GENERALIZED STUMMEL CLASS AND MORREY SPACES OF NONHOMOGENEOUS TYPE

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Abstract. In the context of the spaces of nonhomogeneous type, in this paper we study a relation between the generalized Stummel class and the generalized Morrey spaces. The stummel class is a class of functions related to local behavior of mapping by fractional integral operators. Meanwhile, the generalized Morrey spaces are classes of functions related to local behavior of Hardy-Littlewood maximal function. Our results employ the doubling condition of functions under consideration.

Key words: Generalized Stummel class, generalized Morrey spaces, spaces of nonhomogeneous type.

Abstrak. Dalam konteks ruang bertipe tak homogen, pada makalah ini dibahas hubungan antara kelas Stummel yang diperumum dan ruang Morrey yang diperumum. Kelas Stummel yang diperumum adalah kelas fungsi yang berkaitan dengan sifat lokal pemetaan oleh operator integral, sedangkan ruang Morrey yang diperumum adalah kelas fungsi yang berkaitan dengan sifat lokal dari fungsi maksimal Hardy-Littlewood. Hasil pada makalah ini diperoleh dengan memanfaatkan kondisi *doubling* dari fungsi-fungsi yang digunakan.

Kata kunci: Kelas Stummel yang diperumum, ruang Morrey yang diperumum, ruang bertipe tak homogen.

1. Introduction

Let $B(x, r)$ be a ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$; and for $k > 0$, let us set $B(x, kr)$ as the ball concentric to $B(x, r)$ with radius kr. The Euclidean space $\mathbb{R}^d$, endowed with a positive Radon measure μ, is a homogeneous space if μ satisfies the doubling condition, that is there exists a constant $C > 0$ such that for every

ball $B(x, r)$ we have

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

We refer to [1] for a pioneer work on homogeneous spaces. The space $\mathbb{R}^d$ equipped with the usual Lebesgue measure is an example of homogeneous space.

Since 1970's, the doubling condition has been an important assumption in developing some theories in harmonic analysis. However, some recent results, see [4, 7, 8, 12, 13], show that doubling condition assumption is not as significant as thought before, that is some results can be obtained without doubling condition. The space $(\mathbb{R}^d, \mu)$ where the Radon measure μ does not satisfy the doubling condition is known as the nonhomogeneous space. In this paper, we will particularly work with μ which satisfies *the growth condition*:

$$\mu(B(x, r)) \leq C^* r^n \tag{1}$$

for every ball $B(x, r)$. In the growth condition (1), n is a fixed real number such that $0 < n \leq d$ and the constant C^* is independent of x and r. Without loss of generality, we will assume that $C^* \geq 1$.

In the spaces of homogeneous type, Eridani and Gunawan [2] introduced the generalized Stummel class $S_\psi := S_\psi(\mathbb{R}^d)$, for $\psi : (0, \infty) \to (0, \infty)$, which is defined by

$$S_\psi := \{f \in L^1_{\text{loc}}(\mu) : \lim_{r \to 0} \zeta_\psi f(r) = 0\}.$$

Here,

$$\zeta_\psi f(r) := \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{|x-y|^n} \, dy$$

is called the Stummel modulus of f. The class S_ψ is a generalization of the Stummel class $S_p := S_p(\mathbb{R}^d)$ (for $1 < p < d$) where

$$S_p := \left\{ f \in L^1_{\text{loc}}(\mu) : \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{|f(y)|)}{|x-y|^{d-p}} \, dy = 0 \right\}.$$

This class was introduced by Ragusa and Zamboni [9]. If $\psi(t) = t^p$ and $1 < p < d$, where d is the dimension of the Euclidean space $\mathbb{R}^d$, then we have $S_\psi = S_p$; meanwhile if $\psi(t) = t^2$, the class S_ψ is reduced to the classical Stummel-Kato class. The relevance of the classical Stummel-Kato class in the study of Schrödinger operators can be found for example in [14]. For homogeneous type spaces, Ragusa and Zamboni [9] verified a relation between the Morrey spaces and the Stummel class S_p; meanwhile Eridani and Gunawan [2] as well as Gunawan *et al.* [6] established relations between the generalized Morrey spaces and the generalized Stummel class S_ψ (of homogeneous type spaces). The Stummel class is a class of functions related to local behavior of mapping by generalized fractional integral operators; and the generalized Morrey spaces are classes of functions related to local behavior of Hardy-Littlewood maximal function. We may see [5, 10] for recent results in fractional integral operators and maximal operators. In this paper, we extend the results of [2] and [6] in nonhomogeneous setting.

2. Main Results

For $\psi : (0, \infty) \to (0, \infty)$, we define the generalized Stummel class $S_\psi(\mu) := S_\psi(\mathbb{R}^d, \mu)$ of nonhomogeneous type by

$$S_\psi(\mu) := \{f \in L^1_{\mathrm{loc}}(\mu) : \lim_{r \to 0} \eta_\psi f(r) = 0\}.$$

Here, the nondecreasing function

$$\eta_\psi f(r) := \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{|x-y|^n} \, d\mu(y), \qquad 0 < n \le d, \tag{2}$$

is the Stummel modulus of f.

Now, for $\phi : (0, \infty) \to (0, \infty)$ and $k > 1$, we define the generalized Morrey space of nonhomogeneous type $\mathcal{M}^{1,\phi}(k, \mu) = \mathcal{M}^{1,\phi}(\mathbb{R}^d, k, \mu)$ to be the space of all functions $f \in L^1_{\mathrm{loc}}(\mu)$ for which

$$\|f : \mathcal{M}^{1,\phi}(k, \mu)\| := \sup_{B(x,r)} \frac{1}{\phi(\mu(B(x, kr)))} \frac{1}{\mu(B(x, kr))} \int_{B(x,r)} |f(y)| \, d\mu(y) < \infty.$$

We may refer [3] for definition of the analog spaces in the homogeneous setting.

Throughout this paper, for $t > 0$, we will always assume that $\phi(t)$ is an almost decreasing function and $t\phi(t)$ is an almost increasing function. This also means that we always assume *the doubling condition* on ϕ. The function ϕ is doubling if there exists a constant $C > 0$ such that

$$\frac{1}{C} \le \frac{\phi(s)}{\phi(t)} \le C \quad \text{whenever} \quad 1 \le \frac{s}{t} \le 2. \tag{3}$$

Unless otherwisely stated, for the rest of the paper, we denote by C a positive constant which may vary from line to line. Recall that ϕ is an almost decreasing [increasing] function if there exists C such that $\phi(s) \ge C\phi(t)$ $[\phi(s) \le C\phi(t)]$ whenever $s < t$. Moreover, our generalized Morrey spaces are independent of k since for $k_1 > k_2 > 1$ we have

$$C_1 \|f : \mathcal{M}^{1,\phi}(k_1, \mu)\| \le \|f : \mathcal{M}^{1,\phi}(k_2, \mu)\| \le C_2 \|f : \mathcal{M}^{1,\phi}(k_1, \mu)\|$$

where $C_1, C_2 > 0$.

Our first result states the inclusion of the generalized Morrey spaces into the Stummel class.

Theorem 2.1. *Assume that $t^{-n}\psi(t)$ is almost decreasing. If*

$$\int_0^1 \frac{\psi(t)\phi(t^n)}{t} \, dt < \infty,$$

then $\mathcal{M}^{1,\phi}(k, \mu) \subseteq S_\psi(\mu)$.

Proof. Let $f \in \mathcal{M}^{1,\phi}(2, \mu)$. (This suffices as our generalized Morrey spaces $\mathcal{M}^{1,\phi}(k, \mu)$ are independent of k.) By using the almost decreasing of $t^{-n}\psi(t)$ and

the doubling condition of ϕ, we have

$$\int_{2^j r \le |x-y| < 2^{j+1} r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n} \, d\mu(y)$$

$$\le C \frac{\psi(2^j r)}{(2^j r)^n} \int_{2^j r \le |x-y| < 2^{j+1} r} |f(y)| \, d\mu(y)$$

$$\le C \frac{\psi(2^j r)}{(2^j r)^n} \phi(\mu(B(x, 2^{j+2} r))) \mu(B(x, 2^{j+2} r)) \|f : \mathcal{M}^{1,\phi}(2, \mu)\|$$

$$\le C \frac{\psi(2^j r)}{(2^j r)^n} \phi(C^*(2^{j+2} r)^n)(2^{j+2} r)^n \|f : \mathcal{M}^{1,\phi}(2, \mu)\|$$

$$\le C \int_{2^{j-1} r}^{2^j r} \frac{\psi(t)}{t^n} \phi(t^n) t^n \frac{dt}{t} \|f : \mathcal{M}^{1,\phi}(2, \mu)\|.$$

Then

$$\int_{|x-y| < r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n} \, d\mu(y)$$

$$= \sum_{j=-\infty}^{-1} \int_{2^j r \le |x-y| < 2^{j+1} r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n} \, d\mu(y)$$

$$\le C \sum_{j=-\infty}^{-1} \int_{2^{j-1} r}^{2^j r} \frac{\psi(t)}{t^n} \phi(t^n) t^n \frac{dt}{t} \|f : \mathcal{M}^{1,\phi}(2, \mu)\|$$

$$= C \|f : \mathcal{M}^{1,\phi}(2, \mu)\| \int_0^{r/2} \frac{\psi(t)\phi(t^n)}{t} \, dt.$$

Since $\int_0^1 \frac{\psi(t)\phi(t^n)}{t} \, dt < \infty$, we obtain $\int_0^{r/2} \frac{\psi(t)\phi(t^n)}{t} \, dt \to 0$ whenever $r \to 0$. This proves our assertion.∎

Our next result is the inclusion of the Stummel class into the generalized Morrey spaces. We will present it in the following theorem.

Theorem 2.2. *Assume that $t^{-n}\psi(t)$ is almost decreasing. If $f \in S_\psi(\mu)$ and there exists a positif constant C such that*

$$\int_0^r \frac{t^{n-1} \eta_\psi f(t)}{\psi(t)} \, dt \le C\mu(B(x, r))\phi(\mu(B(x, r)))$$

for all ball $B(x, r)$, then $f \in \mathcal{M}^{1,\phi}(k, \mu)$.

PROOF. Given ball $B(x, r)$ where $x \in \mathbb{R}^d$ and $r > 0$ and for $f \in S_\psi(\mu)$, we have

$$\int_{|x-y|<r} |f(y)| \, d\mu(y) = \sum_{j=-\infty}^{-1} \int_{2^j r \le |x-y| < 2^{j+1} r} |f(y)| \, d\mu(y)$$

$$\le C \sum_{j=-\infty}^{-1} \frac{(2^{j+1} r)^n}{\psi(2^{j+1} r)} \int_{|x-y| < 2^{j+1} r} \frac{|f(y)| \psi(|x-y|)}{|x-y|^n} \, d\mu(y)$$

$$\le C \sum_{j=-\infty}^{-1} \frac{(2^{j+1} r)^n \eta_\psi f(2^{j+1} r)}{\psi(2^{j+1} r)}$$

$$= C \sum_{j=-\infty}^{0} \frac{(2^j r)^n \eta_\psi f(2^j r)}{\psi(2^j r)}.$$

Now, we apply the almost decreasing of $t^{-n} \psi(t)$ to get

$$\int_{|x-y|<r} |f(y)| \, d\mu(y) \le C \sum_{j=-\infty}^{0} \frac{(2^j r)^n \eta_\psi f(2^j r)}{\psi(2^j r)}$$

$$\le C \sum_{j=-\infty}^{0} \int_{2^j r}^{2^{j+1} r} \frac{t^{n-1} \eta_\psi f(t)}{\psi(t)} \, dt$$

$$= C \int_0^{2r} \frac{t^{n-1} \eta_\psi f(t)}{\psi(t)} \, dt$$

$$\le C \mu(B(x, 2r)) \phi(\mu(B(x, 2r)).$$

The last inequality lead us to $f \in \mathcal{M}^{1,\phi}(2, \mu)$. Therefore, the desired result follows. ∎

3. FURTHER RESULTS

For $\psi : \mathbb{R}^d \times (0, \infty) \to (0, \infty)$, define $S_\psi(\mu)$ by the same way in Section 2 with

$$\eta_\psi f(r) := \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} |f(y)| \frac{\psi(x, |x-y|)}{|x-y|^n} \, d\mu(y),$$

see also Section 4 in [6]. For $\phi : \mathbb{R}^d \times (0, \infty) \to (0, \infty)$, let $\mathcal{M}^{1,\phi}(k, \mu)$ be the space of all functions f such that

$$\|f : \mathcal{M}^{1,\phi}(k, \mu)\| := \sup_{B(x,r)} \frac{1}{\phi(x, \mu(B(x, kr)))} \frac{1}{\mu(B(x, kr))} \int_{B(x,r)} |f(y)| \, d\mu(y) < \infty.$$

We always assume that there exists a positive constant C such that

$$\phi(x, t) \le C\phi(x, s), \quad s\phi(x, s) \le Ct\phi(x, t)$$

whenever $x \in \mathbb{R}^d, 0 < s \le t < \infty$. Then, we can prove the following by the same way as Theorems 2.1 and 2.2:

Theorem 3.1. *Assume that there exists a positive constant C such that*

$$t^{-n}\psi(x,t) \leq Cs^{-n}\psi(x,s)$$

whenever $x \in \mathbb{R}^d, 0 < s \leq t < \infty$. If

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_0^r \frac{\psi(x,t)\phi(x,t^n)}{t}\, dt = 0,$$

then $\mathcal{M}^{1,\phi}(k,\mu) \subseteq S_\psi(\mu)$. Conversely, if $f \in S_\psi(\mu)$ and there exists a positif constant C such that

$$\int_0^r \frac{t^{n-1}\eta_\psi f(t)}{\psi(x,t)}\, dt \leq C\mu(B(x,r))\phi(x,\mu(B(x,r)))$$

for all ball $B(x,r)$, then $f \in \mathcal{M}^{1,\phi}(k,\mu)$.

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