J. Indones. Math. Soc.
Vol. 20, No. 2 (2014), pp. 77–94.

ON WEAK CONVERGENCE OF THE PARTIAL SUMS PROCESSES OF THE LEAST SQUARES RESIDUALS OF MULTIVARIATE SPATIAL REGRESSION

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Abstract. A weak convergence of the sequence of partial sums processes of the residuals (PSPR) when the observations are obtained from a multivariate spatial linear regression model (SLRM) is established. The result is then applied in constructing the rejection region of an asymptotic test of hypothesis based on a type of Cramér-von Mises functional of the PSPR. When the null hypothesis is true, we get the limit process as a projection of the multivariate Brownian sheet, whereas under the alternative it is given by a signal plus multivariate noise model. Examples of the limit process under the null hypothesis are also studied.

Key words: Multivariate spatial linear regression model, multivariate Brownian sheet, partial sums process, least squares residual, model-check.

Abstrak. Suatu kekonvergenan lemah dari barisan proses jumlah parsial dari sisaan jika pengamatan diperoleh dari suatu model regresi linear spasial multivariat ditemukan. Hasilnya kemudian diterapkan pada pengonstruksian daerah penolakan dari suatu uji hipotesis secara pendekatan yang berbasis pada suatu tipe dari fungsional Cramér-von Mises dari proses jumlah parsial dari sisaan. Jika hipotesis nol berlaku, proses limit yang diperoleh merupakan proyeksi dari lembaran Brown multivariat, sedangkan dibawah alternatif proses limit diberikan oleh suatu model yang terdiri dari suatu sinyal ditambah pengganggu multivariat. Contoh-contoh proses limit di bawah hipotesis nol juga dipelajari.

Kata kunci: Model regresi linear spasial multivariat, lembaran Brown multivariate, proses jumlah parsial, sisaan kuadrat terkecil, pemeriksaan model.

2000 Mathematics Subject Classification: 60F05, 60G15, 62G08, 62J05, 62M30.
Received: 25-03-2013, revised: 30-01-2014, accepted: 17-02-2014.

1. Introduction

Asymptotic model check (change point check) for linear regressions based on the PSPR has been studied in many literatures [see e.g. MacNail [14, 15], Bischoff [6, 7, 8], and the references cited therein]. The correctness of the assumed models and the existence of a change in the regression function or in the parameters of the model were detected by means of the Kolmogorov, Kolmogorov-Smirnov, and the Cramé-von Mises functionals of the PSPR. In MacNeill [16] and Xie [22], the application of the PSPR has been extended to the problem of boundary detection for SLRM, whereas in Bischoff and Somayasa [9], and Somayasa [18] and [19], it has been investigated from the perspective of model check for the SLRM.

In the literatures mentioned above the attention was restricted to the univariate linear regression models only. However in the practice it is frequent to encounter a situation in which the responses consist of a simultaneous measurement of two or more correlated variables (multivariate observations). Because of such an inherent existence of the correlations within the response variables, more effort will be needed in establishing the limit process of the PSPR.

In this paper we investigate an asymptotic model-check for multivariate SLRM. To see the problem in detail let us consider a regression model

$$\mathbf{Y}(t,s) = \mathbf{g}(t,s) + \mathcal{E}(t,s),\ (t,s) \in \mathbf{E} := [a,b] \times [c,d] \subset \mathbb{R}^2,\ a < b \text{ and } c < d,$$

where $\mathbf{g} : (g^{(1)}, \ldots, g^{(p)})^\top : \mathbf{E} \mapsto \mathbb{R}^p$ is the true-unknown vector valued regression function, $\mathbf{Y} := (Y^{(1)}, \ldots, Y^{(p)})^\top$ is the p-dimensional vector of observations, and $\mathcal{E} := (\varepsilon^{(1)}, \ldots, \varepsilon^{(p)})^\top$ is the p-dimensional vector of random errors with $\mathbb{E}(\mathcal{E}) = \mathbf{0} \in \mathbb{R}^p$, and $Cov(\mathcal{E}) = \mathbf{\Sigma} := (\sigma_{ij})_{i=1,j=1}^{p,p}$ which is assumed to be unknown and positive definite, with $\sigma_{ij} := Cov(\varepsilon^{(i)}, \varepsilon^{(j)})$. Thereby $\mathbf{0}$ is the p-dimensional zero vector. In this paper the Euclidean vector $\mathbf{x}$ is considered as a column vector, while $\mathbf{x}^\top$ is its corresponding row vector. Furthermore we assume that for $i \in \{1, \ldots, p\}$, $g^{(i)}$ is a function of bounded variation on $\mathbf{E}$ in the sense of Vitali, i.e. $g^{(i)} \in BVV(\mathbf{E})$ [see Clarkson and Adams [11] for the definition of $BVV(\mathbf{E})$]. For $n \geq 1$, let Ξ_n be the experimental condition given by a regular lattice on $\mathbf{E}$. That is

$$\Xi_n := \left\{ (t_{n\ell}, s_{nk}) \in \mathbf{E} :\ t_{n\ell} := a + \frac{\ell}{n}(b-a),\ s_{nk} := c + \frac{k}{n}(d-c),\ 1 \leq \ell, k \leq n \right\}.$$

Then $\{\mathbf{Y}_{n\ell k} = (Y_{n\ell k}^{(1)}, \ldots, Y_{n\ell k}^{(p)})^\top := \mathbf{Y}(t_{n\ell}, s_{nk}) : 1 \leq \ell, k \leq n,\ n \geq 1\}$ is a sequence of p-dimensional vector of observations that satisfies the condition

$$\mathbf{Y}_{n\ell k} = \mathbf{g}(t_{n\ell}, s_{nk}) + \mathcal{E}_{n\ell k},\ 1 \leq \ell, k \leq n, \tag{1}$$

where $\{\mathcal{E}_{n\ell k} = (\varepsilon_{n\ell k}^{(1)}, \ldots, \varepsilon_{n\ell k}^{(p)})^\top := \mathcal{E}(t_{n\ell}, s_{nk}) : 1 \leq \ell, k \leq n,\ n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) p-dimensional vector of random errors with $\mathbb{E}(\mathcal{E}_{n\ell k}) = \mathbf{0}$ and $Cov(\mathcal{E}_{n\ell k}) = \mathbf{\Sigma}$.

For a function $h : \mathbf{E} \mapsto \mathbb{R}$, let $h(\Xi_n) := (h(t_{n\ell}, s_{nk}))_{k=1,\ell=1}^{n,n}$ is an $n \times n$ matrix whose entry in the k-th row and ℓ-th column is given by $h(t_{n\ell}, s_{nk})$. Let

$$\mathbf{Y}_{n\times n\times p} := \begin{pmatrix} Y^{(1)}(\Xi_n) \\ \vdots \\ Y^{(p)}(\Xi_n) \end{pmatrix}, \mathbf{g}_{n\times n\times p} := \begin{pmatrix} g^{(1)}(\Xi_n) \\ \vdots \\ g^{(p)}(\Xi_n) \end{pmatrix}, \mathcal{E}_{n\times n\times p} := \begin{pmatrix} \mathcal{E}^{(1)}(\Xi_n) \\ \vdots \\ \mathcal{E}^{(p)}(\Xi_n) \end{pmatrix}$$

be the $n \times n \times p$ dimensional array of observations, the $n \times n \times p$ dimensional array of the mean of $\mathbf{Y}_{n\times n\times p}$, and the $n \times n \times p$ dimensional array of random errors, respectively. Then Model (1) can also be written as

$$\mathbf{Y}_{n\times n\times p} = \mathbf{g}_{n\times n\times p} + \mathcal{E}_{n\times n\times p},\ n \geq 1.$$

It is clear by the construction that for every $n \geq 1$, $\mathcal{E}_{n\times n\times p}$ consists exactly of the set of i.i.d. random vectors $\{\mathcal{E}_{n\ell k} : 1 \leq \ell, k \leq n\}$ with $\mathbb{E}(\mathcal{E}_{n\ell k}) = \mathbf{0}$ and $Cov(\mathcal{E}_{n\ell k}) = \mathbf{\Sigma}$.

By the transformation $(x, y) \mapsto ((x-a)/(b-a), (y-c)/(d-c))$, the experimental condition Ξ_n in the experimental region $[a, b]\times[c, d]$ can always be converted to a regular lattice $\{(\ell/n, k/n) :\ 1 \leq \ell, k \leq n\}$ in the unit rectangle $[0, 1] \times [0, 1]$. Therefore, without loss of generality and also for technical reason we consider in this paper the space $\mathbf{E} = [0, 1]\times[0, 1] =: \mathbf{I}$, and $(t_{n\ell}, s_{nk}) = (\ell/n, k/n)$, $1 \leq \ell, k \leq n$, $n \geq 1$. As a comparison study the reader is referred to Somayasa [18] to see how the limit process of the PSPR of univariate spatial linear regression model was established without transforming the experimental region to the unit rectangle. It was shown therein that the limit process was obtained as a projection of the Brownian sheet on $\mathbf{E}$.

Let $f_1, \ldots, f_d : \mathbf{I} \mapsto \mathbb{R}$ be known real valued regression functions which are linearly independent as functions in $\mathcal{C}(\mathbf{I}) \cap BV_H(\mathbf{I})$, where $\mathcal{C}(\mathbf{I})$ is the space of continuous functions on $\mathbf{I}$, and $BV_H(\mathbf{I})$ is the space of functions of bounded variation on $\mathbf{I}$ in the sense of Hardy [see [11] for the definition of $BV_H(\mathbf{I})$]. As usual $\mathcal{C}(\mathbf{I})$ is endowed with the uniform topology. Furthermore, let $\mathbf{W} := [f_1, \ldots, f_d]$ be the linear subspace generated by $f_1, \ldots, f_d$. Note that $\mathbf{W} \subset L_2(\lambda, \mathbf{I})$, where $L_2(\lambda, \mathbf{I})$ is the space of square Lebesgue integrable function on $\mathbf{I}$. The model-check studied in this paper concerns with the hypotheses

$$H_0 : g^{(i)} \in \mathbf{W},\ \forall i \in \{1, \ldots, p\} \text{ versus } H_1 : g^{(i)} \notin \mathbf{W} \text{ for some } i \in \{1, \ldots, p\}.$$

Equivalently, the hypotheses can also be presented by

$$H_0 : \mathbf{g} \in \mathbf{W}^p\ vs.\ H_1 : \mathbf{g} \notin \mathbf{W}^p,$$

where $\mathbf{W}^p := \mathbf{W}\times\cdots\times\mathbf{W}$ is the product of p copies of $\mathbf{W}$. Clearly $\mathbf{W}^p \subset L_2^p(\lambda, \mathbf{I})$, where $L_2^p(\lambda, \mathbf{I})$ is the product of p copies of $L_2(\lambda, \mathbf{I})$. In this paper the topology in $L_2^p(\lambda, \mathbf{I})$ is induced by the inner product defined by

$$\langle \mathbf{w}, \mathbf{v}\rangle_{L_2^p(\lambda,\mathbf{I})} := \int_{\mathbf{I}} \mathbf{w}^\top \mathbf{v}\ d\lambda = \sum_{i=1}^{p} \left\langle w^{(i)}, v^{(i)} \right\rangle_{L_2(\lambda,\mathbf{I})},$$

for any $\mathbf{w} := (w^{(1)}, \ldots, w^{(p)})^\top$, and $\mathbf{v} = (v^{(1)}, \ldots, v^{(p)})^\top$ in $L_2^p(\lambda, \mathbf{I})$.

Let $\mathbf{W}_n := [f_1(\Xi_n), \ldots, f_d(\Xi_n)]$ be a linear subspace of $\mathbb{R}^{n\times n}$. The hypotheses described above can in practice be realized by testing

$$H_0 : \mathbf{g}_{n\times n\times p} \in \mathbf{W}_n^p \ \ vs. \ H_1 : \mathbf{g}_{n\times n\times p} \notin \mathbf{W}_n^p, \tag{2}$$

where $\mathbf{W}_n^p$ is the product of p copies of $\mathbf{W}_n$, which is furnished in this paper with the inner product defined by

$$\langle \mathbf{A}_{n\times n\times p}, \mathbf{B}_{n\times n\times p}\rangle_{\mathbf{W}_n^p} := \sum_{i=1}^{p} \langle \mathbf{A}_{n\times n}^{(i)}, \mathbf{B}_{n\times n}^{(i)}\rangle_{\mathbb{R}^{n\times n}} := \sum_{i=1}^{p} tr(\mathbf{A}_{n\times n}^{(i)\top}\mathbf{B}_{n\times n}^{(i)}),$$

for any $n \times n \times p$ dimensional arrays

$$\mathbf{A}_{n\times n\times p} = \begin{pmatrix} \mathbf{A}_{n\times n}^{(1)} \\ \vdots \\ \mathbf{A}_{n\times n}^{(p)} \end{pmatrix}, \text{ and } \mathbf{B}_{n\times n\times p} = \begin{pmatrix} \mathbf{B}_{n\times n}^{(1)} \\ \vdots \\ \mathbf{B}_{n\times n}^{(p)} \end{pmatrix} \in \mathbf{W}_n^p.$$

The least squares residuals of the model under H_0 is given by

$$\widehat{\mathbf{R}}_{n\times n\times p} = P_{\mathbf{W}_n^{p}\perp}\mathbf{Y}_{n\times n\times p} = P_{\mathbf{W}_n^{p}\perp}\mathcal{E}_{n\times n\times p} = \begin{pmatrix} P_{\mathbf{W}_n^\perp}\mathcal{E}^{(1)}(\Xi_n) \\ \vdots \\ P_{\mathbf{W}_n^\perp}\mathcal{E}^{(p)}(\Xi_n) \end{pmatrix},$$

(cf. Arnold[2], and Christensen [10], p.8-9), where the last equation follows from the definition of the component wise projection. Here and throughout the paper $P_{\mathbf{V}}$ and $P_{\mathbf{V}^\perp} = Id - P_{\mathbf{V}}$ denote the orthogonal projectors onto a linear subspace $\mathbf{V}$ and onto the orthogonal complement of $\mathbf{V}$, respectively.

If in addition $\{\mathcal{E}_{n\ell k} : 1 \le \ell, k \le n, \ n \ge 1\}$ are assumed to be i.i.d $N_p(\mathbf{0}, \mathbf{\Sigma})$, where N_p is the p-variate normal distribution, then under H_0 we have a normal multivariate linear model which has been studied in many standard text books of multivariate analysis [see e.g. [10], p.2-68]. In contrast to the classical treatment, for our results we assume neither normal nor other distributions for the observations since the test problem considered here will be handled asymptotically based on the sequence of the p-variate PSPR $\left\{\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p}) : n \ge 1\right\}$, where $\mathbf{T}_{n\times n\times p} : (\mathbb{R}^{n\times n})^p \to \mathcal{C}^p(\mathbf{I})$ is a linear operator defined by

$$\mathbf{T}_{n\times n\times p}(\mathbf{A}_{n\times n\times p}) := \begin{pmatrix} \mathbf{T}_n(\mathbf{A}_{n\times n}^{(1)}) \\ \vdots \\ \mathbf{T}_n(\mathbf{A}_{n\times n}^{(p)}) \end{pmatrix} \in \mathcal{C}^p(\mathbf{I}), \ \forall \mathbf{A}_{n\times n\times p} := \begin{pmatrix} \mathbf{A}_{n\times n}^{(1)} \\ \vdots \\ \mathbf{A}_{n\times n}^{(p)} \end{pmatrix},$$

where for every matrix $\mathbf{A}_{n\times n}^{(i)} := (A_{n\ell k}^{(i)})_{k=1,\ell=1}^{n,n} \in \mathbb{R}^{n\times n}$, and $(t,s) \in \mathbf{I}$,

$$\begin{aligned}\mathbf{T}_n(\mathbf{A}_{n\times n}^{(i)})(t,s) := {} & \frac{1}{n}\sum_{k=1}^{[ns]}\sum_{\ell=1}^{[nt]} A_{n\ell k}^{(i)} + \frac{(nt-[nt])}{n}\sum_{k=1}^{[ns]} A_{n[nt]+1,k}^{(i)} \\ & + \frac{(ns-[ns])}{n}\sum_{\ell=1}^{[nt]} A_{n\ell,[ns]+1}^{(i)} + \frac{(nt-[nt])(ns-[ns])}{n} A_{n[nt]+1,[ns]+1}^{(i)}.\end{aligned} \tag{3}$$

Thereby $\mathbf{T}_n(\mathbf{A}^{(i)}_{n\times n})(t,s) := 0$, for at least $t = 0$ or $s = 0$. [see also Park [17] and [9] for the definition of $\mathbf{T}_n$]. The space $\mathcal{C}^p(\mathbf{I})$ and $(\mathbb{R}^{n\times n})^p$ are the product of the p copies of $\mathcal{C}(\mathbf{I})$ and $\mathbb{R}^{n\times n}$, respectively, where $\mathcal{C}^p(\mathbf{I})$ is endowed with a topology induced by a metric ρ, defined by

$$\rho(\mathbf{w},\mathbf{u}) := \sum_{i=1}^{p} \left\| w^{(i)} - u^{(i)} \right\|_{\infty}, \mathbf{w} = (w^{(1)},\dots,w^{(p)})^\top, \mathbf{u} = (u^{(1)},\dots,u^{(p)})^\top \in \mathcal{C}^p(\mathbf{I}).$$

Every $\mathbf{w} \in \mathcal{C}^p(\mathbf{I})$ is further assumed to satisfy $\mathbf{w}(t,s) = 0$, for $t = 0$, or $s = 0$.

To test (2) the functional of the sequence of p-variate PSPR is frequently observed. For example, a type of Cramér-von Misses statistic defined by

$$CM_n := \left\| \frac{1}{n^2}\mathbf{\Sigma}^{-1/2} \sum_{\ell=1}^{n}\sum_{k=1}^{n} \mathbf{T}_{n\times n\times p}(P_{\mathbf{W}_n^{p\perp}} \mathbf{Y}_{n\times n\times p})(\ell/n, k/n) \right\|_{\mathbb{R}^p}$$

where $\|\cdot\|_{\mathbb{R}^p}$ is the usual Euclidean norm on $\mathbb{R}^p$, is reasonable for constructing the rejection region of the test. It is worth noting that adequateness of the assumed model to the sample depends heavily on the length of the residuals in a sense that the larger the residual is the worst the model will be. Therefore in the classical theory of model check for multivariate linear regression model the question whether the assumed model holds true is tasted based on the length of the residuals (cf. Arnold [2], and Christensen [10], p.9-20). Since the partial sums operator is one-to-one on $(\mathbb{R}^{n\times n})^p$, instead of investigating the length of the residuals we observe the length of the partial sums of the residuals such as CM_n. Analogously, based on this statistic H_0 will be rejected for a large value of CM_n. Hence an asymptotic size α-test, $\alpha \in (0,1)$, will reject H_0 if and only if $CM_n \geq k_\alpha$. Thereby k_α is a constant satisfying the equation $\mathbb{P}\{CM_n \geq k_\alpha \mid H_0\} = \alpha$. For a given α, k_α is approximated by the $(1-\alpha)$-th quantiles of the asymptotic distributions of CM_n under H_0.

The rest of the paper is organized as follows. In Section 2 we present in detail the limit process of the p-variate PSPR under the H_0 as well as under the K. The proof of the invariance principle for the p-variate Brownian sheet is presented in the same section, beforehand we discuss examples of the limit process under H_0. The paper is closed with some conclusions, see Section 3.

2. Main Results

A stochastic process $B^p := \{B^{(1)}(t,s),\dots,B^{(p)}(t,s) : (t,s) \in \mathbf{I}\}$ is called a p-variate Brownian sheet, if it satisfies the conditions: $B^p(t,s) \sim N_p(\mathbf{0}, ts\mathbf{I}_p)$, $\forall (t,s) \in \mathbf{I}$, where $\mathbf{I}_p$ is the $p\times p$ identity matrix, and

$$Cov(B^p(t,s), B^p(t',s')) = p(t\wedge t')(s\wedge s'),\ \forall (t,s),(t',s') \in \mathbf{I},$$

where $t \wedge s$ is the minimum between t and s. This is a simple extension of the definition of the multivariate Brownian motion on the unit interval $[0,1]$ studied e.g. in Kiefer [12].

Theorem 2.1. *Let* $\{\mathbf{E}_{n\times n\times p}: \ n \geq 1\}$ *be a sequence of* $n \times n \times p$ *arrays of random variables such that for every* $n \geq 1$, $\mathbf{E}_{n\times n\times p}$ *consists of the set* $\{\mathcal{E}_{n\ell k} = (\varepsilon_{n\ell k}^{(1)}, \ldots, \varepsilon_{n\ell k}^{(p)})^\top : 1 \leq \ell, k \leq n\}$ *of i.i.d. random vectors with* $\mathbb{E}(\mathcal{E}_{111}) = \mathbf{0}$ *and* $Cov(\mathcal{E}_{111}) = \mathbf{\Sigma}$. *Then as* $n \to \infty$, $\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p}) \xrightarrow{\mathcal{D}} B^p$. *Here and throughout this paper* "$\xrightarrow{\mathcal{D}}$" *stands for the convergence in distribution (weakly), and* $\mathbf{\Sigma}^{-1/2}$ *is the root square of* $\mathbf{\Sigma}^{-1}$, *i.e.* $\mathbf{\Sigma}^{-1} = \mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{-1/2}$.

Proof. Let $\mathbf{U}_{n,p} := \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p})$. By Theorem 8.2 in [4] it suffices to show the sequence of finite dimensional distributions of $\mathbf{U}_{n,p}$ converges to that of B^p, and the sequence of the distributions of $\mathbf{U}_{n,p}$ is tight. To prove the first condition we define $S_{o,u}^{q,v} := \sum_{\ell=o}^{q}\sum_{k=u}^{v}\mathcal{E}_{n\ell k}$, with $\mathcal{E}_{n\ell k} = (\varepsilon_{n\ell k}^{(1)}, \cdots, \varepsilon_{n\ell k}^{(p)})^\top$, for $o, q, u, v \in \mathbb{N}$, $o \leq q$ and $u \leq v$. Hence by Equation (3), for every $(x, y) \in \mathbf{I}$, we have a decomposition

$$\mathbf{U}_{n,p}(x, y) = \frac{1}{n}\mathbf{\Sigma}^{-1/2}S_{1,1}^{[nx],[ny]} + \mathbf{\Sigma}^{-1/2}\Psi_{[nx][ny]},$$

where

$$\Psi_{[nx][ny]} := \frac{(nx - [nx])}{n}\sum_{k=1}^{[ny]}\mathcal{E}_{n[nx]+1,k} + \frac{(ny - [ny])}{n}\sum_{k=1}^{[nx]}\mathcal{E}_{n\ell,[ny]+1} + \frac{(nx - [nx])(ny - [ny])}{n}\mathcal{E}_{n[nx]+1,[ny]+1}$$

Let us consider an arbitrary fixed point $(t, s) \in \mathbf{I}$. It will be shown

$$\mathbf{U}_{n,p}(t, s) \xrightarrow{\mathcal{D}} B^p(t, s), \text{ as } n \to \infty.$$

Since the Lindeberg-Levy multivariate central limit theorem (cf. [21], p.16) guarantees that

$$\frac{1}{n}\mathbf{\Sigma}^{-1/2}S_{1,1}^{[nt],[ns]} \xrightarrow{\mathcal{D}} N_p(\mathbf{0}, ts\mathbf{I}_p), \text{ as } n \to \infty,$$

then by recalling Theorem 4.1 in [4], the assertion will follow if we show

$$\left\|\mathbf{U}_{n,p}(t, s) - \frac{1}{n}\mathbf{\Sigma}^{-1/2}S_{1,1}^{[nt],[ns]}\right\|_{\mathbb{R}^p} \xrightarrow{\mathcal{P}} 0, \text{ as } n \to \infty,$$

where $\xrightarrow{\mathcal{P}}$ denotes the convergence in probability (stochastically). But this is straightforward, since by the preceding decomposition we get

$$\left\|\mathbf{U}_{n,p}(t, s) - \frac{1}{n}\mathbf{\Sigma}^{-1/2}S_{1,1}^{[nt],[ns]}\right\|_{\mathbb{R}^p} = \left\|\mathbf{\Sigma}^{-1/2}\Psi_{[nt][ns]}\right\|_{\mathbb{R}^p},$$

and by using the well known Chebyshev's inequality, $\forall \mathbf{v} \in \mathbb{R}^p$, and $\forall \varepsilon > 0$, it holds

$$\mathbb{P}\left\{\left|\mathbf{v}^\top\mathbf{\Sigma}^{-1/2}\Psi_{[nt][ns]}\mathbf{v}\right| \geq \epsilon\right\} \leq \frac{Var(\mathbf{v}^\top\Sigma^{-1/2}\Psi_{[nt][ns]}\mathbf{v})}{\varepsilon^2} \leq \frac{1}{\varepsilon^2}\left(\frac{2\mathbf{v}^\top\mathbf{v}}{n} + \frac{\mathbf{v}^\top\mathbf{v}}{n^2}\right) \longrightarrow 0, \text{ as } n \to \infty.$$

Next we consider two arbitrary different points $(t,s),(t',s')\in\mathbf{I}$. It must be shown that $(\mathbf{U}_{n,p}(t,s),\mathbf{U}_{n,p}(t',s'))^\top \xrightarrow{\mathcal{D}} (B^p(t,s),B^p(t',s'))^\top$, as $n\to\infty$. For that we consider in this stage two possible cases only, for which $t<t'$ and $s<s'$, and $t<t'$ and $s'<s$. In the first situation we have

$$(\mathbf{U}_{n,p}(t,s),\mathbf{U}_{n,p}(t',s'))^\top = (\mathbf{U}_{n,p}(t,s),(\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s))+\mathbf{U}_{n,p}(t,s))^\top,$$

where $\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s)$ and $\mathbf{U}_{n,p}(t,s)$ are stochastically independent. Hence by the preceding result and the well known Cramér-Wold device, it suffices to show

$$\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s)\xrightarrow{\mathcal{D}} N_p(\mathbf{0},(t's'-ts)\mathbf{I}_p).$$

Since $\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt'],[ns']}-S_{1,1}^{[nt],[ns]}\right)=\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{[nt]+1,1}^{[nt'],[ns']}+S_{1,[ns]+1}^{[nt],[ns']}\right)$, and

$$\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{[nt]+1,1}^{[nt'],[ns']}+S_{1,[ns]+1}^{[nt],[ns']}\right)\xrightarrow{\mathcal{D}} N_p(\mathbf{0},(t's'-ts)\mathbf{I}_p),$$

then by using the same reason as in the case of a single point, it is enough to show

$$\left\|\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s)-\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt'],[ns']}-S_{1,1}^{[nt],[ns]}\right)\right\|_{\mathbb{R}^p}\xrightarrow{\mathcal{P}}0.$$

But this is also an immediate consequence of the fact

$$\begin{aligned}\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s)-&\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt'],[ns']}-S_{1,1}^{[nt],[ns]}\right)\\&=\mathbf{\Sigma}^{-1/2}\left(\Psi_{[nt'][ns']}-\Psi_{[nt][ns]}\right)\xrightarrow{\mathcal{P}}0.\end{aligned}$$

In the second case we can rewrite $(\mathbf{U}_{n,p}(t,s),\mathbf{U}_{n,p}(t',s'))^\top$ as

$$\begin{aligned}(\mathbf{U}_{n,p}(t,s),\mathbf{U}_{n,p}(t',s'))^\top=(&\mathbf{U}_{n,p}(t,s)-\mathbf{U}_{n,p}(t,s')+\mathbf{U}_{n,p}(t,s'),\\&\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s')+\mathbf{U}_{n,p}(t,s'))^\top.\end{aligned}$$

It is clear by the definition, $\mathbf{U}_{n,p}(t,s)-\mathbf{U}_{n,p}(t,s')$ and $\mathbf{U}_{n,p}(t,s')$, and $\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s')$ and $\mathbf{U}_{n,p}(t,s')$ are stochastically independent, respectively. Hence by the same argument, it suffices to show $\mathbf{U}_{n,p}(t,s)-\mathbf{U}_{n,p}(t,s')\xrightarrow{\mathcal{D}} N_p(\mathbf{0},(ts-ts')\mathbf{I}_p)$ and $\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s')\xrightarrow{\mathcal{D}} N_p(\mathbf{0},(t's'-ts')\mathbf{I}_p)$. Since by applying the Lindeberg-Levy multivariate central limit theorem (cf. [21], p.16), we get

$$\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt],[ns]}-S_{1,1}^{[nt],[ns']}\right)=\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,[ns']+1}^{[nt],[ns]}\right)\xrightarrow{\mathcal{D}} N_p(\mathbf{0},t(s-s')\mathbf{I}_p),$$

and

$$\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt'],[ns']}-S_{1,1}^{[nt],[ns']}\right)=\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{[nt]+1,1}^{[nt'],[ns']}\right)\xrightarrow{\mathcal{D}} N_p(\mathbf{0},(t'-t)s'\mathbf{I}_p).$$

then once again by using Theorem 4.1 in [4], the assertion will follow, if both

$$\left\|\mathbf{U}_{n,p}(t,s)-\mathbf{U}_{n,p}(t,s')-\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt],[ns]}-S_{1,1}^{[nt],[ns']}\right)\right\|_{\mathbb{R}}^{p}\xrightarrow{\mathcal{P}}0,$$

and

$$\left\|\mathbf{U}_{n,p}(t',s')-\mathbf{U}_{n,p}(t,s')-\frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt'],[ns']}-S_{1,1}^{[nt],[ns']}\right)\right\|_{\mathbb{R}}^{p}\xrightarrow{\mathcal{P}}0.$$

But these can be directly obtained from the equations

$$\mathbf{U}_{n,p}(t,s) - \mathbf{U}_{n,p}(t,s') - \frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt],[ns]} - S_{1,1}^{[nt],[ns']}\right) = \mathbf{\Sigma}^{-1/2}\Psi_{[nt][ns']},$$

and

$$\mathbf{U}_{n,p}(t',s') - \mathbf{U}_{n,p}(t,s') - \frac{1}{n}\mathbf{\Sigma}^{-1/2}\left(S_{1,1}^{[nt'],[ns']} - S_{1,1}^{[nt],[ns']}\right) = \mathbf{\Sigma}^{-1/2}\Psi_{[nt'][ns']},$$

where both $\mathbf{\Sigma}^{-1/2}\Psi_{[nt][ns']}$ and $\mathbf{\Sigma}^{-1/2}\Psi_{[nt'][ns']}$ converge in probability to 0, as $n \to \infty$. The proof for the cases of a set of three and more points on $\mathbf{I}$ can be handled analogously.

Since $\mathbf{U}_{n,p}(t,s) = 0$, for either $t = 0$ or $s = 0$, in order to prove the tightness of the sequence of the distributions of $\mathbf{U}_{n,p}$, it is sufficient to show

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\{W(\mathbf{T}_{n\times n\times p}(\mathcal{E}_{n\times n\times n});\delta) \geq \varepsilon\} = 0,\ \forall\varepsilon > 0,$$

(c.f. Theorem 8.2 in [4]), where for every $\mathbf{x} = (x^{(1)}, \ldots, x^{(p)})^\top \in \mathcal{C}^p(\mathbf{I})$, $W(\mathbf{x};\delta)$ is the modulus of continuity of $\mathbf{x}$, defined by

$$W(\mathbf{x};\delta) := \sup_{\|(t,s)-(t',s')\|\leq\delta} \|\mathbf{x}(t,s) - \mathbf{x}(t',s')\|_{\mathbb{R}^p},\ 0 < \delta < 1.$$

The proof will be finished, if for $i = 1, \ldots, p$ we show

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\{W(\mathbf{U}_{n,p}^{(i)};\delta) \geq \varepsilon\} = 0,\ \forall\varepsilon > 0,$$

where $\mathbf{U}_{n,p}^{(i)} := \mathbf{T}_n(\mathcal{E}^{(i)}(\Xi_n))$, by the reason $W(\mathbf{x};\delta) \leq \sum_{i=1}^p W(x^{(i)};\delta)$.

Let $\{\mathbf{I}_{\ell k} := [t_{\ell-1}, t_\ell] \times [s_{k-1}, s_k] : 1 \leq \ell \leq p, 1 \leq k \leq q\}$ be a partition on $\mathbf{I}$, where $0 = t_0 < t_1 < \ldots < t_p = 1$, $0 = s_0 < s_1 < \ldots < s_q = 1$, such that

$$\min_{1\leq\ell\leq p}(t_\ell - t_{\ell-1}) \geq \delta,\ \text{and}\ \min_{1\leq k\leq q}(s_k - s_{k-1}) \geq \delta,\ \delta \in (0,1).$$

Then for every $\varepsilon > 0$, it holds

$$\mathbb{P}\left\{W(\mathbf{U}_{n,p}^{(i)};\delta\sqrt{2}) \geq 3\varepsilon\right\} \leq \sum_{k=1}^{q}\sum_{\ell=1}^{p}\mathbb{P}\left\{\sup_{(t,s)\in\mathbf{I}_{\ell k}}\left|\mathbf{U}_{n,p}^{(i)}(t,s) - \mathbf{U}_{n,p}^{(i)}(t_{\ell-1}, s_{k-1})\right| \geq \varepsilon\right\}.$$

To be easier in analyzing the last inequality, let us chose $t_\ell = m_\ell/n$, and $s_k = m'_k/n$, for $0 \leq \ell \leq p$, and $0 \leq k \leq q$, where m_ℓ and m'_k are integers that satisfy the condition

$$0 = m_0 < m_1 < \ldots < m_{\ell-1} < m_\ell < \ldots < m_p = n,$$
$$0 = m'_0 < m'_1 < \ldots < m'_{k-1} < m'_k < \ldots < m'_q = n.$$

Let $S_{o,u}^{(i)q,v} := \sum_{\ell=o}^{q}\sum_{k=u}^{v}\varepsilon_{n\ell k}^{(i)}$ be the i-th component of $S_{o,u}^{q,v}$. Then by the polygonal property of the partial sums, we get

$$\mathbb{P}\left\{W(\mathbf{U}_{n,p}^{(i)};\delta\sqrt{2}) \geq 3\varepsilon\right\} \leq \sum_{k=1}^{q}\sum_{\ell=1}^{p}\mathbb{P}\left\{\max_{\substack{m_{\ell-1}\leq i_1\leq m_\ell \\ m'_{k-1}\leq i_2\leq m'_k}}\left|S_{1,1}^{(i)i_1,i_2} - S_{1,1}^{(i)m_{\ell-1},m'_{k-1}}\right| \geq \varepsilon n\right\},$$

whenever

$$\frac{m_\ell}{n} - \frac{m_{\ell-1}}{n} \geq \delta, \text{ and } \frac{m'_k}{n} - \frac{m'_{k-1}}{n} \geq \delta, \text{ for } 1 \leq \ell \leq p, 1 \leq k \leq q.$$

Furthermore, the stochastically independence of $\{\varepsilon^{(i)}_{n\ell k} : 1 \leq \ell, k \leq n\}$ implies

$$\mathbb{P}\left\{W(\mathbf{U}^{(i)}_{n,p};\delta\sqrt{2}) \geq 3\varepsilon\right\} \leq \sum_{k=1}^{q}\sum_{\ell=1}^{p}\mathbb{P}\left\{\max_{\substack{0\leq i_1\leq m_\ell - m_{\ell-1}\\ 0\leq i_2 \leq m'_k - m'_{k-1}}} \left|S^{(i)i_1,i_2}_{1,1}\right| \geq \varepsilon n\right\}.$$

For further simplification we chose $m_\ell = \ell m$ and $m'_k = km'$, for some integers m and m' that satisfy $m_\ell - m_{\ell-1} = m \geq n\delta$ and $m'_k - m'_{k-1} = m' \geq n\delta$, for $0 \leq \ell < p$ and $0 \leq k < q$. Since the indexes p and q must satisfy $(p-1)m < n \leq pm$ and $(q-1)m' < n \leq qm'$, then we have $p = \lceil n/m \rceil \overset{n\to\infty}{\longrightarrow} 1/\delta < 2/\delta$ and $q = \lceil n/m' \rceil \overset{n\to\infty}{\longrightarrow} 1/\delta < 2/\delta$, where $\lceil x \rceil := \min\{z \in \mathbb{N} : \ x \leq z\}$. Moreover, $n/m \overset{n\to\infty}{\longrightarrow} 1/\delta > 1/2\delta$ and $n/m' \overset{n\to\infty}{\longrightarrow} 1/\delta > 1/2\delta$. Hence, for large n and for every $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}\left\{W(\mathbf{U}^{(i)}_{n,p};\delta\sqrt{2}) \geq 3\varepsilon\right\} \leq& \frac{4}{\delta^2}\ \mathbb{P}\left\{\max_{\substack{0\leq i_1\leq m\\ 0\leq i_2\leq m'}}\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon\sqrt{mm'}}{2\delta}\right\}\\ \leq& \frac{12}{\delta^2}\max_{\substack{0\leq i_1\leq m\\ 0\leq i_2\leq m'}}\mathbb{P}\left\{\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon\sqrt{mm'}}{6\delta}\right\},\end{aligned}$$

where the last inequality is obtained by applying Etemadi's inequality (cf. Theorem 22.5 in [5]). Consequently we get

$$\begin{aligned}&\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left\{W(\mathbf{U}^{(i)}_{n,p};\delta\sqrt{2}) \geq 3\varepsilon\right\}\\ &\leq \lim_{\delta\to 0}\limsup_{m,m'\to\infty}\frac{12}{\delta^2}\max_{\substack{0\leq i_1\leq m\\ 0\leq i_2\leq m'}}\mathbb{P}\left\{\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon\sqrt{mm'}}{6\delta}\right\}\\ &= \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{12}{\delta^2}\max_{\substack{0\leq i_1\leq n\\ 0\leq i_2\leq n}}\mathbb{P}\left\{\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon n}{6\delta}\right\}. \qquad (4)\end{aligned}$$

Let ℓ_δ and k_δ be large enough such that $\ell_\delta \leq i_1 \leq n$ and $k_\delta \leq i_2 \leq n$, then by the central limit theorem and Markov's inequality (cf. Athreya and Lahiri [3], p.83),

$$\begin{aligned}\mathbb{P}\left\{\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon n}{6\delta}\right\} \leq& \mathbb{P}\left\{\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon\sqrt{\ell_\delta k_\delta}}{6\delta}\right\} = \mathbb{P}\left\{|Z| \geq \frac{\varepsilon}{6\delta\sigma}\right\} \leq \frac{\mathbb{E}\,|Z|^4}{(\frac{\varepsilon}{6\delta\sigma})^4}\\ =& \frac{\left(var\,|Z|^2 + \left[\mathbb{E}\,|Z|^2\right]^2\right)6^4\delta^4\sigma^4}{\varepsilon^4} = \frac{(2+1)6^4\delta^4\sigma^4}{\varepsilon^4} = \frac{3888\delta^4\sigma^4}{\varepsilon^4}.\end{aligned}$$

For $i_1 \leq \ell_\lambda \leq n$ and $i_2 \leq k_\lambda \leq n$, we can apply Chebyshev's inequality to get

$$\mathbb{P}\left\{\left|S^{(i)i_1,i_2}_{1,1}\right| \geq \frac{\varepsilon n}{6\delta}\right\} \leq \frac{36(i_1 i_2)\delta^2\sigma^2}{\varepsilon^2 n^2} \leq \frac{36(\ell_\delta k_\delta)\delta^2\sigma^2}{\varepsilon^2 n^2}.$$

Thus, the maximum on the right side of (4) is dominated by

$$\max\left\{\frac{3888\delta^4\sigma^4}{\varepsilon^4}, \frac{36(\ell_\delta k_\delta)\delta^2\sigma^2}{\varepsilon^2 n^2}\right\}.$$

Hence, we finally get

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left\{W(\mathbf{U}_{n,p}^{(i)};\delta\sqrt{2})\geq 3\varepsilon\right\}\leq \lim_{\delta\to 0}\limsup_{n\to\infty}\max\left\{\frac{46656\delta^2\sigma^4}{\varepsilon^4}, \frac{432(\ell_\delta k_\delta)\sigma^2}{\varepsilon^2 n^2}\right\}$$

which is clearly equal to 0. This leads us to the conclusion that the sequence of the distributions of $\mathbf{U}_{n,p}$ is tight. □

To be able to project B^p, the so-called reproducing kernel Hilbert space (RKHS) of B^p is decisive. It can be obtained by applying the method proposed in Lifshits [13], p.93. According to this method, the RKHS of B^p, denoted by $\mathcal{H}_{B^p}$, is given by

$$\mathcal{H}_{B^p} := \left\{\mathbf{h}:\mathbf{I}\to\mathbb{R}^p,\ \exists\mathbf{f}\in L_2^p(\lambda,\mathbf{I}), \mathbf{h}(t,s)=\int_{[0,t]\times[0,s]}\mathbf{f}d\lambda,\ (t,s)\in\mathbf{I}\right\}.$$

It is understood that the integral considered here and throughout this paper is defined component wise. We furnish $\mathcal{H}_{B^p}$ with the inner product and the norm defined by $\langle\mathbf{h}_1,\mathbf{h}_2\rangle_{\mathcal{H}_{B^p}} := \langle\mathbf{f}_1,\mathbf{f}_2\rangle_{L_2^p(\lambda,\mathbf{I})}$, and $\|\mathbf{h}_i\|_{\mathcal{H}_{B^p}} := \|\mathbf{f}_i\|_{L_2^p(\lambda,\mathbf{I})}$, for which $\mathbf{h}_i(t,s) = \int_{[0,t]\times[0,s]}\mathbf{f}_i d\lambda$, $(t,s)\in\mathbf{I}$, $\mathbf{f}_i\in L_2^p(\lambda,\mathbf{I})$, $i=1,2$. Under such defined inner product and norm, $\mathcal{H}_{B^p}$ becomes a Hilbert space, since it is isometry with the Hilbert space $L_2^p(\lambda,\mathbf{I})$. For a fixed $(t',s')\in\mathbf{I}$ the function $K(\cdot;(t',s')):\mathbf{I}\to\mathbb{R}$, defined by $K((t,s);(t',s')) := p(t\wedge t')(s\wedge s')$, for $(t,s)\in\mathbf{I}$ is an element of $\mathcal{H}_{B^p}$. It describes the covariance of B^p. In the work of Somayasa [20] the role of RKHS of the one dimensional Brownian sheet was demonstrated in analyzing the power of a test based on the Kolmogorov functional of the PSPR of univariate regression.

Associated with the regression functions $f_1,\ldots,f_d$, let $\mathbf{W}_{\mathcal{H}_B} := [h_1,\ldots,h_d]$ be a linear subspace generated by $\{h_1,\ldots,h_d\}$, where $h_j(t,s) := \int_{[0,t]\times[0,s]} f_j d\lambda$, for $(t,s)\in\mathbf{I}$, $j=1,\ldots,d$. Furthermore, let $\mathbf{W}_{\mathcal{H}_{B^p}} := [\mathbf{h}_1,\ldots,\mathbf{h}_d]\subset\mathcal{H}_{B^p}$, where $\mathbf{h}_j:\mathbf{I}\to\mathbb{R}^p$, with $\mathbf{h}_j(t,s) := \int_{[0,t]\times[0,s]}(f_j,\ldots,f_j)^\top d\lambda$, for $(t,s)\in\mathbf{I}$, $j=1,\ldots,d$. It can be seen that $\mathbf{W}_{\mathcal{H}_B}^p\subseteq\mathbf{W}_{\mathcal{H}_{B^p}}$, where $\mathbf{W}_{\mathcal{H}_B}^p := \mathbf{W}_{\mathcal{H}_B}\times\cdots\times\mathbf{W}_{\mathcal{H}_B}$. The equality $\mathbf{T}_n(P_{\mathbf{W}_n}\mathbf{B}_{n\times n}) = P_{\mathbf{W}_{n\mathcal{H}_B}}\mathbf{T}_n(\mathbf{B}_{n\times n})$, for every $\mathbf{B}_{n\times n}\in\mathbb{R}^{n\times n}$, was investigated in Proposition 2.2 of [9], where $\mathbf{W}_{n\mathcal{H}_B} := \{\mathbf{T}_n(\mathbf{A}_{n\times n}):\mathbf{A}_{n\times n}\in\mathbf{W}_n\}$. Hence by using the definition of $\mathbf{T}_{n\times n\times p}$ and the component wise projection, it holds

$$\begin{aligned}\mathbf{T}_{n\times n\times p}(P_{\mathbf{W}_n^p}\mathbf{A}_{n\times n\times p}) &= \mathbf{T}_{n\times n\times p}\begin{pmatrix} P_{\mathbf{W}_n}\mathbf{A}_{n\times n}^{(1)} \\ \vdots \\ P_{\mathbf{W}_n}\mathbf{A}_{n\times n}^{(p)}\end{pmatrix} = \begin{pmatrix}\mathbf{T}_n(P_{\mathbf{W}_n}\mathbf{A}_{n\times n}^{(1)}) \\ \vdots \\ \mathbf{T}_n(P_{\mathbf{W}_n}\mathbf{A}_{n\times n}^{(p)})\end{pmatrix} \\ &= (P_{\mathbf{W}_{n\mathcal{H}_B}}\mathbf{T}_n(\mathbf{A}_{n\times n}^{(1)}),\ldots,P_{\mathbf{W}_{n\mathcal{H}_B}}\mathbf{T}_n(\mathbf{A}_{n\times n}^{(p)}))^\top \\ &= P_{\mathbf{W}_{n\mathcal{H}_B}^p}\mathbf{T}_{n\times n\times p}(\mathbf{A}_{n\times n\times p}),\ \forall\mathbf{A}_{n\times n\times p}\in(\mathbb{R}^{n\times n})^p.\end{aligned}$$

The linearity of $P_{\mathbf{W}_{n\mathcal{H}_B}}$ further implies

$$\mathbf{C}\mathbf{T}_{n\times n\times p}(P_{\mathbf{W}_n^p}\mathbf{A}_{n\times n\times p}) = P_{\mathbf{W}_{n\mathcal{H}_B}^p}\mathbf{C}\mathbf{T}_{n\times n\times p}(\mathbf{A}_{n\times n\times p}), \tag{5}$$

for every $p \times p$ dimensional matrix $\mathbf{C}$, and $\mathbf{A}_{n\times n\times p} \in (\mathbb{R}^{n\times n})^p$. Furthermore, Lemma A.15 in [9] guarantees the existence of a projection $P^*_{\mathbf{W}_{\mathcal{H}_B}} : \mathcal{C}(\mathbf{I}) \to \mathbf{W}_{\mathcal{H}_B}$ with the property

$$\left\| P^*_{\mathbf{W}_{n\mathcal{H}_B}} u_n - P^*_{\mathbf{W}_{\mathcal{H}_B}} u \right\|_\infty \to 0, \text{ in } \mathcal{C}(\mathbf{I}), \text{ as } n \to \infty, \tag{6}$$

whenever $\|u_n - u\|_\infty$ converges to 0, as $n \to \infty$, where $P_{\mathbf{W}_{n\mathcal{H}_B}}$ and $P_{\mathbf{W}_{\mathcal{H}_B}}$ constitute the restrictions of $P^*_{\mathbf{W}_{n\mathcal{H}_B}}$ and $P^*_{\mathbf{W}_{\mathcal{H}_B}}$ on the reproducing kernel Hilbert space of the Brownian sheet, respectively. Thereby $P^*_{\mathbf{W}_{\mathcal{H}_B}} u := \sum_{j=1}^d \langle h_j, u\rangle h_j$, where

$$\begin{aligned}\langle h_j, u\rangle := \Delta_{\mathbf{I}}(uf_j) &- \int_{[0,1]}^{(R)} u(t,1)df_j(t,1) - \int_{[0,1]}^{(R)} u(1,s)df_j(1,s) \\ &+ \int_{[0,1]}^{(R)} u(t,0)df_j(t,0) + \int_{[0,1]}^{(R)} u(0,s)df_j(0,s) + \int_{\mathbf{I}}^{(R)} u(t,s)df_j(t,s),\end{aligned}$$

$\Delta_{\mathbf{I}}(w) := w(1,1) - w(1,0) - w(0,1) + w(0,0)$, for $w : \mathbf{I} \mapsto \mathbb{R}$, $\int^{(R)}$ is the Riemann-Stieltjes integral, and $\{f_1, \ldots, f_d\}$ is assumed to build an orthonormal basis (ONB) for $\mathbf{W}$. We refer the reader to Lemma A.15 in [9] for a complete investigation regarding the properties of this bilinear form.

Now we are ready to state the limit process of the p-variate PSPR for the model specified in the preceding section.

Theorem 2.2. *Let $\{f_1, \ldots, f_d\}$ be an ONB of $\mathbf{W} \subset \mathcal{C}(\mathbf{I}) \cap BV_H(\mathbf{I})$. If H_0 holds true, then we have*

$$\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p}) \xrightarrow{\mathcal{D}} B^p_{(\mathbf{f}_1,\ldots,\mathbf{f}_d)} := B^p - P^*_{\mathbf{W}^p_{\mathcal{H}_B}} B^p, \text{ as } n \to \infty,$$

where $P^*_{\mathbf{W}^p_{\mathcal{H}_B}} B^p = (P^*_{\mathbf{W}_{\mathcal{H}_B}} B^{(1)}, \cdots, P^*_{\mathbf{W}_{\mathcal{H}_B}} B^{(p)})^\top$, and $\mathbf{f}_j$ in the index $(\mathbf{f}_1, \ldots, \mathbf{f}_d)$ stand for the p-copies of the regression function f_j. Moreover $B^p_{\mathbf{f}_1,\ldots\mathbf{f}_d}$ is a centered Gaussian process with the covariance function $K_{(\mathbf{f}_1,\ldots,\mathbf{f}_d)} : \mathbf{I} \times \mathbf{I} \to \mathbb{R}$, defined by

$$K_{(\mathbf{f}_1,\ldots,\mathbf{f}_d)}((t,s),(t',s')) := p[(t\wedge t')(s\wedge s') - \sum_{j=1}^d h_j(t,s)h_j(t',s')],$$

where for $j = 1, \ldots, d$, $h_j(t,s) = \int_{[0,t]\times[0,s]} f_j \, d\lambda$, $(t,s) \in \mathbf{I}$.

Proof. By the linearity of $\mathbf{T}_{n\times n\times p}$ and the definition of $\widehat{\mathbf{R}}_{n\times n\times p}$, we get under H_0

$$\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p}) = \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p}) - \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(P_{\mathbf{W}^p_n}\mathbf{E}_{n\times n\times p})$$

Furthermore, by using Equation (5) and Proposition 2.2 in [9], it holds for the second term in the right-hand side

$$\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(P_{\mathbf{W}^p_n}\mathbf{E}_{n\times n\times p}) = P^*_{\mathbf{W}^p_{n\mathcal{H}_B}}\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p}).$$

The term $\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p})$ converges to B^p in $\mathcal{C}^p(\mathbf{I})$, by Theorem 2.1. To get the limit of the second term, suppose for the moment $(\mathbf{u}_n)_{n\geq 1}$, $\mathbf{u}_n := (u_n^{(1)},\ldots,u_n^{(p)})^\top$ and $\mathbf{u} := (u^{(1)},\ldots,u^{(p)})^\top$ are functions in $\mathcal{C}^p(\mathbf{I})$, such that $\rho(\mathbf{u}_n,\mathbf{u})\to 0$ as $n\to\infty$. Then (6) and the definition of component wise projection imply

$$\rho(P^*_{\mathbf{W}^p_{n\mathcal{H}_B}}\mathbf{u}_n, P^*_{\mathbf{W}^p_{\mathcal{H}_B}}\mathbf{u}) = \sum_{i=1}^{p}\left\|P^*_{\mathbf{W}_{n\mathcal{H}_B}}u_n^{(i)} - P^*_{\mathbf{W}_{\mathcal{H}_B}}u^{(i)}\right\|_\infty \to 0, \text{ as } n\to\infty.$$

This shows that $P^*_{\mathbf{W}^p_{n\mathcal{H}_B}}$ has the property $P^*_{\mathbf{W}^p_{n\mathcal{H}_B}}\mathbf{u}_n$ converges to $P^*_{\mathbf{W}^p_{\mathcal{H}_B}}\mathbf{u}$ in $\mathcal{C}^p(\mathbf{I})$, whenever $\mathbf{u}_n$ converges to $\mathbf{u}$ in $\mathcal{C}^p(\mathbf{I})$. On the other hand it is known from the definition of $\mathbf{T}_{n\times n\times p}$ presented in Section 1, that $\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p})$ has the sample path in $\mathcal{C}^p(\mathbf{I})$. Hence, by applying Theorem 2.1 and the mapping theorem of Rubin (Theorem 5.5 in Billingsley [4]), $P^*_{\mathbf{W}^p_{n\mathcal{H}_B}}\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p})$ converges weakly to $P^*_{\mathbf{W}^p_{\mathcal{H}_B}}B^p$ in $\mathcal{C}^p(\mathbf{I})$. □

What follows is a direct consequence of the well-known continuous mapping theorem [see e.g. [4], p.29].

Corollary 2.3. *Suppose the condition of Theorem 2.2 is satisfied. Then under H_0 it holds*

$$\left\|\frac{1}{n^2}\mathbf{\Sigma}^{-1/2}\sum_{\ell=1}^{n}\sum_{k=1}^{n}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p})(\ell/n,k/n)\right\|_{\mathbb{R}^p} \xrightarrow{\mathcal{D}} \left\|\int_{\mathbf{I}} B^p_{(\mathbf{f}_1,\ldots,\mathbf{f}_d)}\,d\lambda\right\|_{\mathbb{R}^p}.$$

Proof. We note that $\frac{1}{n^2}\mathbf{\Sigma}^{-1/2}\sum_{\ell=1}^n\sum_{k=1}^n\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p})(\ell/n,k/n)$ can also be written as the Lebesgue integral $\int_{\mathbf{I}}\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p})(t,s)\lambda(dt,ds)$. Since the mapping $\mathcal{C}^p(\mathbf{I})\ni\mathbf{f}\mapsto\int_{\mathbf{I}}\mathbf{f}\,d\lambda\in\mathbb{R}^p$ is continuous, the continuous mapping theorem and Theorem 2.2 imply

$$\frac{1}{n^2}\mathbf{\Sigma}^{-1/2}\sum_{\ell=1}^{n}\sum_{k=1}^{n}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p})(\ell/n,k/n) \xrightarrow{\mathcal{D}} \int_{\mathbf{I}} B^p_{(\mathbf{f}_1,\ldots,\mathbf{f}_d)}\,d\lambda \text{ as } n\to\infty.$$

Similarly, we get for $n\to\infty$,

$$\left\|\frac{1}{n^2}\mathbf{\Sigma}^{-1/2}\sum_{\ell=1}^{n}\sum_{k=1}^{n}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p})(\ell/n,k/n)\right\|_{\mathbb{R}^p} \xrightarrow{\mathcal{D}} \left\|\int_{\mathbf{I}} B^p_{(\mathbf{f}_1,\ldots,\mathbf{f}_d)}\,d\lambda\right\|_{\mathbb{R}^p},$$

by the fact $\|\cdot\|_{\mathbb{R}^p}$ is continuous on $\mathbb{R}^p$. □

To get the asymptotic distribution of the PSPR and the CM_n under the alternative, we consider a localized model, defined by

$$\mathbf{Y}_{n\times n\times p} = \mathbf{g}^{loc}_{n\times n\times p} + \mathbf{E}_{n\times n\times p}, \text{where } \mathbf{g}^{loc}_{n\times n\times p} := \begin{pmatrix}\frac{1}{n}g^{(1)}(\Xi_n)\\ \vdots \\ \frac{1}{n}g^{(p)}(\Xi_n)\end{pmatrix}.$$

Theorem 2.4. *Suppose* $\{f_1, \ldots, f_d\}$ *constitutes an ONB of* $\mathbf{W} \subset \mathcal{C}(\mathbf{I}) \cap BV_H(\mathbf{I})$. *Then under the alternative* $H_1 : \mathbf{g}_{n\times n\times p} \notin \mathbf{W}_n^p$, *we have*

$$\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p}) \xrightarrow{\mathcal{D}} P^*_{\mathbf{W}^{p\perp}_{\mathcal{H}_B}} \mathbf{\Sigma}^{-1/2}\mathbf{h}_g + P^*_{\mathbf{W}^{p\perp}_{\mathcal{H}_B}} B^p,$$

$$CM_n \xrightarrow{\mathcal{D}} \left\| \int_{\mathbf{I}} (P^*_{\mathbf{W}^{p\perp}_{\mathcal{H}_B}} \mathbf{\Sigma}^{-1/2}\mathbf{h}_g + P^*_{\mathbf{W}^{p\perp}_{\mathcal{H}_B}} B^p)\, d\lambda \right\|_{\mathbb{R}^p}$$

where $\mathbf{h}_g := (h_{g^{(1)}}, \cdots, h_{g^{(p)}})^\top : \mathbf{I} \to \mathbb{R}^p$, *with* $h_{g^{(i)}}(t,s) := \int_{[0,t]\times[0,s]} g^{(i)}\, d\lambda$.

Proof. For $n \in \mathbb{N}$, if $\mathbf{g}_{n\times n\times p} \notin \mathbf{W}_n$, then $\mathbf{g}^{loc}_{n\times n\times p} \notin \mathbf{W}_n$. Hence the $n\times n\times p$ array of the residuals of the localized model under H_1 is given by

$$\begin{aligned}\widehat{\mathbf{R}}_{n\times n\times p} =& P_{\mathbf{W}_n^{p\perp}}(\mathbf{g}^{loc}_{n\times n\times p} + \mathbf{E}_{n\times n\times p})\\ =& \mathbf{g}^{loc}_{n\times n\times p} - P_{\mathbf{W}_n^p}\mathbf{g}^{loc}_{n\times n\times p} + \mathbf{E}_{n\times n\times p} - P_{\mathbf{W}_n^p}\mathbf{E}_{n\times n\times p}.\end{aligned}$$

The linearity of $\mathbf{T}_{n\times n\times p}$, Equation (5) and Proposition 2.2 of [9] further imply

$$\begin{aligned}\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\widehat{\mathbf{R}}_{n\times n\times p}) &= \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p}) - \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(P_{\mathbf{W}_n^p}\mathbf{g}^{loc}_{n\times n\times p})\\ &\quad + \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p}) - \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(P_{\mathbf{W}_n^p}\mathbf{E}_{n\times n\times p})\\ &= \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p}) - P_{\mathbf{W}^p_{n\mathcal{H}_B}}\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p})\\ &\quad + \mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p}) - P_{\mathbf{W}^p_{n\mathcal{H}_B}}\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{E}_{n\times n\times p}). \qquad (7)\end{aligned}$$

Let us consider first the term $\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p})$ in (7), whose i-th component is given by $\mathbf{T}_n(g^{(i)}(\Xi_n)/n) = \frac{1}{n}\mathbf{T}_n(g^{(i)}(\Xi_n))$, where $\mathbf{T}_n(g^{(i)}(\Xi_n))$ is defined by Equation (3), and $g^{(i)}$ is assumed in $BVV(\mathbf{I})$, $\forall i \in \{1, \ldots, p\}$. By Theorem 5 in Adams and Clarkson [1], there exist non decreasing functions $g_1^{(i)}$ and $g_2^{(i)}$ on the compact set $\mathbf{I}$, such that $g^{(i)} = g_1^{(i)} - g_2^{(i)}$, for all i. Hence $g_1^{(i)}$ and $g_2^{(i)}$ are bounded respectively by $M_{i1} := \left|g_1^{(i)}(1,1)\right|$ and $M_{i2} := \left|g_2^{(i)}(1,1)\right|$ on $\mathbf{I}$. Let us define a sequence of step functions $S_n^{(i)}(t,s) := \sum_{\ell=1}^n \sum_{k=1}^n g^{(i)}(\ell/n, k/n)\mathbf{1}_{\mathbf{C}_{\ell k}}(t,s)$, for $(t,s) \in \mathbf{I}$, $n \geq 1$, where $\mathbf{1}_{\mathbf{A}}$ stands for the indicator of $\mathbf{A} \subseteq \mathbf{I}$, and $\mathbf{C}_{\ell k}$ is the half-open rectangle $((\ell-1)/n, \ell/n] \times ((k-1)/n, k/n]$, for $1 \leq \ell, k \leq n$. Similarly, for $n \geq 1$, let

$$S_{n1}^{(i)} := \sum_{\ell=1}^n \sum_{k=1}^n g_1^{(i)}(\ell/n, k/n)\mathbf{1}_{\mathbf{C}_{\ell k}}, \text{ and } S_{n2}^{(i)} := \sum_{\ell=1}^n \sum_{k=1}^n g_2^{(i)}(\ell/n, k/n)\mathbf{1}_{\mathbf{C}_{\ell k}}$$

be the sequence of step functions associated with $g_1^{(i)}$ and $g_2^{(i)}$, respectively. It is clear by the definition that $\left|S_{n1}^{(i)}\right| \leq M_{i1}$ and $\left|S_{n2}^{(i)}\right| \leq M_{i2}$, for all $n \geq 1$, and both $S_{n1}^{(i)}$ and $S_{n2}^{(i)}$ converge uniformly to $g_1^{(i)}$ and $g_2^{(i)}$, respectively, as $n \to \infty$. Hence, by using Lebesgue dominated convergence theorem (cf. Athreya and Lahiri [3] p.57),

$$\lim_{n\to\infty}\int_{[0,t]\times[0,s]} S_{n1}^{(i)}(x,y)\ \lambda(dx,dy) = \int_{[0,t]\times[0,s]} g_1^{(i)}(x,y)\lambda(dx,dy)$$
$$\lim_{n\to\infty}\int_{[0,t]\times[0,s]} S_{n2}^{(i)}(x,y)\lambda(dx,dy) = \int_{[0,t]\times[0,s]} g_2^{(i)}(x,y)\lambda(dx,dy).$$

On the other hand the computation of the Lebesgue integral of $S_n^{(i)}$ over the rectangle $[0,t]\times[0,s]$ results in the equality $\mathbf{T}_n(g^{(i)}(\Xi_n)/n)(t,s) = \int_{[0,t]\times[0,s]} S_n^{(i)}\ d\lambda$. This implies

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{T}_n(g^{(i)}(\Xi_n)/n)(t,s) = \lim_{n\to\infty}\int_{[0,t]\times[0,s]} S_n^{(i)}(x,y)\lambda(dx,dy)\\
&= \lim_{n\to\infty}\int_{[0,t]\times[0,s]} S_{n1}^{(i)}(x,y)\lambda(dx,dy) - \lim_{n\to\infty}\int_{[0,t]\times[0,s]} S_{n2}^{(i)}(x,y)\lambda(dx,dy)\\
&= \int_{[0,t]\times[0,s]} g_1^{(i)}(x,y)\lambda(dx,dy) - \int_{[0,t]\times[0,s]} g_2^{(i)}(x,y)\lambda(dx,dy)\\
&= \int_{[0,t]\times[0,s]} g^{(i)}(x,y)\lambda(dx,dy) = h_{g^{(i)}}(t,s).
\end{aligned}$$

We note that both $\mathbf{T}_n(g^{(i)}(\Xi_n)/n)$ and $h_{g^{(i)}}$ are absolutely continuous on $\mathbf{I}$ having the densities $S_n^{(i)}$ and $g^{(i)}$, respectively. Actually the convergence of $\mathbf{T}_n(g^{(i)}(\Xi_n)/n)$ to $h_{g^{(i)}}$ is uniformly convergence, since we have

$$\begin{aligned}
&\left\|\mathbf{T}_n(g^{(i)}(\Xi_n)/n) - h_{g^{(i)}}\right\|_\infty\\
&= \sup_{0\le t,s\le 1}\left|\int_{[0,t]\times[0,s]} S_n^{(i)}(x,y)\lambda(dx,dy) - \int_{[0,t]\times[0,s]} g^{(i)}(x,y)\lambda(dx,dy)\right|\\
&\le \sup_{0\le t,s\le 1}\int_{[0,t]\times[0,s]}\left|S_n^{(i)}(x,y) - g^{(i)}(x,y)\right|\lambda(dx,dy)\\
&\le \int_{\mathbf{I}}\left|(S_{n1}^{(i)} - S_{n2}^{(i)}) - (g_1^{(i)} - g_2^{(i)})\right| d\lambda\\
&\le \left\|S_{n1}^{(i)} - g_1^{(i)}\right\|_\infty + \left\|S_{n2}^{(i)} - g_2^{(i)}\right\|_\infty \to 0,\ \text{as } n\to\infty.
\end{aligned}$$

Next let $\mathbf{C} := (\mathbf{c}_1,\ldots,\mathbf{c}_p)^\top \in \mathbb{R}^{p\times p}$, where for $1\le i\le p$, $\mathbf{c}_i := (c_{i1},\ldots,c_{ip})^\top \in \mathbb{R}^p$. By the last result, and the condition that

$$\mathbf{C}\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p}) = \left(\sum_{i=1}^p c_{1i}\mathbf{T}_n(g^{(i)}(\Xi_n)/n),\ldots,\sum_{i=1}^p c_{pi}\mathbf{T}_n(g^{(i)}(\Xi_n)/n)\right)^\top$$
$$\mathbf{C}\mathbf{h}_g = \left(\sum_{i=1}^p c_{1i}h_{g^{(i)}},\ldots,\sum_{i=1}^p c_{pi}h_{g^{(i)}}\right)^\top,$$

then we further get for $n \to \infty$,

$$\rho\left(\mathbf{C}\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p}) - \mathbf{C}\mathbf{h}_g\right) = \sum_{i=1}^{p}\left\|\sum_{j=1}^{p} c_{ij}\mathbf{T}_n(g^{(j)}(\Xi_n)/n) - \sum_{j=1}^{p} c_{ij}h_{g^{(j)}}\right\|_\infty$$
$$\leq \sum_{i=1}^{p}\sum_{j=1}^{p}|c_{ij}|\left\|\mathbf{T}_n(g^{(j)}(\Xi_n)/n) - h_{g^{(j)}}\right\|_\infty \to 0.$$

This result leads us to the conclusion that

$$\mathbf{\Sigma}^{-1/2}\mathbf{T}_{n\times n\times p}(\mathbf{g}^{loc}_{n\times n\times p}) \to \mathbf{\Sigma}^{-1/2}\mathbf{h}_g, \text{ uniformly in } \mathcal{C}^p(\mathbf{I}) \text{ as } n \to \infty.$$

The rest terms of (7) can be handled by applying the similar technique as in the proof of Theorem 2.2. The second assertion of the theorem is a direct consequence of the continuous mapping theorem. It can be proven analogously with the proof of Corollary 2.3. □

Remark 2.5. *In the application* $\mathbf{\Sigma}$ *is sometimes unknown. In such a case* $\mathbf{\Sigma}$ *can be directly replaced with a consistent estimator without altering the asymptotic results, for example with that defined in* [2].

2.1. **Examples.** We present examples of the limit processes corresponding to the model under the null hypothesis.

2.1.1. *Constant regression model.* Let us consider $H_0 : g^{(i)} \in \mathbf{W} = [f_1]$, $i = 1, \ldots, p$, where $f_1(t,s) = 1$, for $(t,s) \in \mathbf{I}$. Then we get $\mathbf{W}^p = [\mathbf{f}_1]$, $\mathbf{W}^p_{\mathcal{H}_B} = [\mathbf{h}_1]$, where $\mathbf{f}_1 = \mathbf{1} := (1,\ldots,1)^\top \in \mathbb{R}^p$, $\mathbf{h}_1(t,s) = (ts,\ldots,ts)^\top$, for $(t,s) \in \mathbf{I}$. By using the definition of $P^*_{\mathbf{W}_{\mathcal{H}_B}}$, we have for every $\mathbf{u} \in \mathcal{C}^p(\mathbf{I})$, $P^*_{\mathbf{W}^p_{\mathcal{H}_B}}\mathbf{u} = \mathbf{u}(1,1)\mathbf{h}_1$, by the assumption $\mathbf{u}(t,s) = 0$, for $t = 0$ or $s = 0$. Hence, the limit of the sequence of the PSPR of this model is given by

$$B^p_{(\mathbf{f}_1)}(t,s) := B^p(t,s) - tsB^p(1,1),\ (t,s) \in \mathbf{I},$$

which is the p-variate Brownian pillow, having the covariance function

$$K_{(\mathbf{f}_1)}((t,s),(t',s')) = p\left[(t\wedge t')(s\wedge s') - tst's'\right], (t,s),(t',s') \in \mathbf{I}.$$

2.1.2. *First-order regression model.* Suppose under H_0 a first-order regression model is assumed, i.e. for $i = 1,\ldots,p$, $g^{(i)} \in \mathbf{W} = [f_1, f_2, f_3]$, where $f_1(t,s) = 1$, $f_2(t,s) = t$, and $f_3(t,s) = s$. The Gram-Schmidt ONB of $\mathbf{W}$ is $\widetilde{f}_1(t,s) = 1$, $\widetilde{f}_2(t,s) = \sqrt{3}(2t-1)$, and $\widetilde{f}_3(t,s) = \sqrt{3}(2s-1)$. So we have $\mathbf{W}^p = [\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2, \widetilde{\mathbf{f}}_3]$, where $\widetilde{\mathbf{f}}_1(t,s) = \mathbf{1}$, $\widetilde{\mathbf{f}}_2(t,s) = (\sqrt{3}(2t-1),\ldots,\sqrt{3}(2t-1))^\top$, and $\widetilde{\mathbf{f}}_3(t,s) = (\sqrt{3}(2s-1),\ldots,\sqrt{3}(2s-1))^\top$. Consequently $\mathbf{W}^p_{\mathcal{H}_B}$ is generated by $\mathbf{h}_1(t,s) = (ts,\ldots,ts)^\top$, $\mathbf{h}_2(t,s) = (\sqrt{3}ts(t-1),\ldots,\sqrt{3}ts(t-1))^\top$, and $\mathbf{h}_3(t,s) = (\sqrt{3}ts(s-1),\ldots,\sqrt{3}ts(s-1))^\top$, $(t,s) \in \mathbf{I}$. By the definition of the integral componentwise, we get for every

$\mathbf{u} \in \mathcal{C}^p(\mathbf{I})$,

$$P^*_{\mathbf{W}^p_{\mathcal{H}_B}} \mathbf{u} = \widetilde{\mathbf{f}}_1(1,1)\mathbf{u}(1,1)\mathbf{h}_1 + \left(\widetilde{\mathbf{f}}_2(1,1)\mathbf{u}(1,1) - 2\sqrt{3}\int_{[0,1]} \mathbf{u}(t,1)dt\right)\mathbf{h}_2 + \left(\widetilde{\mathbf{f}}_3(1,1)\mathbf{u}(1,1) - 2\sqrt{3}\int_{[0,1]} \mathbf{u}(1,s)ds\right)\mathbf{h}_3.$$

Thus the limit of the p-variate PSPR under H_0 is given by

$$B^p_{(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3)}(t,s) := B^p_{(\mathbf{f}_1)}(t,s) - 3ts(t+s-2)B^p(1,1) + 6ts(t-1)\int_{[0,1]} B^p(t,1)dt + 6ts(s-1)\int_{[0,1]} B^p(1,s)ds,$$

with the covariance function

$$K_{(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3)}((t,s),(t',s')) = p\left[(t\wedge t')(s\wedge s') - tst's' - 3tst's'(t-1)(t'-1) - 3tst's'(s-1)(s'-1)\right].$$

2.1.3. *Second-order regression model.* For the last example we consider $H_0 : g^{(i)} \in \mathbf{W} = [f_1, f_2, f_3, f_4, f_5, f_6]$, for $i = 1, \ldots, p$, where $f_1(t,s) = 1$, $f_2(t,s) = t$, $f_3(t,s) = s$, $f_4(t,s) = t^2$, $f_5(t,s) = ts$, and $f_6(t,s) = s^2$. The associated Gram-Schmidt ONB of $\mathbf{W}$ is $\widetilde{f}_1(t,s) = 1$, $\widetilde{f}_2(t,s) = \sqrt{3}(2t-1)$, $\widetilde{f}_3(t,s) = \sqrt{3}(2s-1)$, $\widetilde{f}_4(t,s) = \sqrt{5}(6t^2-6t+1)$, $\widetilde{f}_5(t,s) = \frac{1}{3}(4ts-2t-2s+1)$, $\widetilde{f}_6(t,s) = \sqrt{5}(6s^2-6s+1)$. Hence $\mathbf{W}^p = [\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2, \widetilde{\mathbf{f}}_3, \widetilde{\mathbf{f}}_4, \widetilde{\mathbf{f}}_5, \widetilde{\mathbf{f}}_6]$, and $\mathbf{W}^p_{\mathcal{H}_B} = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6]$, where $\widetilde{\mathbf{f}}_j = (\widetilde{f}_j, \ldots, \widetilde{f}_j)^\top$, and $\mathbf{h}_j(t,s) = \int_{[0,t]\times[0,s]} \widetilde{\mathbf{f}}_j \, d\lambda$, $j = 1, \ldots, 6$. Thus the limit process of the p-variate PSPR associated to this model is given by

$$\begin{aligned} B^p_{(\mathbf{f}_1,\ldots,\mathbf{f}_6)}(t,s) &= B^p_{(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3)}(t,s) \\ &- (10t^3s + t^2s^2/9 - 136t^2s/9 - 136ts^2/9 + 10ts^3 + 91ts)B^p(t,s) \\ &+ (120t^3s - 180t^2s + 60ts)\int_{[0,1]} B^p(x,1)xdx \\ &+ (2t^2s^2/9 - 60t^3s + 808t^2s/9 - 2ts^2/9 - 268ts/9)\int_{[0,1]} B^p(x,1)dx \\ &+ (2t^2s^2/9 - 2t^2s/9 + 808ts^2/9 - 60ts^3 - 268ts/9)\int_{[0,1]} B^p(1,y)dy \\ &+ (120ts^3 - 180ts^2 + 60ts)\int_{[0,1]} B^p(1,y)ydy \\ &- 4/9(t^2s^2 - t^2s - ts^2 + ts)\int_{\mathbf{I}} B^p(x,y)dxdy, \end{aligned}$$

with the covariance function

$$\begin{aligned}K_{(\mathbf{f}_1,\ldots,\mathbf{f}_6)}((t,s),(t',s')) = p\,[&(t\wedge t')(s\wedge s')\\ &-tst's' - 3tst's'(t-1)(t'-1) - 3tst's'(s-1)(s'-1)\\ &-5(2t^3s-3t^2s+ts)(2t'^3s'-3t'^2s'+t's')\\ &-\frac{1}{9}(t^2s^2-t^2s-ts^2+ts)(t'^2s'^2-t'^2s'-t's'^2+t's')\\ &-5(2ts^3-3ts^2+ts)(2t's'^3-3t's'^2+t's')\big]\end{aligned}$$

3. **Concluding Remarks**

The limit process of the sequence of the PSPR for multivariate linear regression model assumed under H_0 has been derived by applying the method proposed in [9]. As a by product, the limit process is the component-wise projection of the p-variate Brownian sheet onto its reproducing kernel Hilbert space. The experimental design under which the results have been determined sofar is given by a sequence of regular lattices. Our results however can also be immediately extended to a more general sampling scheme such as under the one proposed in [6]. The partial sums deal with in this paper are indexed by a family of rectangles with the origin $(0,0)$ as an essential corner of each rectangle. In a forthcoming paper by Somayasa, the limit of the multivariate PSPR indexed by a family of Borel sets is investigated [see also [19] for the univariate case].

Acknowledgement The author is grateful to the General Directorate of Higher Education of the Republic of Indonesia (DIKTI) for the financial support.

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