," *Malaysian Journal of Mathematical Sciences*, vol. 18, no. 3, pp. 491–500, 2024. https://doi.org/10.47836/MJMS.18.3.02.

[25] S. Pirzada and I. Gutman, "Energy of a graph is never an odd integer," *Bulletin of Kerala Mathematics Association*, vol. 1, pp. 129–132, 2004.

[26] S. Pirzada and I. Gutman, "Energy of a graph is never the square root of an odd integer," *Applicable Analysis and Discrete Mathematics*, vol. 2, no. 1, pp. 118–121, 2008. https://pefmath.etf.rs/vol2num1/AADM-Vol2-No1-118-121.pdf.