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The Exponentiated of Modifying Hyperbolic Tangent Distribution: Model, Properties, and Applications

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Abstract. This article introduces a new two-parameter continuous probability distribution, namely, the Exponentiated of Modifying Hyperbolic Tangent (EMHT) distribution. It is derived by modifying hyperbolic tangent function. Several probability functions and distributional quantities of the EMHT distribution are derived. Maximum likelihood estimation is assigned to find estimators of the EMHT distribution's parameters. Numerical experiments are then conducted to examine the performance of the proposed estimator. The results show that the average estimate for each parameter approaches its actual value as the sample size increases. The final section of this article presents applications of the EMHT distribution to real datasets and performs a comparative study with some existing distributions to exhibit its potential as an alternative model for non-negative continuous data.

Key words and phrases: hazard rate, maximum likelihood estimation, Monte Carlo, moment generating function.

1. INTRODUCTION

Recently, there have been numerous attempts to construct new continuous distributions with the aim to provide various curves of hazard/failure rate functions. Basically, there are two types of hazard curves, that is monotone and non-monotone hazard curves. The monotone hazard rate functions could be in the form of constant or nonincreasing or nondecreasing hazard curves. However, the most recent interest amongst researchers is to chase the non-monotone hazard functions which will appear in the form of bathtub and unimodal (upside-down) curves. In practice, the bathtub and the unimodal hazard rates which typically appear in the future lifetime of human populations and insurance claims studies cannot be achieved

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by common continuous distributions, such as exponential, Weibull, and Pareto distribution.

An approach has been used by several researchers to create new distributions, that is by exponentiating the Cumulative Distribution Function (CDF) or Survival Function (SF) of the common distributions. By this approach, the original distribution eventually will get an addition of one parameter and will provide more variety of hazard rate forms. The exponentiated Gamma distribution by Gupta et al. [1] has added the bathtub hazard rate curve while the original Gamma distribution only provides monotone hazard rate function. The Exponentiated Pareto (EP) distribution by Gupta et al. [1] has added unimodal hazard rate while the original Pareto distribution only provides decreasing hazard rate. The Exponentiated Exponential (EE) distribution by Gupta and Kundu [2] has added the increasing and decreasing hazard rates while the original exponential distribution only provides a constant hazard rate. The Exponentiated Weibull (EW) distribution by Mudholkar and Srivastava [3] has added the bathtub and unimodal hazard rates while the original Weibull distribution only provides monotone hazard rate functions. The Exponentiated Lindley (EL) distribution by Nadarajah et al. [4] has added decreasing and bathtub hazard rates while the original Lindley distribution only provides increasing hazard rate. The exponentiated Lomax distribution by Abdul-Moniem and Abdel-Hameed [5] has added the unimodal and decreasing hazard rate curves while the original Lomax distribution only provides increasing hazard rate. The exponentiated Gompertz distribution by El-Gohary et al. [6] has added decreasing and bathtub hazard rate curves while the original Gompertz distribution only provides increasing hazard rate. The exponentiated Bilal by Abd-Elrahman [7] has added the decreasing and unimodal hazard rates while the original Bilal distribution only provides an increasing hazard rate. The Exponentiated Rayleigh (ER) distribution by Mahmoud and Ghazal [8] has added decreasing hazard rate curves while the original Rayleigh distribution only provides an increasing hazard rate. The Exponentiated Gumbel (EG) distribution by Khazaei and Nanvapishes [9] has added an increasing hazard rate curve while the original Gumbel distribution only provides decreasing hazard rate.

In this study, a new two-parameter distribution is introduced, called the Exponentiated of Modifying Hyperbolic Tangent (EMHT) distribution. This distribution is established by inducing two parameters to the Hyperbolic Tangent (HT) distribution. The term "hyperbolic tangent distribution" was first introduced by Mohammad and Mendoza [10]. However, the HT distribution discussed in this study has only positive support and does not contain any parameter which is very different with HT mentioned in [10]. The basic foundation of the EMHT distribution comes from the first quadrant of HT function, which eventually forms the CDF of HT distribution. The main purpose for introducing the EMHT distribution is to enhance the flexibility of the baseline HT distribution while also holding the parsimony principle. The HT distribution is only able to form monotonically increasing hazard rate. With a smaller number of parameters, the EMHT distribution is able to produce monotonically increasing and decreasing, bathtub, and unimodal hazard rates. Various statistical properties are explored, and real data applications are provided to exhibit the performance of the EMHT distribution.

2. EXPONENTIATED OF MODIFYING HYPERBOLIC TANGENT DISTRIBUTION

Hyperbolic Tangent (HT) is one of the trigonometric functions defined on the unit hyperbola and mathematically can be expressed by Equation (1).

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}. \tag{1}$$

The basic idea of constructing the proposed distribution came up with investigating the graph of the hyperbolic tangent function as given by Figure 1.

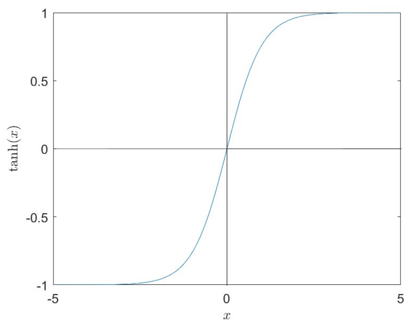

FIGURE 1. The hyperbolic tangent function on the interval [-5, 5].

According to Figure 1, the first quadrant of HT function resembles the shape of a CDF. In this region, HT function is bounded below and above, respectively, by zero and one. Hence, the HT itself can be a prospective CDF for a continuous random variable. However, HT distribution produces no flexibility since there is no parameter contained in this distribution.

Herein, the HT function is modified to derive the CDF of a distribution, namely, the exponentiated of modifying hyperbolic tangent (EMHT) distribution which has flexibility. The CDF of EMHT distribution is constructed in the following way.

(1) Consider hyperbolic tangent function defined in Equation (1) under positive real numbers. By adding parameter $\theta > 0$, Equation (1) becomes

$$\tanh(x^\theta) = \frac{e^{x^\theta} - e^{-x^\theta}}{e^{x^\theta} + e^{-x^\theta}}, \quad x \geq 0; \ \theta > 0. \tag{2}$$

The value of θ must be positive in order to preserve the CDF curve as already formed by the first quadrant of HT function.

(2) By exponentiating the two-hand side of Equation (2) by a positive real number λ, Equation (2) transforms into Equation (3)

$$(\tanh(x^\theta))^\lambda = \left(\frac{e^{x^\theta} - e^{-x^\theta}}{e^{x^\theta} + e^{-x^\theta}} \right)^\lambda, \quad x \geq 0;\ \theta > 0. \tag{3}$$

The value of λ must be positive in order to preserve the CDF curve formed by a function in Equation (2).

The Equation (3) represents the CDF of EMHT consisting of two parameters $\theta > 0$ and $\lambda > 0$. A positive random variable X has the EMHT distribution, denoted by EMHT(θ, λ), if its CDF takes the form

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (\tanh(x^\theta))^\lambda & \text{if } x > 0 \end{cases}, \tag{4}$$

where θ and λ are both shape parameters.

Theorem 2.1. *The function given in Equation (4) satisfies the CDF properties.*

Proof. First, it will be shown that $0 \leq F_X(x) \leq 1$, for all $x \geq 0$. By considering the hyperbolic function given in Equation (1), the following inequality holds for $x > 0$

$$0 \leq \tanh(x) \leq 1 \Leftrightarrow 0 \leq (\tanh(x^\theta))^\lambda \leq 1.$$

Thus, $F_X(x) \in [0, 1]$ for all $x > 0; \theta > 0; \lambda > 0$.

Second, it will be shown that $F_X(x)$ is a non-decreasing function. Let x_1 and x_2 be non-negative real numbers such that $x_1 < x_2$. Since hyperbolic tangent is a non-decreasing function over the field of a non-negative real number, then the following inequality holds

$$\tanh(x_1^\theta) \leq \tanh(x_2^\theta) \Leftrightarrow (\tanh(x_1^\theta))^\lambda \leq (\tanh(x_2^\theta))^\lambda \Leftrightarrow F_X(x_1) \leq F_X(x_2).$$

Since $F_X(x_1) \leq F_X(x_2)$ for any $x_1 < x_2$, then $F_X(x)$ is a non-decreasing function.

Third, it will be shown that $F_X(x)$ is a right-continuous function. Consider the limit from the right of $F_X(x)$ as $x \to a$, that is

$$\lim_{x \to a^+} F_X(x) = \lim_{x \to a^+} (\tanh(x^\theta))^\lambda = \left(\lim_{x \to a^+} \tanh(x^\theta) \right)^\lambda = (\tanh(a^\theta))^\lambda = F_X(a).$$

Since $\lim_{x \to a^+} F_X(x) = F_X(a)$, then $F_X(x)$ is a right-continuous function.

Fourth, it will be shown that $\lim_{x \to \infty} F_X(x) = 1$ and $\lim_{x \to -\infty} F_X(x) = 0$.

$$\lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} (\tanh(x^\theta))^\lambda = \left(\lim_{x \to \infty} \tanh(x^\theta) \right)^\lambda = (1)^\lambda = 1,$$

$$\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} 0 = 0.$$

As the function given in Equation (4) meets four CDF properties, the finding $F_X(x)$ is a valid CDF of X. $\square$

The following Figure 2 and Figure 3 present the CDF plots of EMHT distribution.

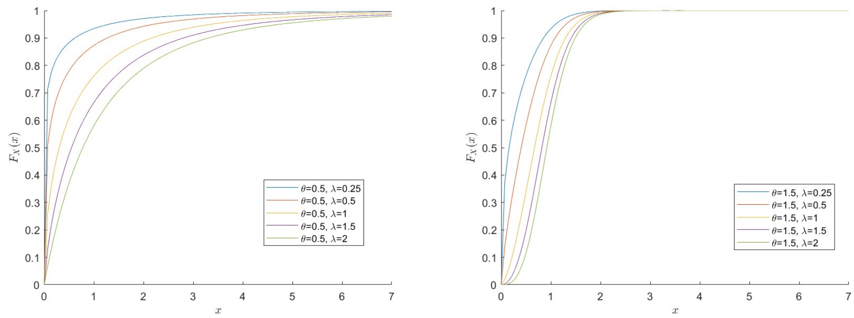

Figure 2. The CDF plots of EMHT distribution with a fixed value of θ and different values of λ.

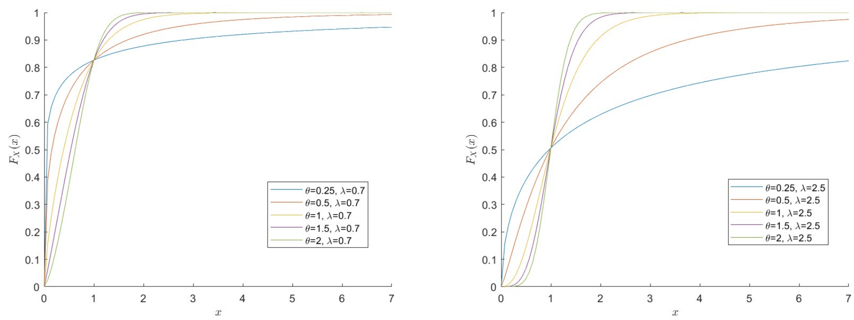

Figure 3. The CDF plots of EMHT distribution with a fixed value of λ and different values of θ.

Figure 2 shows that for a fixed value of θ, the CDF tends to one faster as the value of λ gets smaller. On the other hand, Figure 3 shows that for a fixed value of λ, the CDF tends to one faster as the value of θ gets larger. With a CDF of the EMHT distribution given in Equation (4), the corresponding probability density function (PDF) of the EMHT distribution is obtained as follows

$$\begin{aligned} f_X(x) = \frac{d}{dx}[F_X(x)] &= \frac{d}{dx}\left[(\tanh(x^\theta))^\lambda\right] \\ &= \theta\ \lambda\ x^{\theta-1}\ \operatorname{sech}^2(x^\theta)\ \tanh^{\lambda-1}(x^\theta) \\ &= \theta\ \lambda\ x^{\theta-1}\ \operatorname{csch}(x^\theta)\ \operatorname{sech}(x^\theta)\ \tanh^\lambda(x^\theta), \quad x > 0. \end{aligned} \tag{5}$$

Figure 4 presents the plots of EMHT density function in Equation (5) and indicates that the EMHT distribution is right skewed because it has a long right tail. In the

EMHT distribution, the probabilities grow quickly and taper off slowly for larger values.

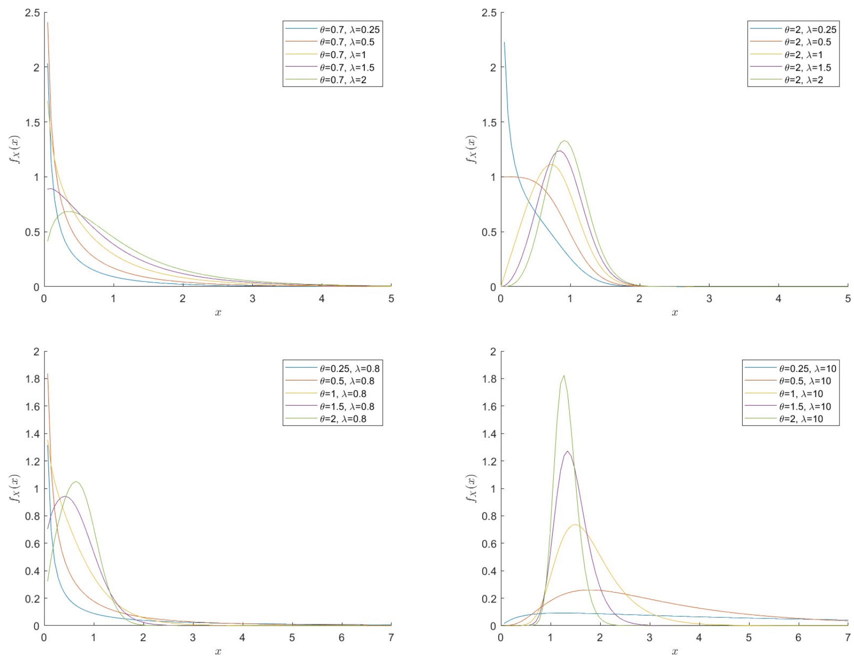

FIGURE 4. The PDF plots of EMHT distribution.

The survival and hazard rate functions of EMHT distribution are given, respectively, by Equations (6) and (7).

$$S_X(x) = 1 - F_X(x) = 1 - (\tanh(x^\theta))^\lambda, \quad x > 0, \tag{6}$$

$$h_X(x) = \frac{f_X(x)}{S_X(x)} = \frac{\theta\ \lambda\ x^{\theta-1}\ \mathrm{csch}(x^\theta)\ \mathrm{sech}(x^\theta)\ \tanh^\lambda(x^\theta)}{1 - \tanh^\lambda(x^\theta)}, \quad x > 0. \tag{7}$$

The hazard rate function (HRF) of the EMHT distribution, given in Equation (7), can appear in many forms. The behavior of the EMHT hazard rate is shown case by case.

Case 1: $\lambda = 1$ and $\theta = 1$ (EMHT(1,1)). In this case, Equation (7) simplifies to

$$h_X(x) = \frac{\mathrm{csch}(x)\ \mathrm{sech}(x)\ \tanh(x)}{1 - \tanh(x)} = 1 + \tanh(x). \tag{8}$$

Since $\tanh(x)$ is a monotonically increasing function for $x > 0$, then $1 + \tanh(x)$ is also monotonically increasing function. Figure 5 presents the plot of hazard rate function in Equation (8). The hazard rate function of EMHT(1,1) is bounded below and above by one and two, respectively. In this case, the EMHT distribution can be called as the Hyperbolic Tangent (HT) distribution having no parameter and with the CDF $F_X(x) = \tanh(x),\ x \geq 0$. However, HT distribution does not have flexibility since it only produces monotonically increasing HF.

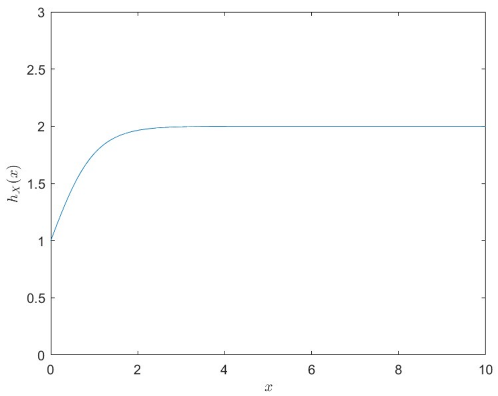

FIGURE 5. The hazard rate of EMHT(1,1).

Case 2: $\lambda = 1$ and $\theta > 0$ (EMHT$(\theta, 1)$). In this case, Equation (7) simplifies to

$$h_X(x) = \frac{\theta x^{\theta-1}\ \mathrm{csch}(x^\theta)\ \mathrm{sech}(x^\theta)\ \tanh(x^\theta)}{1 - \tanh(x^\theta)} = \theta x^{\theta-1}(1 + \tanh(x^\theta)). \tag{9}$$

Figures 6 and 7 present the plot of hazard rate function in Equation (9).

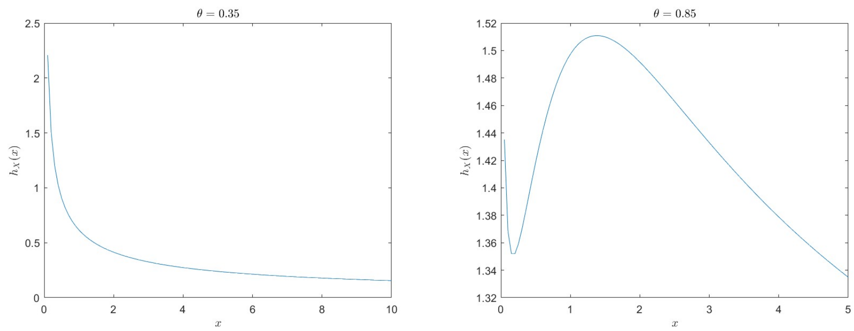

FIGURE 6. The hazard rate of EMHT$(\theta, 1)$ for $\theta < 1$.

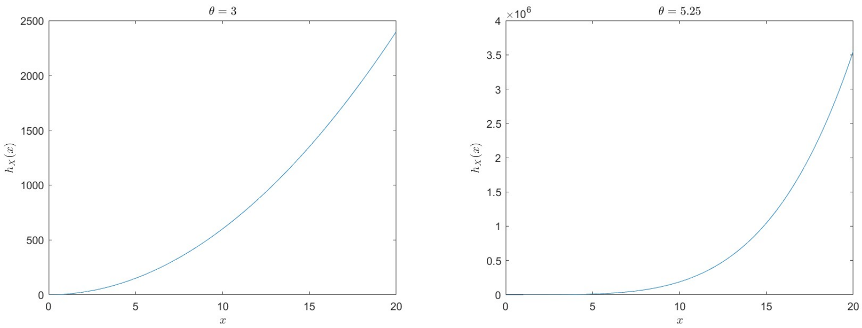

FIGURE 7. The hazard rate of EMHT$(\theta, 1)$ for $\theta > 1$.

Here is the behavior of hazard rate function (9)

$$h_X(x;\theta) = \begin{cases} \infty & \text{if } x \to \infty \text{ and } \theta > 1 \\ 0 & \text{if } x \to \infty \text{ and } \theta < 1 \\ 0 & \text{if } x \to 0 \text{ and } \theta > 1 \\ \infty & \text{if } x \to 0 \text{ and } \theta < 1 \\ \infty & \text{if } \theta \to \infty \\ 0 & \text{if } \theta \to 0. \end{cases}$$

In this case, the EMHT distribution can be called as the Modified Hyperbolic Tangent (MHT) distribution having parameter $\theta > 0$. For $\theta < 1$, the hazard rate function can be in the shape of decreasing or decreasing-increasing-decreasing (bathtub-decreasing) curves. Moreover, for large values of x, the hazard rates are small. For $\theta > 1$, as the value of x increases, the hazard rate increases without bound.

Case 3: $\theta = 1$ and $\lambda > 0$ (EMHT$(1, \lambda)$). In this case, Equation (7) simplifies to

$$h_X(x) = \frac{\lambda \, \text{csch}(x) \, \text{sech}(x) \, \tanh^{\lambda}(x)}{1 - \tanh^{\lambda}(x)}. \tag{10}$$

Figures 8 and 9 present the plot of hazard rate function (10). In this case, the EMHT distribution can be called as the Exponentiated Hyperbolic Tangent (EHT) distribution having parameter $\lambda > 0$. The EHT distribution has a hazard rate function with two different shapes:

- Monotonically increasing for $\lambda \geq 1$.
- Bathtub for $\lambda < 1$.

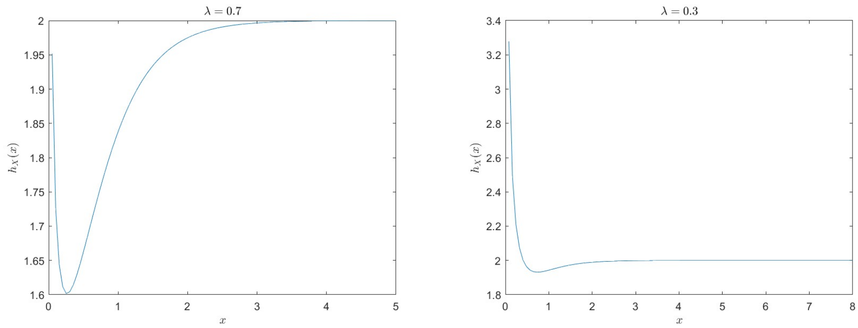

FIGURE 8. The hazard rate of EMHT$(1, \lambda)$ for $\lambda < 1$.

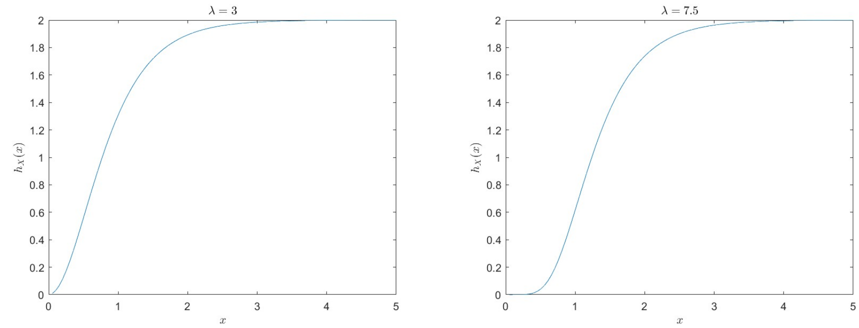

FIGURE 9. The hazard rate of EMHT$(1, \lambda)$ for $\lambda \geq 1$.

Here is the behavior of hazard rate function (10)

$$h_X(x;\lambda) = \begin{cases} 2 & \text{if } x \to \infty \text{ and } \lambda > 1 \\ 0 & \text{if } x \to 0 \text{ and } \lambda > 1 \\ 0 & \text{if } \lambda \to \infty \\ \frac{\operatorname{csch}(x)\operatorname{sech}(x)}{-\ln(\tanh(x))} & \text{if } \lambda \to 0. \end{cases}$$

For $\lambda > 1$, the hazard rate increases and is bounded above by 2. For $\lambda < 1$, the hazard rate function is in the shape of bathtub.

Case 4: $\theta > 0$ and $\lambda > 0$ (EMHT(θ, λ)). In this case, Equation (7) remains the same, and the hazard curves generated from this equation are given in Figures 10 and 11. This distribution has hazard rate functions with four different shapes:

- Increasing for $\theta > 1$ and $\lambda > 1$.
- Decreasing for $\theta < 1$ and $\lambda < 1$.
- Decreasing or unimodal for $\theta < 1$ and $\lambda > 1$.
- Increasing or bathtub for $\theta > 1$ and $\lambda < 1$.

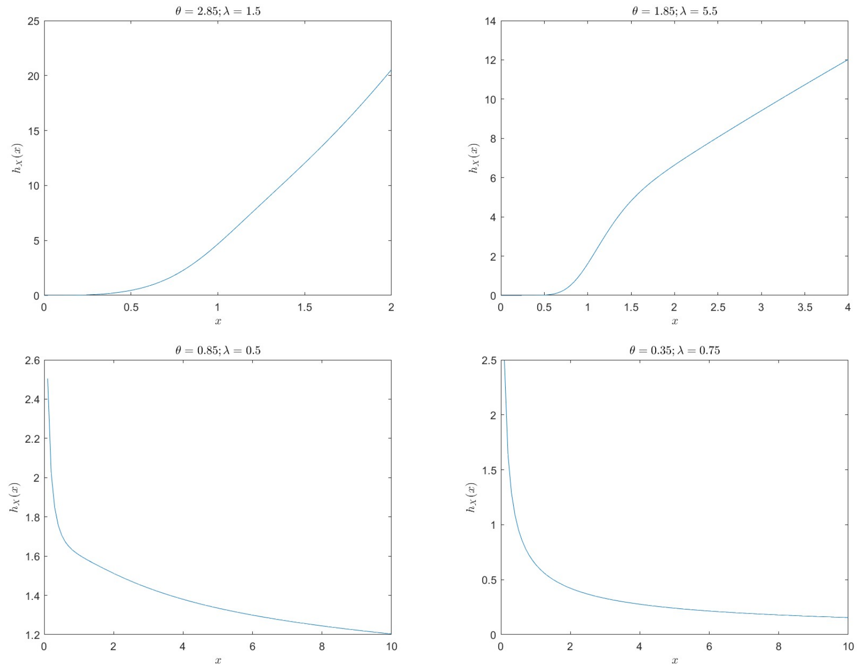

FIGURE 10. The hazard rate of EMHT(θ, λ) for $\theta, \lambda > 1$ and $\theta, \lambda < 1$.

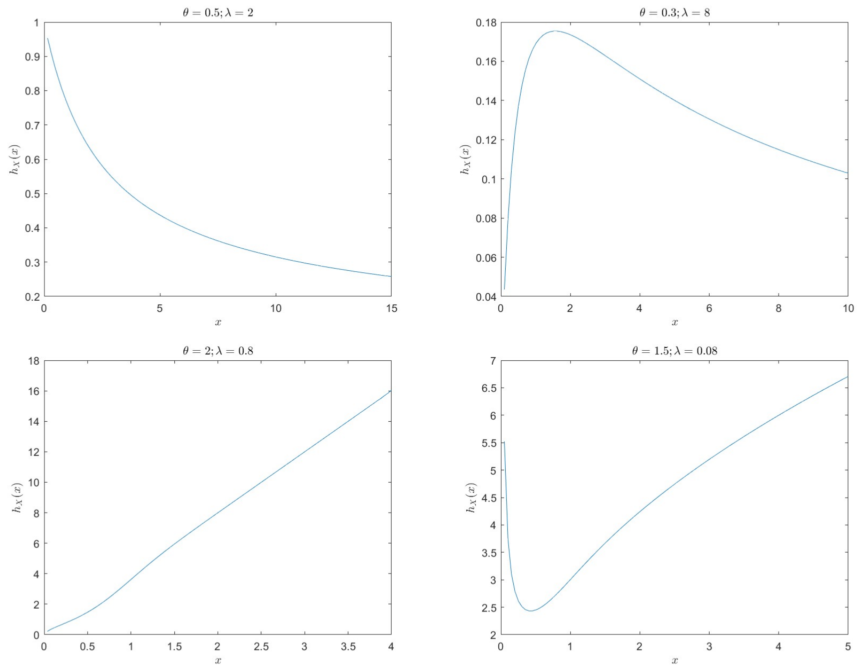

FIGURE 11. The hazard rate of EMHT(θ, λ) for $\theta < 1$ and $\lambda > 1$, and $\theta > 1$ and $\lambda < 1$.

Thefore, based on all cases, the hazard rate function of the EMHT distribution can be in the shapes of increasing, decreasing, bathtub, unimodal, or decreasing-increasing-decreasing.

3. MEAN AND VARIANCE OF EMHT DISTRIBUTION

Let $X \sim \text{EMHT}(\theta, \lambda)$, then mean and variance of EMHT distribution can be obtained as follows

$$E(X) = \int_0^\infty x\theta\lambda x^{\theta-1} \operatorname{csch}(x^\theta) \operatorname{sech}(x^\theta) \tanh^\lambda(x^\theta)dx = \theta\lambda \int_0^\infty x^\theta \frac{\sinh^{\lambda-1}(x^\theta)}{\cosh^{\lambda+1}(x^\theta)}dx \tag{11}$$

$$\operatorname{Var}(X) = E\left([X-\mu]^2\right) = \int_0^\infty (x-\mu)^2\theta\lambda x^{\theta-1} \operatorname{csch}(x^\theta) \operatorname{sech}(x^\theta) \tanh^\lambda(x^\theta)dx \tag{12}$$

where μ is the mean (first moment) of the EMHT distribution. However, the numerical approach is needed to calculate the mean in Equation (11) and the variance in Equation (12), using the mathematical software, such as WolframAlpha, Mathematica or Matlab. Nevertheless, this study provides the mean and variance of the EMHT distribution for selected values of θ and λ as presented in Table 1.

Table 1. The mean, variance, and MGF of the EMHT distribution for some combinations of θ and λ.

t	θ	λ	Mean	Variance	$M_X(t)$
-50	0.5	5	2.48913	5.870031843	0.000167988
-10	5	0.5	0.734836	0.0564280531	0.499771
-5	5	5	1.06006	0.0086627964	0.00556042
-5	0.5	0.5	0.458893	1.250277215	0.583864
-1	0.85	1.3	0.815946	0.5141221251	0.84738
-0.5	3	0.08	0.199689	0.07937330328	0.91333

4. MOMENT GENERATING FUNCTION OF EMHT DISTRIBUTION

Let $X \sim \text{EMHT}(\theta, \lambda)$, then the Moment Generating Function (MGF) of EMHT distribution can be obtained as follows

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\, \theta\lambda x^{\theta-1} \operatorname{csch}(x^\theta) \operatorname{sech}(x^\theta) \tanh^\lambda(x^\theta)dx. \tag{13}$$

The integral in Equation (13) does not have a closed-form solution, but it can be evaluated numerically using mathematical software, such as WolframAlpha, Mathematica and Matlab. Based on simulation, for any combinations of θ and λ, the MGF of EMHT distribution is only defined on $t < 0$, and some possible values of $M_X(t)$ are given in Table 1.

5. QUANTILES OF EMHT DISTRIBUTION

As a CDF of the EMHT distribution has a closed-form solution, the quantile of this distribution can be obtained by direct calculation. Theorem 5.1 presents the p quantile of the EMHT distribution.

Theorem 5.1. *Let $X \sim EMHT(\theta, \lambda)$, then the p quantile of X is*

$$x_p = \left(arctanh \left(p^{\frac{1}{\lambda}} \right) \right)^{\frac{1}{\theta}},$$

where $0 < p < 1$.

Proof. The p quantile of X can be obtained as follows:

$$\begin{aligned} F_X(x_p) = p &\Leftrightarrow (\tanh(x_p^\theta))^\lambda = p \\ &\Leftrightarrow \tanh(x_p^\theta) = p^{\frac{1}{\lambda}} \\ &\Leftrightarrow x_p^\theta = \operatorname{arctanh}\left(p^{\frac{1}{\lambda}}\right), \end{aligned}$$

Hence, the p quantile (100p-th percentile) of X is

$$x_p = \left(\operatorname{arctanh} \left(p^{\frac{1}{\lambda}} \right) \right)^{\frac{1}{\theta}}.$$

By Theorem 5.1, the median (50-th percentile) of the EMHT distribution is $x_{0.5} = \left(\operatorname{arctanh} \left(0.5^{\frac{1}{\lambda}} \right) \right)^{\frac{1}{\theta}}$. This quantile function can be used to generate data from the EMHT distribution by setting the values of parameters θ and λ. □

6. MODE OF EMHT DISTRIBUTION

The following Theorem 6.1 presents the mode of the EMHT distribution.

Theorem 6.1. *Let $X \sim EMHT(\theta, \lambda)$, then the mode of X is the value of x satisfying the following equation*

$$(\theta - 1)\tanh(x^\theta) - (1 + \lambda)\theta x^\theta \tanh^2(x^\theta) - \theta x^\theta (1 - \lambda) = 0.$$

In particular, if $\theta = 1$, the mode of X is

$$x_{mode} = arctanh \left(\sqrt{\frac{\lambda - 1}{\lambda + 1}} \right),$$

and only valid for $\lambda > 1$.

Proof. The mode of EMHT distribution is the value x that satisfies the following equation

$$\arg\max_{x>0} f_X(x). \tag{14}$$

The value of x that maximizes Equation (14) can be obtained by solving the following equation.

$$\frac{d}{dx}f_X(x) = 0,$$

or equivalent to

$$\theta\lambda x^{\theta-2}\operatorname{csch}(x^\theta)\operatorname{sech}(x^\theta)\tanh^{\lambda-1}(x^\theta)\times$$
$$[(\theta-1)\tanh(x^\theta) - \theta x^\theta\coth(x^\theta)\tanh(x^\theta) - \theta x^\theta\tanh^2(x^\theta) + \theta\lambda x^\theta\operatorname{sech}^2(x^\theta)] = 0. \tag{15}$$

Since the expression $\theta\lambda x^{\theta-2}\operatorname{csch}(x^\theta)\operatorname{sech}(x^\theta)\tanh^{\lambda-1}(x^\theta)$ is non-zero, then Equation (15) can be rewritten as follows

$$\begin{aligned}
(\theta-1)\tanh(x^\theta) - \theta x^\theta\coth(x^\theta)\tanh(x^\theta) - \theta x^\theta\tanh^2(x^\theta) + \theta\lambda x^\theta\operatorname{sech}^2(x^\theta) &= 0\\
\Leftrightarrow (\theta-1)\tanh(x^\theta) - \theta x^\theta - \theta x^\theta\tanh^2(x^\theta) + \theta\lambda x^\theta[1-\tanh^2(x^\theta)] &= 0\\
\Leftrightarrow (\theta-1)\tanh(x^\theta) - (1+\lambda)\theta x^\theta\tanh^2(x^\theta) - \theta x^\theta(1-\lambda) &= 0.
\end{aligned}$$

The last equation cannot be solved analytically and thereby requires a numerical approach such as Newton-Raphson. However, for a specified value of θ, the mode of the EMHT can be obtained analytically. If $\theta = 1$, the last expression can be rewritten as follows

$$\begin{aligned}
(1+\lambda)x\tanh^2(x) + x(1-\lambda) &= 0\\
\Leftrightarrow (1+\lambda)\tanh^2(x) &= \lambda - 1\\
\tanh(x) &= \pm\sqrt{\frac{\lambda-1}{\lambda+1}}.
\end{aligned}$$

Since $x > 0$, the negative root can be neglected, as $\tanh(x)$ is a positive function with respect to x.Thus,

$$\tanh(x) = \sqrt{\frac{\lambda-1}{\lambda+1}}. \tag{16}$$

Now, take the inverse hyperbolic tangent (i.e., arctanh) of both sides of Equation (16), yields

$$x = \operatorname{arctanh}\left(\sqrt{\frac{\lambda-1}{\lambda+1}}\right),$$

and it exists only for $\lambda > 1$. Therefore, for $\theta = 1$, the mode of EMHT distribution is $\operatorname{arctanh}\left(\sqrt{\frac{\lambda-1}{\lambda+1}}\right)$. □

7. MAXIMUM LIKELIHOOD ESTIMATION

In this research, the maximum likelihood estimation (MLE) is applied to estimate the parameters of EMHT distribution as the MLE will result in unbiased estimator for large sample size. The estimation process is conducted case by case.

The following is the process to construct the likelihood and log-likelihood functions of the EMHT distribution.

$$\begin{aligned}
L(\lambda,\theta;x_1,x_2,\ldots,x_n) &= \prod_{i=1}^{n} f_X(x_i;\lambda,\theta) \\
&= \prod_{i=1}^{n} \theta\lambda x_i^{\theta-1}\ \mathrm{csch}(x_i^\theta)\ \mathrm{sech}(x_i^\theta)\ \tanh^\lambda(x_i^\theta) \\
&= (\theta\lambda)^n \prod_{i=1}^{n} x_i^{\theta-1}\ \mathrm{csch}(x_i^\theta)\ \mathrm{sech}(x_i^\theta)\ \tanh^\lambda(x_i^\theta).
\end{aligned}$$

Hence, its log-likelihood function is given as follows:

$$\begin{aligned}
l(\lambda,\theta;x_1,x_2,\ldots,x_n) &= n\ln(\theta\lambda) + \sum_{i=1}^{n} \ln\left[x_i^{\theta-1}\ \mathrm{csch}(x_i^\theta)\ \mathrm{sech}(x_i^\theta)\ \tanh^\lambda(x_i^\theta)\right] \\
&= n\ln(\theta\lambda) + (\theta-1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln[\mathrm{csch}(x_i^\theta)] \\
&+ \sum_{i=1}^{n}\ln[\mathrm{sech}(x_i^\theta)] + \lambda\sum_{i=1}^{n}\ln[\tanh(x_i^\theta)]
\end{aligned} \tag{17}$$

Case 1: Both λ and θ are not given. The value of λ and θ that maximize Equation (17) can be obtained by taking the first partial derivative of Equation (17) with respect to λ and θ, and equating to zero respectively, so that the following equations are obtained.

$$\begin{aligned}
\frac{\partial l(\lambda,\theta)}{\partial\lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n}\ln\left[\tanh(x_i^\theta)\right] = 0 \\
\frac{\partial l(\lambda,\theta)}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n} x_i^\theta\ln(x_i)\coth(x_i^\theta) \\
&\qquad - \sum_{i=1}^{n} x_i^\theta\ln(x_i)\tanh(x_i^\theta) + 2\lambda\sum_{i=1}^{n} x_i^\theta\ln(x_i)\mathrm{csch}(2x_i^\theta) = 0.
\end{aligned} \tag{18}$$

The solution of system of nonlinear equations (18) does not have a closed-form. Hence, a numerical approach is required to find the MLE of λ and θ. This study uses the Newton-Raphson method to find estimated value of both parameters.

Case 2: Estimate θ when λ is given. The value of θ that maximizes Equation (17) can be obtained by taking the first derivative of Equation (17) with respect to θ, and equating to zero, so that the following equation is obtained.

$$\frac{dl(\lambda,\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} x_i^{\theta} \ln(x_i) \coth(x_i^{\theta}) - \sum_{i=1}^{n} x_i^{\theta} \ln(x_i) \tanh(x_i^{\theta}) + 2\lambda \sum_{i=1}^{n} x_i^{\theta} \ln(x_i) \operatorname{csch}(2x_i^{\theta}) = 0. \tag{19}$$

As in the first case, the Newton-Raphson method is necessary aim to support the MLE for solving nonlinear Equation (19).

Case 3: Estimate λ when θ is given. The value of λ that maximizes Equation (17) can be obtained by taking the first partial derivative of Equation (17) with respect to λ, and equating to zero, so that the following equation is obtained.

$$\frac{dl(\lambda,\theta)}{d\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \ln\left[\tanh(x_i^{\theta})\right] = 0. \tag{20}$$

The solution of nonlinear Equation (20) is

$$\hat{\lambda} = -\frac{n}{\sum_{i=1}^{n} \ln[\tanh(x_i^{\theta})]}.$$

8. SIMULATION STUDIES

As the parameters of the EMHT distribution cannot be derived analytically, this section presents a series of numerical simulations to evaluate the performance of the proposed estimators for selected parameter values across various sample sizes. The Monte Carlo simulation is performed with 10,000 replications for each specified parameter configuration. Parameter estimation is carried out using the maximum likelihood estimation (MLE) method, as outlined in the preceding section. The performance of the MLE is assessed using two metrics: the average estimate and the mean squared error (MSE), which are computed as follows:

$$\text{Average estimate} = \frac{\sum_{j=1}^{10,000} \hat{\tau}_j}{10,000}; \quad \text{MSE} = \frac{\sum_{j=1}^{10,000} (\hat{\tau}_j - \tau)^2}{10,000}$$

where $\hat{\tau}_j$ is the estimated value for the considered parameter on the j-th experiment and τ is the actual value of the considered parameter.

The Monte Carlo algorithm for generating data of size n from the EMHT distribution using its quantile $x_p = \left(\operatorname{arctanh}\left(p^{\frac{1}{\lambda}}\right)\right)^{\frac{1}{\theta}}$ is presented as follows:

Step 1: Set the value of parameters θ and λ.
Step 2: Generate the value of p from the uniform distribution defined on $(0, 1)$ for

n times and assign it as p_i, where $i = 1, 2, \ldots, n$.
Step 3: Calculate x_{p_i} using p_i alongside θ and λ set on Step 1 through an Equation

$$x_{p_i} = \left(\operatorname{arctanh}\left(p^{\frac{1}{\lambda}}\right)\right)^{\frac{1}{\theta}}.$$

The Monte Carlo algorithm will produce the values of $x_{p_1}, x_{p_2}, \ldots, x_{p_n}$ generated from the EMHT distribution. In this study, we generate the data of size 100, 500, and 1000 from the EMHT distribution for cases of $\theta < \lambda$, $\theta > \lambda$ and $\theta = \lambda$. The selected values of θ and λ produce data with increasing, decreasing, bathtub, unimodal, and decreasing-increasing-decreasing HRF. The numerical results are presented in Table 2.

Table 2. Numerical results of EMHT distribution parameter estimation for generated data of size n.

Parameter	Average Estimate			MSE		
	$n = 100$	$n = 500$	$n = 1000$	$n = 100$	$n = 500$	$n = 1000$
$\theta = 0.5$	0.5044	0.5008	0.5012	0.0010	0.0002	0.0001
$\lambda = 5$	5.0790	5.0268	4.9993	0.2983	0.0558	0.0251
$\theta = 5$	5.2789	5.0497	5.0404	1.0019	0.1227	0.0706
$\lambda = 0.5$	0.4952	0.4971	0.4977	0.0091	0.0016	0.0008
$\theta = 5$	5.0675	5.0112	5.0084	0.1052	0.0187	0.0098
$\lambda = 5$	5.0416	5.0304	5.0096	0.2425	0.0572	0.0252
$\theta = 0.5$	0.5301	0.5050	0.5049	0.0090	0.0013	0.0007
$\lambda = 0.5$	0.4903	0.4985	0.4963	0.0080	0.0016	0.0008
$\theta = 0.85$	0.8786	0.8545	0.8515	0.0126	0.0021	0.0010
$\lambda = 1$	0.9897	1.0026	1.0012	0.0219	0.0040	0.0021
$\theta = 1.25$	2.8059	1.5386	1.3699	6.4406	0.5804	0.1565
$\lambda = 0.03$	0.0232	0.0281	0.0290	0.0003	0.0001	0.0000

Based on Table 2, the average estimates of the parameters are generally close to their actual values, with the distance never exceeding 0.5. Moreover, the average estimates have a closer gap with the actual parameters as the sample size increases. It is also confirmed by its MSE, as the sample size gets larger, the MSE gets smaller. Therefore, it is reasonable to conclude that the finding estimators are relevant to be used as the point estimation of each parameter.

9. APPLICATIONS TO REAL-WORLD DATA

This section provides the application of the EMHT distribution to four datasets to show the usefulness of the EMHT distribution and to be compared with some common non-negative continuous distributions alongside the Exponentiated Weibull (EW) distribution. The choice of the competing distributions is in accordance to the similarity in hazard rates characteristics and the number of parameters in each competing distribution that is not more or less far with the parameters contained in the EMHT distribution. These competing models are:

Exponentiated Weibull(α, β, θ) with PDF

$$f_X(x) = \alpha\beta\theta^\beta x^{\beta-1} e^{-(\theta x)^\beta} \left(1 - e^{-(\theta x)^\beta}\right)^{\alpha-1}, \quad \text{where } \alpha > 0;\ \beta > 0;\ \theta > 0.$$

Gamma(α, β) with PDF

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right), \quad \text{where } \alpha > 0;\ \beta > 0.$$

Weibull(α, β) with PDF

$$f_X(x) = \alpha\beta^{-\alpha} x^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^\alpha\right), \quad \text{where } \alpha > 0;\ \beta > 0.$$

Rayleigh(θ) with PDF

$$f_X(x) = 2\theta^2 x e^{-(\theta x)^2}, \quad \text{where } \theta > 0.$$

Lindley(l) with PDF

$$f_X(x) = \frac{l^2(1+x)e^{-lx}}{l+1}, \quad \text{where } l > 0.$$

Bilal(b) with PDF

$$f_X(x) = \frac{6e^{-\frac{2x}{b}}\left(1 - e^{-\frac{x}{b}}\right)}{b}, \quad \text{where } b > 0.$$

The datasets used in this study are:

- Dataset 1: The average net wage/salary (in million Rupiahs) per month received by the employee from the main job on 17 sectors in Indonesia (source: https://www.bps.go.id/en/statistics-table/2/MTUyMSMy/rata-rata-upah-gaji.html).
- Dataset 2: 264 observations of the maximum monthly wind speed (in mph) recorded in Palm Beach, Florida, from January 1984 to December 2005 (source: http://www.ncdc.noaa.gov./).
- Dataset 3: 251 motor insurance claims collected from a survey conducted by an insurance company in Thailand in 2013 [11].
- Dataset 4: The snow accumulation data in inches in the Raleigh-Durham airport, North Carolina, from 1948 to 2000 [9].
- Dataset 5: The time between failures (thousands of hours) of secondary reactor pumps [12].

The results for fitting respective dataset to all considered distributions are evaluated by considering the score-based approaches such as the log-likelihood value, Akaike's information criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov-Smirnov (KS) goodness of fit statistic with its p-values, as presented in Tables 3, 4, 5, 6, and 7, respectively.

Table 3. Fitting result for dataset 1.

Distribution	KS statistic (p-value)	Log-likelihood	AIC	BIC	Fitted parameters
EMHT	0.10981 (0.9723)	-24.66248	53.32496	54.99139	$\hat{\theta} = 0.7516$ $\hat{\lambda} = 43.6301$
Exponentiated Weibull	0.12562 (0.9211)	-24.2335	54.467	56.9666	$\hat{\alpha} = 2.75706$ $\hat{\beta} = 2.22897$ $\hat{\theta} = 0.35511$
Weibull	0.13219 (0.8907)	-24.34364	52.6873	54.3537	$\hat{\alpha} = 3.75801$ $\hat{\beta} = 3.82859$
Gamma	0.12229 (0.9344)	-24.21907	52.4381	54.1046	$\hat{\alpha} = 11.06635$ $\hat{\beta} = 0.31179$
Rayleigh	0.26149 (0.163)	-28.47745	58.9549	59.7881	$\hat{\theta} = 0.2779$
Lindley	0.35619 (0.0195)	-34.84812	71.6962	72.5295	$\hat{l} = 0.48499$
Bilal	0.3132 (0.0558)	-32.45683	66.9137	67.7469	$\hat{b} = 4.1861$

Table 4. Fitting result for dataset 2.

Distribution	KS statistic (p-value)	Log-likelihood	AIC	BIC	Fitted parameters
EMHT	0.07887 (0.07492)	-896.4984	1796.997	1804.149	$\hat{\theta} = 0.50251$ $\hat{\lambda} = 93822.92$
Exponentiated Weibull	0.077144 (0.0864)	-898.3182	1802.636	1813.364	$\hat{\alpha} = 631.5777$ $\hat{\beta} = 0.91762$ $\hat{\theta} = 0.20841$
Weibull	0.17042 (0.0000)	-983.8561	1971.712	1978.864	$\hat{\alpha} = 3.87469$ $\hat{\beta} = 43.68567$
Gamma	0.11958 (0.00105)	-923.1104	1850.221	1857.373	$\hat{\alpha} = 24.54758$ $\hat{\beta} = 1.63412$
Rayleigh	0.37324 (0.0000)	-1073.859	2149.718	2153.294	$\hat{\theta} = 0.02433$
Lindley	0.40944 (0.0000)	-1147.995	2297.989	2301.565	$\hat{l} = 0.04870$
Bilal	0.4042 (0.0000)	-1145.274	2292.548	2296.124	$\hat{b} = 48.719$

Table 5. Fitting result for dataset 3.

Distribution	KS statistic (p-value)	Log-likelihood	AIC	BIC	Fitted parameters
EMHT	0.057048 (0.3875)	-2633.721	5271.442	5278.493	$\hat{\theta} = 0.14427$ $\hat{\lambda} = 474.284$
Exponentiated Weibull	0.061919 (0.291)	-2631.394	5268.788	5279.364	$\hat{\alpha} = 20.1156$ $\hat{\beta} = 0.31379$ $\hat{\theta} = 0.00613$
Weibull	0.12528 (0.00076)	-2661.247	5326.494	5333.545	$\hat{\alpha} = 0.88828$ $\hat{\beta} = 13987.7$
Gamma	0.15147 (0.0000)	-2664.617	5333.234	5340.285	$\hat{\alpha} = 0.93130$ $\hat{\beta} = 16137$
Rayleigh	0.51076 (0.0000)	-2934.364	5870.728	5874.253	$\hat{\theta} = 0.00004$
Lindley	0.26246 (0.0000)	-2724.981	5451.962	5455.487	$\hat{l} = 0.00013$
Bilal	0.24793 (0.0000)	-2715.97	5433.94	5437.465	$\hat{b} = 17516.1$

Table 6. Fitting result for dataset 4.

Distribution	KS statistic (p-value)	Log-likelihood	AIC	BIC	Fitted parameters
EMHT	0.10909 (0.4415)	-107.4513	218.903	223.189	$\hat{\theta} = 0.45287$ $\hat{\lambda} = 3.12852$
Exponentiated Weibull	0.11805 (0.3437)	-107.4566	220.913	227.343	$\hat{\alpha} = 4.04895$ $\hat{\beta} = 0.48777$ $\hat{\theta} = 2.91166$
Weibull	0.099341 (0.563)	-109.8276	223.655	227.942	$\hat{\alpha} = 0.90257$ $\hat{\beta} = 2.00786$
Gamma	0.11383 (0.3879)	-110.3484	224.697	228.983	$\hat{\alpha} = 0.91892$ $\hat{\beta} = 2.31294$
Rayleigh	0.4707 (0.0000)	-174.9177	351.8355	353.9786	$\hat{\theta} = 0.27403$
Lindley	0.15087 (0.1136)	-114.5881	231.1761	233.3193	$\hat{l} = 0.7408$
Bilal	0.19728 (0.01483)	-124.2624	250.5248	252.668	$\hat{b} = 2.492$

Table 7. Fitting result for dataset 5.

Distribution	KS statistic (p-value)	Log-likelihood	AIC	BIC	Fitted parameters
EMHT	0.11097 (0.9101)	-32.32359	68.64717	70.91816	$\hat{\theta} = 0.44981$ $\hat{\lambda} = 2.18660$
Exponentiated Weibull	0.096743 (0.9682)	-31.83198	69.66397	73.07045	$\hat{\alpha} = 10.3097$ $\hat{\beta} = 0.29985$ $\hat{\theta} = 40.0950$
Weibull	0.11839 (0.8667)	-32.51392	69.02784	71.29883	$\hat{\alpha} = 0.80774$ $\hat{\beta} = 1.39150$
Gamma	0.1379 (0.7237)	-32.75918	69.51835	71.78934	$\hat{\alpha} = 0.74588$ $\hat{\beta} = 2.11544$
Rayleigh	0.48281 (0.0000)	-56.46189	114.9238	116.0593	$\hat{\theta} = 0.40639$
Lindley	0.24406 (0.1084)	-35.30539	72.6108	73.7463	$\hat{l} = 0.9575$
Bilal	0.32757 (0.01066)	-42.51443	87.0289	88.1644	$\hat{b} = 1.8505$

According to Tables 3, 4, 5, 6, and 7, the p-values of the KS goodness-of-fit test for the EMHT distribution are greater than the significance level of 0.05 for all datasets. This indicates that all datasets are convenient with the EMHT distribution. Furthermore, the EMHT distribution is the most suitable, as it produces the highest value of the log-likelihood function at its maximum and also the lowest AIC and BIC values, particularly for datasets 2, 3, 4 and 5. Although the EMHT distribution is not the most appropriate model for dataset 1, its performance is able to compete with Gamma and Weibull distributions as the differences between these three distributions in all measures are very small.

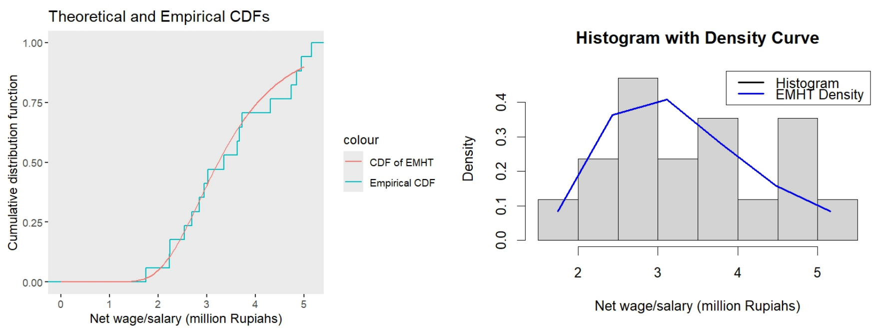

Figure 12. The fitted CDF and PDF of the EMHT distribution and empirical CDF for dataset 1.

Figures 12, 13, 14, 15, and 16 display the fitted CDF curves of the EMHT distribution alongside the empirical CDFs, as well as the corresponding fitted PDFs for each dataset. For all datasets, the theoretical CDFs closely align with the empirical CDFs. Additionally, the fitted PDFs closely resemble the histograms of the datasets. These results strongly support the goodness-of-fit results, indicating that the EMHT distribution provides an appropriate model for all five datasets. This confirms the EMHT distribution as a viable and legit option for modeling datasets 1 through 5.

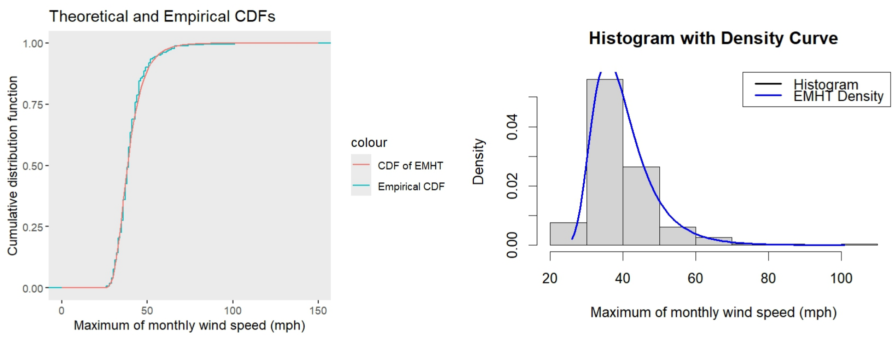

FIGURE 13. The fitted CDF and PDF of the EMHT distribution and empirical CDF for dataset 2.

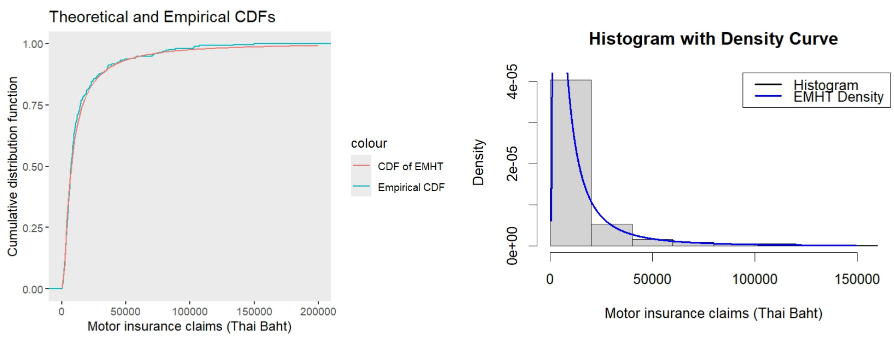

FIGURE 14. The fitted CDF and PDF of the EMHT distribution and empirical CDF for dataset 3.

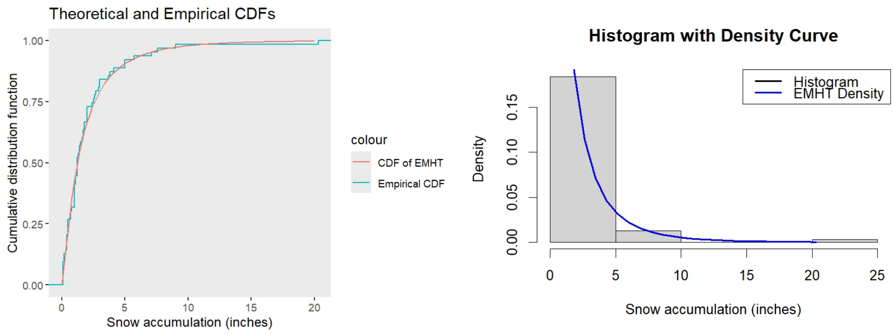

FIGURE 15. The fitted CDF and PDF of the EMHT distribution and empirical CDF for dataset 4.

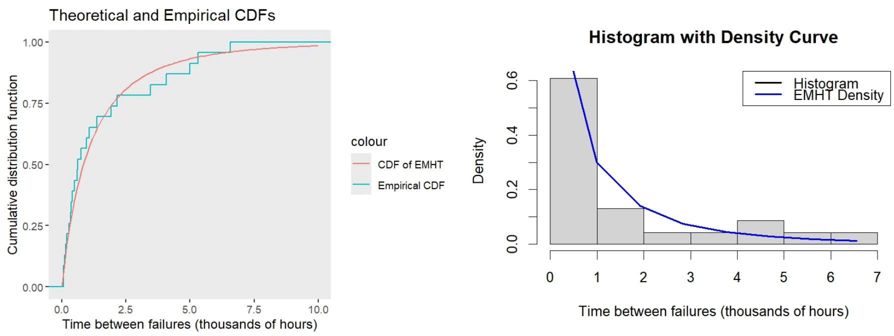

FIGURE 16. The fitted CDF and PDF of the EMHT distribution and empirical CDF for dataset 5.

10. CONCLUSION

This research proposes the novel EMHT distribution, constructed by powering a positive real number to the CDF of the MHT distribution. As confirmed by its hazard rate function, the EMHT distribution offers flexibility, as it can generate data with monotone and non-monotone hazard rates. This study also derived essential probability properties of the proposed distribution, including its CDF and PDF. The mean, variance, and some other statistical measures can be calculated using mathematical software despite their closed-form is not being available.

Parameter estimation for the EMHT distribution is performed using MLE with a numerical method in some cases because the fitted parameter cannot be obtained analytically. The performance of the MLE is examined by conducting Monte Carlo simulations with various parameter values and sample sizes. The simulation results show that the estimated parameters are very close to the actual parameter values, particularly for large sample sizes.

Furthermore, the EMHT distribution compares favorably with some classic non-negative continuous distributions, and exponentiated Weibull distribution. The compatibility of the EMHT distribution with datasets 1, 2, 3, 4, and 5 is confirmed by the p-values of the Kolmogorov-Smirnov goodness-of-fit test. The EMHT distribution also provides the smallest AIC and BIC values for datasets 2, 3, 4, and 5. Therefore, the EMHT distribution can be a promising alternative to other existing distributions for modeling non-negative continuous data.

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