J. Indones. Math. Soc.
Vol. 31, No. 2 (2025), pp. 1–8.

The Zero Product Probability of Some Finite Ring of Matrices Based on the Order of the Annihilator

Nurhidayah Zaid[1*], Nor Haniza Sarmin[2,3], and Sanhan Muhammad Salih Khasraw[4]

[1]Department of Mathematical Sciences Studies, Universiti Teknologi MARA, Malaysia.
[2]Department of Mathematical Sciences, Universiti Teknologi Malaysia, Malaysia.
[3]Department of Mathematics, Universitas Airlangga, Indonesia.
[4]Department of Mathematics, College of Education, Salahaddin University-Erbil, Iraq.
nurhidayahzaid@uitm.edu.my[1], nhs@utm.my[2], sanhan.khasraw@su.edu.krd[4]

Abstract. An annihilator is defined as the set of pairs of elements in a ring R in which the product of the elements in the pair is the zero element of R. In this paper, we aim to determine the order of the annihilator for the finite ring of matrices of dimension two over integers modulo prime, $M_2(\mathbb{Z}_p)$. Furthermore, the zero product probability of a finite ring is also computed. The zero product probability is the probability that two elements of a finite ring have product zero. Based on the order of the annihilator, the general formula of the zero product probability of $M_2(\mathbb{Z}_p)$ is determined.

Key words and phrases: annihilator of a ring, ring of matrices, noncommutative ring, probability in rings.

1. INTRODUCTION

For over five decades, probability theory has been a topic widely investigated by various researchers in the field of algebra, including group theorists. This topic is studied to describe various characteristics of finite groups and give knowledge on the structure of its elements. One of the well-known examples is the commutativity degree of a finite group, which is the probability that two random elements selected from a finite group commute. Erdos and Turan introduced this concept [1], where the authors worked on determining the abelianness of symmetric groups. The idea has now sparked the interest of the ring theorists, where it is studied on some finite rings. To begin, MacHale [2] determined the probability that two random elements of a ring commute for noncommutative rings and found several bounds related to

*Corresponding author
2020 Mathematics Subject Classification: 13A70, 15B33, 60B20
Received: 17-10-2024, accepted: 13-04-2025.

the subject. The probability is written as $P(R) = \frac{\sum_{r \in R} |C_R(r)|}{|R|^2}$, where $C_R(r)$ is the subring $\{x \in R | xr = rx\}$.

Many studies have been done on the commuting probability of finite rings, e.g., see [3, 4, 5, 6]. Afterwards, researchers started to study the probability of the product of elements in a finite ring. For example, Rehman et al. [7] studied the probability of product in finite commutative rings, specifically the ring of integers $\mathbb{Z}_n$, where the aim was to obtain a desirable product in $\mathbb{Z}_n$. The probability is mathematically written as

$$P_{\overline{m}}(\mathbb{Z}_n) = \frac{|\{(\overline{x}, \overline{y}) \in \mathbb{Z}_n \times \mathbb{Z}_n | \overline{x} \cdot \overline{y} = \overline{m}\}|}{|\mathbb{Z}_n \times \mathbb{Z}_n|},$$

where $\overline{x}, \overline{y}$ and $\overline{m}$ are the elements in $\mathbb{Z}_n$.

Khasraw introduced another probability associated with the product of ring elements in [8]. The author defined the probability that the product of two randomly chosen elements in a finite ring is zero. The study was done on finite commutative rings with identity. Afterward, several researchers studied the probability that two elements of a finite ring have product zero in specific rings, including Mohammed Salih [9], who studied the probability of finite group rings. Besides that, Dolžan [10] investigated the probability that two elements of a finite ring have product zero in semisimple rings. In addition, Zai et al. [11] extended the probability that two elements of a finite ring have product zero, focusing on the noncommutative ring, specifically on the ring $\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_2)$. The authors in [11] officially named the probability as the zero product probability of noncommutative rings. Since the zero product probability is a new notion in rings, not many types of noncommutative rings have been explored. Hence, this paper studies the zero product probability of the ring of matrices, which is noncommutative.

This paper investigates the zero product probability for the ring of 2×2 matrices over integers modulo prime p. From this point on, the ring is denoted by $M_2(\mathbb{Z}_p)$. To obtain the zero product probability, the order of the annihilator of $M_2(\mathbb{Z}_p)$ is first determined by using the properties of the determinant of matrices. The results in this paper primarily contribute to the theoretical results in ring theory, especially on the annihilators of a ring, which have not yet been found in the existing literature.

This paper is structured into four main sections. The first section serves as an introduction to the study. The second section presents the definitions and concepts used in this study. The third section provides the results obtained in this study. Finally, the fourth section concludes the study.

2. PRELIMINARIES

This section provides the definitions and concepts used in this study. One of the crucial terms used in this study is the zero-divisors of a finite ring. The zero divisor of a finite ring R is defined as a nonzero element x in R such that there exists another nonzero element y in R, where the product of x and y is equal to zero [12].

Remarkably, McCoy [13] discovered a technique for identifying the zero-divisors of the finite ring of matrices by using the determinants of the matrices. The finding is given in the following proposition.

Proposition 2.1. [13] *Let A be a given element of the ring of $n \times n$ matrices, M_n with elements in the commutative ring R. Then A is a zero divisor of M_n if and only if the determinant of A is a divisor of zero in R.*

In other words, the author stated that a matrix A in a ring of matrices M_n over integers modulo m is a zero divisor if and only if its determinant is zero $(\text{mod } m)$ [13]. Subsequently, some basic properties of a matrix with zero determinant are given in the following theorem.

Theorem 2.2. [14] *Let A be a square matrix. Then, the determinant of A, $det(A) = 0$ if:*

(1) all elements of one of the rows or columns of A are zero.
(2) two parallel rows or columns of A are equal.
(3) two parallel lines of A are proportional.

Subsequently, another important term used in this study is the annihilator of a finite ring. Khasraw [8] considered the annihilator of a finite commutative ring R as a set of pairs of elements in R, in which the product of the elements in each pair is zero. In other words, the zero-divisors of R are paired and written in a set called the annihilator. Following that, Zaid et al. [15] extended the definition of the annihilator to finite noncommutative rings. The definition is given as follows:

Definition 2.3. [15] *Let R be a noncommutative ring. Then, the annihilator of R is the set of ordered pairs $(x, y) \in R \times R$ such that $xy = 0$. The set is mathematically written as $Ann(R) = \{(x, y) \in R \times R | xy = 0\}$.*

The order of the annihilator of a ring, denoted by $|Ann(R)|$, is the number of elements in the set. In addition to the annihilator, another notion in the focus of this paper is the zero product probability. The zero product probability of noncommutative rings was introduced by Zai et al. [11], intending to study the zero product attributes of noncommutative rings. The definition of the zero product probability of noncommutative rings is given as follows:

Definition 2.4. [11] *Let R be a noncommutative ring. Then, the zero product probability of R is*

$$P(R) = \frac{|\{(x, y) \in R \times R | xy = 0\}|}{|R \times R|}.$$

One important method used to obtain the main results is the linear Diophantine method, which involves two variables. The method is explained in the following theorem.

Theorem 2.5. [16] *The linear Diophantine equation $ax + by = c$ has a solution if and only if $d|c$ where d is the greatest common divisor of a and b, i.e.,* $\gcd(a, b) = d$*. If x_0 and y_0 are any particular solution of the equation, then all other solutions of the equation are given by*

$$x = x_0 + \left(\frac{b}{d}\right) t, \; y = y_0 - \left(\frac{a}{d}\right) t,$$

where t is an arbitrary integer.

The following section discusses the main results found in this study, which includes the order of the annihilator of $M_2(\mathbb{Z}_p)$, which then leads to the general formula for the zero product probability of $M_2(\mathbb{Z}_p)$.

3. MAIN RESULTS

In this section, the main results of this study are presented. First, the order of the annihilator is determined for $M_2(\mathbb{Z}_p)$ by using its definition. The order of the annihilator of $M_2(\mathbb{Z}_p)$ is given in the following proposition.

Proposition 3.1. *Given $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_p \right\}$ is a noncommutative ring and $R = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_p - \{0\} \right\}$ is a subset of S. Then, the order of the annihilator of R, $|Ann(R)| = p^5 - 2p^4 - 2p^3 + 8p^2 - 7p + 2$.*

Proof. Given $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_p \right\}$ and a subset of S, $R = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_p - \{0\} \right\}$. The elements $X \in R$ and $Y \in S$ are determined using the following matrix multiplication.

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \pmod{p}.$$

The matrix multiplication leads to the following system.

$$x_1y_1 + x_2y_3 \equiv 0 \pmod{p} \tag{1}$$
$$x_1y_2 + x_2y_4 \equiv 0 \pmod{p} \tag{2}$$
$$x_3y_1 + x_4y_3 \equiv 0 \pmod{p} \tag{3}$$
$$x_3y_2 + x_4y_4 \equiv 0 \pmod{p}. \tag{4}$$

First, the number of possible elements of X is determined. The calculations are divided into two cases:

(Case 1) When X is a zero divisor.

(Case 2) When X is not a zero divisor.

Case 1: When X is a zero divisor.
If X is a zero divisor of R, based on Proposition 2.1, the determinant of X, $\det(X) = 0$. According to the properties of determinant as stated in Theorem 2.2, there are two conditions where $\det(X) = 0$, which are:

(1) when two rows (or columns) of X are identical; and
(2) when the rows of X are proportional.

For the first condition, the number of possible elements in X with identical rows is $(p-1)(p-1) = p^2 - 2p + 1$ since there are $p-1$ possible nonzero values for elements x_1 and x_2 in the first row. In contrast, the second row has the same element as the first, i.e., $x_1 = x_3$ and $x_2 = x_4$. The same goes for elements X with identical columns. However, $p-1$ elements are removed from the total since the elements which $x_1 = x_2 = x_3 = x_4$ have been considered in the situation where X has identical rows. Therefore, for this condition, $|X| = \big(p^2 - 2p + 1\big) + \big(p^2 - 2p + 1\big) - (p-1) = 2p^2 - 5p + 3$.

Next, for the second condition, the number of possible elements in X with proportional rows, $X = \begin{bmatrix} x_1 & x_2 \\ kx_1 & kx_2 \end{bmatrix} \in R$, where $k \in \mathbb{Z}_p - \{0, 1\}$ is determined. Since all entries must be nonzero, x_1 has $p-1$ possible values. Then, there are $p-2$ possible values for x_2 since $x_1 \neq x_2$ so that both columns are not identical. For the second row, there are $p-2$ possible multiples since $k \in \mathbb{Z}_p - \{0, 1\}$. Hence, $|X| = (p-1)(p-2)(p-2) = p^3 - 5p^2 + 8p - 4$.

Combining both cases, then the total number of possible elements of X when X is a zero divisor in R is $|X| = \big(2p^2 - 5p + 3\big) + \big(p^3 - 5p^2 + 8p - 4\big) = p^3 - 3p^2 + 3p - 1$.

The next calculations are focused in solving for $Y \in S$ where $XY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Only the system formed by congruence (1) and congruence (2) is solved since they are in similar forms as congruence (3) and congruence (4). Congruence (1) and (2) are solved by using linear Diophantine method.

Congruence (1) has a solution if and only if its greatest common divisor, $\gcd(x_1, x_2) = q$ and $q|0$, where q is any positive integer. Since all integers divide 0, thus this argument is true, and congruence (1) has solutions, where the solutions are given by:

$$\begin{aligned} y_1 &= {y_1}' + (\frac{x_2}{q})t \\ y_3 &= {y_3}' - (\frac{x_1}{q})t. \end{aligned}$$

It is found that regardless of the value of $q \in \{1, 2, ..., p-1\}$, y_3 is uniquely dependent on the value of y_1 since $y_3 - {y_3}' = -\frac{x_1}{x_2}(y_1 - {y_1}')$. Therefore, there are p possible solutions for y_1 and y_3 in congruence (1). Similarly, there are p solutions for y_2 and y_4 in congruence (2). This indicates that for $Y \in S$, $|Y| = p \times p = p^2$.

Case 2: When X is not a zero divisor.
Meanwhile, if X is not a zero divisor of R, the number of possible elements X is $(p-1)^4 - \left(p^3 - 3p^2 + 3p - 1\right) = p^4 - 5p^3 + 9p^2 - 7p + 2$. Since X is not a zero divisor, the only $Y \in S$ where the product $XY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero matrix itself.

Therefore by combining both cases, the order of the annihilator,

$$\begin{aligned} |Ann(R)| &= \left[\left(p^3 - 3p^2 + 3p - 1\right) \times p^2\right] + \left[\left(p^4 - 5p^3 + 9p^2 - 7p + 2\right) \times 1\right] \\ &= p^5 - 2p^4 - 2p^3 + 8p^2 - 7p + 2. \end{aligned}$$

□

The following example presents the order of the annihilator of the ring R when $p = 2$.

Example 3.2. *Given a ring* $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_2 \right\}$ *and a subset of* S, $R = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_2 - \{0\} \right\}$. *Based on Definition 2.3, the annihilator of* R *is given in the following set.*

$$\begin{aligned} Ann(R) = \Bigg\{ & \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right), \\ & \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \Bigg\}. \end{aligned}$$

Therefore, $|Ann(R)| = 4$. *The result is consistent with Proposition 3.1, where the order of the annihilator of* R, $|Ann(R)| = (2)^5 - 2(2)^4 - 2(2)^3 + 8(2)^2 - 7(2) + 2 = 4$.

Next, the following theorem gives the zero product probability of the ring R by using Definition 2.4 and the results found in Proposition 3.1.

Theorem 3.3. *Given a ring* $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_p \right\}$ *and the ring* $R = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_p - \{0\} \right\}$ *is a subset of* S. *Then, the zero product probability of* R, $P(R) = \frac{2}{p^8} - \frac{7}{p^7} + \frac{8}{p^6} - \frac{2}{p^5} - \frac{2}{p^4} + \frac{1}{p^3}$.

Proof. Based on Proposition 3.1, the number of the annihilators of R, $Ann(R) = p^5 - 2p^4 - 2p^3 + 8p^2 - 7p + 2$. Therefore, the zero product probability of R,

$$\begin{aligned} P(R) &= \frac{|Ann(R)|}{|S|^2} = \frac{p^5 - 2p^4 - 2p^3 + 8p^2 - 7p + 2}{\left(p^4\right)^2} \\ &= \frac{2}{p^8} - \frac{7}{p^7} + \frac{8}{p^6} - \frac{2}{p^5} - \frac{2}{p^4} + \frac{1}{p^3}. \end{aligned}$$

□

4. CONCLUDING REMARKS

In this paper, the general formula is established for the zero product probability of the ring of 2×2 matrices over integers modulo prime, $M_2(\mathbb{Z}_p)$. To formulate the general formula of the probability, the general formula of the order of its annihilator is first formed. It is found that the general formula of the annihilator of $M_2(\mathbb{Z}_p)$ depends on the value of p. In addition, for future studies, the order of the annihilator can be studied on matrices over all integers, not only limited to the prime integers.

Acknowledgement. The first author would like to acknowledge the support of Universiti Teknologi MARA (UiTM) for providing the resources and facilities necessary to complete this study.

REFERENCES

[1] P. Erdos and P. Turan, "On some problem of a statistical group theory iv," *Acta Mathematica Academiae Scientiaruin Hungaricae*, vol. 19, no. 3-4, pp. 413–435, 1968. https://doi.org/10.1007/BF00536750.

[2] D. MacHale, "Commutativity in finite rings," *The American Mathematical Monthly*, vol. 83, no. 1, pp. 30–32, 1976. https://doi.org/10.1080/00029890.1976.11994032.

[3] S. M. Buckley and D. MacHale, "Commuting probability for subrings and quotient rings," *Journal of Algebra Combinatorics Discrete Structures and Applications*, vol. 4, no. 2, pp. 189–196, 2016. https://dergipark.org.tr/en/download/article-file/267255.

[4] J. Dutta, D. K. Basnet, and R. K. Nath, "On commuting probability of finite rings," *Indagationes Mathematicae*, vol. 28, pp. 372–382, 2017. https://doi.org/10.1016/j.indag.2016.10.002.

[5] P. Dutta and R. K. Nath, "On relative commuting probability of finite rings," *Miskolc Mathematical Notes*, vol. 20, no. 1, pp. 225–232, 2019. https://doi.org/10.48550/arXiv.1708.05142.

[6] P. Dutta and R. K. Nath, "A generalization of commuting probability of finite rings," *Asian-European Journal of Mathematics*, vol. 11, no. 2, pp. 1850023(1–15), 2018. https://doi.org/10.1142/S1793557118500237.

[7] S. Rehman, A. Q. Baig, and K. Haider, "A probabilistic approach toward finite commutative rings," *Southeast Asian Bulletin of Mathematics*, vol. 43, p. 413–418, 2019.

[8] S. M. S. Khasraw, "What is the probability that two elements of a finite ring have product zero?," *Malaysian Journal of Fundamental and Applied Sciences*, vol. 16, no. 4, p. 497–499, 2020. https://doi.org/10.48550/arXiv.1810.13312.

[9] H. M. Mohammed Salih, "On the probability of zero divisor elements in group rings," *International Journal of Group Theory*, vol. 11, no. 4, p. 253–257, 2022. https://ijgt.ui.ac.ir/article_26054.html.

[10] D. Dolžan, "The probability of zero multiplication in finite rings," *Bulletin of the Australian Mathematical Society*, vol. 106, no. 1, p. 83–88, 2022. https://doi.org/10.1017/S0004972721001246.

[11] N. A. F. O. Zai, N. H. Sarmin, and N. Zaid, "The zero product probability of some finite rings," *Menemui Matematik*, vol. 42, no. 2, p. 51–58, 2020. https://myjms.mohe.gov.my/index.php/dismath/article/download/13044/6742.

[12] J. Fraleigh and V. Katz, *A First Course in Abstract Algebra (7th Edition)*. Addison-Wesley, 2003.

[13] N. H. McCoy, "Remarks on divisors of zero," *The American Mathematical Monthly*, vol. 49, no. 5, p. 286–295, 1942. https://doi.org/10.1080/00029890.1942.11991226.

[14] F. E. Hohn, *Elementary Matrix Algebra*. Collier-Macmillan, 1973.

[15] N. Zaid, N. H. Sarmin, and S. M. S. Khasraw, "The zero product probability of ring of matrices based on euler's phi-function," *Kongzhi yu Juece/Control and Decision*, vol. 38, no. 7, pp. 2319–2326, 2023.

[16] D. Burton, *Elementary Number Theory*. McGraw-Hill, 2002.