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Forgotten Topological Index of The Zero Divisor Graph for Some Rings of Integers

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Abstract. A topological index is a numerical value that provides information about the structure of a graph. Among various degree-based topological indices, the forgotten topological index (F-index) is of particular interest in this study. The F-index is calculated for the zero divisor graph of a ring R. In graph theory, the zero divisor graph of R is defined as a graph with vertex set the zero-divisors of R, and for distinct vertices a and b are adjacent if $a \cdot b = 0$. This research focuses on the zero divisor graph of the commutative ring of integers modulo $2\rho^n$ where ρ is an odd prime and n is a positive integer. The objectives are to determine the set of all zero divisors, analyze the vertex degrees of the graph, and then compute the F-index of the zero divisor graph. Using algebraic techniques, we derive the degree of each vertex, the distribution of vertex degrees, and the number of edges in the graph. The general expression for the F-index of the zero divisor graph for the ring is established. The results contribute to understanding topological indices for algebraic structures, with potential applications in chemical graph theory and related disciplines.

Key words and phrases: topological index, forgotten topological index, zero divisor, zero divisor graph, commutative ring

1. INTRODUCTION

There has been an increase in interest among researchers in graphs related to rings, especially commutative rings. Additionally, graphs of finite commutative rings can be used in robotics, information theory, elliptical curve cryptography, and

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physics [1]. In this article, we focus on the commutative ring of integers modulo $2\rho^n$, denoted by $\mathbb{Z}_{2\rho^n}$ where ρ is an odd prime and n is a positive integer. First, the zero-divisors of a ring are generally defined as follows:

Definition 1.1. [2] *A zero divisor in a ring R is an element a such that there exists a nonzero element b in R where $ab = 0$.*

In this research, a simple connected graph Γ of a ring R, denoted as $\Gamma(R)$, consisting of two finite sets, namely the set of vertices and the set of edges, is of concern. The zero divisor graph was proposed by Anderson and Livingston [3] in 1999, defined as follows:

Definition 1.2. [3] *Let R be a commutative ring (with unity), the graph $\Gamma(R)$ is defined as the graph with vertex set consisting of all zero divisors of R where two vertices a and b are adjacent if and only if $a \cdot b = 0$.*

The following lemma is needed in the proof of our result in Section 3.

Lemma 1.3. [4] *The sum of all the degrees of all the vertices of any graph Γ is equal to twice its size.*

We denote the set of all zero divisors in a ring R as $Z(R)$. Anderson and Weber [5] investigated $\Gamma(R)$ when R does not have an identity, and they determined all such zero divisor graphs with 14 or fewer vertices. In 2020, Cherrabi et al. [6] introduced a new graph extension of $\Gamma(R)$. Later, in 2022, the graphical structure and the adjacency spectrum of $\Gamma(\mathbb{Z}_n)$Bajaj and Panigrahi have studied in [7]. Recently, Hanif et al. [8] calculated the entropy measure of $\Gamma(R)$ for various rings.

Meanwhile, the topological index is a numerical value associated with the molecular graph, which exhibits a strong correlation with specific physical properties of a molecule. From that, the topological index is used to model different physicochemical properties and biological activities of chemical compounds [9]. This article focuses on a degree-based topological index, also known as F-index or the forgotten topological index. In 2015, Furtula and Gutman [10] introduced the F-index as follows:

Definition 1.4. [10] *Let Γ be a connected graph. Then, the F-index,*

$$FT(\Gamma) = \sum_{a \in V(\Gamma)} \delta(a)^3,$$

where $\delta(a)$ is the degree of vertex a and $V(\Gamma)$ is the number of vertices of Γ.

The F-index of the line graphs of several popular chemical structures, which is quite common in drug molecular graphs, has been presented by Mehak and Bhatti [11]. Cancan et al. [12] figured out the F-index of triangular cactus manacles, tetragonal cactus restraints, hexagonal cactus restraints, and polyomino restraints. Gursoy et al. [13] calculated the F-index of the ring $\mathbb{Z}_n$ where $n = p^\alpha, pq, p^2q, p^2q^2, pqr$.

2. THE ZERO DIVISORS IN $\mathbb{Z}_{2\rho^n}$

In this section, we show the set of zero divisors and calculate the number of zero divisors in $\mathbb{Z}_{2\rho^n}$, given in the following propositions.

Proposition 2.1. *The set of all zero divisors in $\mathbb{Z}_{2\rho^n}$,*

$$Z\left(\mathbb{Z}_{2\rho^n}\right) = \{2, 4, 6, \ldots, 2\left(\rho^n - 1\right)\} \cup \{\rho, 2\rho, 3\rho, \ldots, \rho\left(2\rho^{n-1} - 1\right)\}.$$

Proof. Let $a \in Z\left(\mathbb{Z}_{2p^n}\right)$ satisfy the given conditions:

(a) $\gcd(a, 2) > 1$ and A_1 is the set of all zero divisors of a. Then, $A_1 = \{2, 4, 6, \ldots, 2\left(\rho^n - 1\right)\}$ with cardinality $\left(\rho^n - 1\right)$.
(b) $\gcd(a, \rho) > 1$ and A_2 is the set of all zero divisors of a. Then, $A_2 = \{\rho, 2\rho, 3\rho, \ldots, \rho\left(2\rho^{n-1} - 1\right)\}$ with cardinality $\left(2\rho^{n-1} - 1\right)$.
(c) $\gcd(a, 2\rho^n) = 2\rho$ and $A_1 \cap A_2$ are the set of all zero divisors of a. Then, $A_1 \cap A_2 = \{2\rho, 4\rho, 6\rho, \ldots, 2\rho\left(\rho^{n-1} - 1\right)\}$ with cardinality $\left(\rho^{n-1} - 1\right)$.

Thus, $Z\left(\mathbb{Z}_{2\rho^n}\right) = A_1 \cup A_2 = \{2, 4, 6, \ldots, 2\left(\rho^n - 1\right)\} \cup \{\rho, 2\rho, 3\rho, \ldots, \rho\left(2\rho^{n-1} - 1\right)\}$. □

Proposition 2.2. *The number of zero divisors in $\mathbb{Z}_{2\rho^n}$, $\left|Z\left(\mathbb{Z}_{2\rho^n}\right)\right| = \rho^n + \rho^{n-1} - 1$.*

Proof. By using the inclusion-exclusion principle, $\left|Z\left(\mathbb{Z}_{2\rho^n}\right)\right| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. Then, using Proposition 2.1 with their cardinalities, $\left|Z\left(\mathbb{Z}_{2\rho^n}\right)\right| = \left(\rho^n - 1\right) + \left(2\rho^{n-1} - 1\right) - \left(\rho^{n-1} - 1\right) = \rho^n + \rho^{n-1} - 1$. □

3. THE CONSTRUCTION OF THE ZERO DIVISOR GRAPH FOR $\mathbb{Z}_{2\rho^n}$

In this section, the degree of each vertex, the number of vertices and the number of edges of the zero divisor graph of $\mathbb{Z}_{2\rho^n}$, denoted by $\Gamma(\mathbb{Z}_{2\rho^n})$, are determined. First, the degree of each vertex in $\Gamma(\mathbb{Z}_{2\rho^n})$ is provided in Proposition 3.1, Proposition 3.2 and Proposition 3.3.

Proposition 3.1. *Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $gcd\left(a, 2\rho^n\right) = 2$,then $deg\left(a\right) = 1$.*

Proof. Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(a, 2\rho^n) = 2$, and let $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(b, 2\rho^n) = \rho^j$ where a and b are adjacent if and only if $j = n$. Since $\gcd(b, 2\rho^n) = \rho^j$ and $j = n$, so $b \in \rho^n\mathbb{Z}_{2\rho^n}$ and $\left|\rho^n\mathbb{Z}_{2\rho^n}\right| = \left|\rho^n\{0, 1, 2, 3, \ldots, (2\rho^n - 1)\}\right| = \frac{2\rho^n}{\rho^n} - 1 = 2 - 1 = 1$. Thus, since $0 \notin Z\left(\mathbb{Z}_{2\rho^n}\right)$, so $\deg\left(a\right) = 1$ with $\gcd(a, 2\rho^n) = 2$. □

Proposition 3.2. *Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $gcd(a, 2\rho^n) = \rho^i$, then $\deg\left(u\right) = \rho^i - 1$ for $i = 1, 2, ..., n$.*

Proof. Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(a, 2\rho^n) = p^i$ and let $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(b, 2\rho^n) = 2\rho^j$, $\gcd(b, 2\rho^n) = \rho^j$ or $\gcd(b, 2\rho^n) = 2$ where a and b are adjacent if and only if $i + j \geq n$. Since $\gcd(b, 2\rho^n) = 2\rho^j$, $\gcd(b, 2\rho^n) = \rho^j$ or $\gcd(b, 2\rho^n) = 2$ where $j \geq n - i$, so $b \in 2\rho^{n-i}\mathbb{Z}_{2\rho^n}$ and $\left|2\rho^{n-i}\mathbb{Z}_{2\rho^n}\right| = \left|2\rho^{n-i}\{0, 1, 2, 3, \ldots, (2\rho^n - 1)\}\right| = \frac{2\rho^n}{2\rho^{n-i}} - 1 = \frac{1}{\rho^{-i}} - 1 = \rho^i - 1$. Thus, $\deg\left(a\right) = \rho^i - 1$ with $\gcd(a, 2\rho^n) = \rho^i$ for $i = 1, 2, ..., n$. □

Proposition 3.3. *Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $gcd(a, 2\rho^n) = 2\rho^i$, then*

$$\deg(a) = \begin{cases} 2\rho^i - 1 & , \text{ for } i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\ 2\left(\rho^i - 1\right) & , \text{ for } i > \left\lfloor \frac{n-1}{2} \right\rfloor. \end{cases}$$

Proof. The proof is divided into two cases:

(1) Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(a, 2\rho^n) = 2\rho^i$ and let $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(b, 2\rho^n) = 2\rho^j$ and $\gcd(b, 2\rho^n) = \rho^j$ where a and b are adjacent if and only if $i + j \geq n$. Since $\gcd(b, 2\rho^n) = 2\rho^j$, $\gcd(b, 2\rho^n) = \rho^j$ and $j \geq n - i$, so $b \in \rho^{n-i}\mathbb{Z}_{2\rho^n}$ and $\left|\rho^{n-i}\mathbb{Z}_{2\rho^n}\right| = \left|\rho^{n-i}\{0, 1, 2, 3, \ldots, (2\rho^n - 1)\}\right| = \frac{2\rho^n}{\rho^{n-i}} - 1 = \frac{2}{\rho^{-i}} - 1 = 2\rho^i - 1$. Thus, $\deg(a) = 2\rho^i - 1$ for $i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

(2) Let $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(a, 2\rho^n) = 2\rho^i$ and let $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ with $\gcd(b, 2\rho^n) = 2\rho^j$, $\gcd(b, 2\rho^n) = \rho^j$ or $\gcd(b, 2p^n) = 2$ where a and b are adjacent if and only if $i + j \geq n$. Since $\gcd(b, 2\rho^n) = 2\rho^j$, $\gcd(b, 2\rho^n) = \rho^j$ or $\gcd(b, 2\rho^n) = 2$ where $j \geq n - i$, so $b \in \rho^{n-i}\mathbb{Z}_{2\rho^n}$ and $\left|\rho^{n-i}\mathbb{Z}_{2\rho^n}\right| = \left|\rho^{n-i}\{0, 1, 2, 3, \ldots, (2\rho^n - 1)\}\right| = \frac{2\rho^n}{\rho^{n-i}} - 2 = \frac{2}{\rho^{-i}} - 2 = 2\left(\rho^i - 1\right)$. Thus, since $0 \notin Z\left(\mathbb{Z}_{2\rho^n}\right)$ and $\gcd\left(2\rho^i, 2\rho^n\right) = 2\rho^i$, so $\deg(a) = 2\left(\rho^i - 1\right)$ for $i > \left\lfloor \frac{n-1}{2} \right\rfloor$.

□

Next, the number of vertices in $\Gamma\left(\mathbb{Z}_{2\rho^n}\right)$ for a given degree is divided into three cases, given in Propositions 3.4, 3.5, and 3.6.

Proposition 3.4. *Let $a \in V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)$, thus $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where $gcd(a, 2\rho^n) = 2$. Then $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^n - \rho^{n-i}\right)$ for $i = 1$.*

Proof. Given $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where $\gcd(a, 2\rho^n) = 2$. Then $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^n - \rho^{n-i}\right)$ for $i = 1$ and $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where gcd $(b, 2) = 1$ then $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = 1$. So $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^k - \rho^{k-i}\right)(1) = \left(\rho^k - \rho^{k-i}\right)$. □

Proposition 3.5. *Let $a \in V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)$, thus $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where $gcd(a, 2\rho^n) = \rho^i$. Then*

$$\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \begin{cases} \left(\rho^{n-i} - \rho^{n-(i+1)}\right), for\ 1 \leq i \leq n - 1, \\ \left(\rho^{n-i}\right), \qquad\qquad\quad for\ i = n. \end{cases}$$

Proof. Given $a \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where $\gcd(a, 2\rho^n) = \rho^i$. Then

(1) $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^{n-i} - \rho^{n-(i+1)}\right)$ for $1 \leq i \leq n - 1$ and $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where $\gcd(b, 2) = 2$. Then $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = 2^1 - 2^0 = 1$. So $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^{n-i} - \rho^{n-(i+1)}\right)(1) = \left(\rho^{n-i} - \rho^{n-(i+1)}\right)$.

(2) $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^{n-i}\right)$ for $i = n$ and $b \in Z\left(\mathbb{Z}_{2\rho^n}\right)$ where $\gcd(b, 2) = 2$. So $\left|V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)\right| = \left(\rho^{n-i}\right)$.

□

Proposition 3.6. *Let $a \in V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)$, thus $a \in Z\left(\mathbb{Z}_{2p^n}\right)$ where $gcd(a, 2p^n) = 2p^i$ for $1 \leq i \leq n - 1$.*

Proof. Given $a \in Z(\mathbb{Z}_{2\rho^n})$ where $\gcd(a, 2\rho^n) = 2\rho^i$. Then $|V(\Gamma(\mathbb{Z}_{2\rho^n}))| = \left(\rho^{n-i} - \rho^{n-(i+1)}\right)$ for $1 \le i \le n-1$ and $b \in \mathbb{Z}_{2\rho^n}$, where $\gcd(b, 2) = 1$. So $|V(\Gamma(\mathbb{Z}_{2\rho^n}))| = \left(\rho^{n-i} - \rho^{n-(i+1)}\right)$. □

The following theorem determines the number of edges in $\Gamma(\mathbb{Z}_{2\rho^n})$.

Theorem 3.7. *The number of edges in* $\Gamma(\mathbb{Z}_{2\rho^n})$,

$$|E(\Gamma(\mathbb{Z}_{2\rho^n}))| = \frac{1}{2}\left[\left(\rho^n - \rho^{n-1}\right)\left(3n - 2 - \frac{2\left(\rho^{n-1}-1\right)}{\rho^{n-1}(\rho-1)} - \frac{\rho^{n-1} - \rho^{\left\lfloor \frac{n-1}{2}\right\rfloor}}{\rho^{\left\lfloor \frac{3(n-1)}{2}\right\rfloor}(\rho-1)}\right) + \rho^n - 1\right].$$

Proof. Using Lemma 1.3 and utilising from Proposition 3.1 to Proposition 3.6, we get

$$\begin{aligned}
|E(\Gamma(\mathbb{Z}_{2\rho^n}))| &= \frac{1}{2}\sum_{a \in V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)} \delta(a) \\
&= \frac{1}{2}\Bigg[\left(\rho^n - \rho^{n-1}\right) + \sum_{i=1}^{n-1}\left(\rho^{n-i} - \rho^{n-(i+1)}\right)\left(\rho^i - 1\right) \\
&\quad + \sum_{i=1+n-1}^{n} \rho^{n-i}\left(\rho^i - 1\right) + \sum_{i=1}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\left(\rho^{n-i} - \rho^{n-(i+1)}\right)\left(2\rho^i - 1\right) \\
&\quad + 2\sum_{i=1+\left\lfloor \frac{n-1}{2}\right\rfloor}^{n-1}\left(\rho^{n-i} - \rho^{n-(i+1)}\right)\left(\rho^i - 1\right)\Bigg].
\end{aligned}$$

By employing summation rules and geometric sequences, a simplified general expression is derived as follows:

$$|E(\Gamma(\mathbb{Z}_{2\rho^n}))| = \frac{1}{2}\left[\left(\rho^n - \rho^{n-1}\right)\left(3n - 2 - \frac{2\left(\rho^{n-1}-1\right)}{\rho^{n-1}(\rho-1)} - \frac{\rho^{n-1} - \rho^{\left\lfloor \frac{n-1}{2}\right\rfloor}}{\rho^{\left\lfloor \frac{3(n-1)}{2}\right\rfloor}(\rho-1)}\right) + \rho^n - 1\right].$$

□

4. MAIN RESULTS

Finally, the following theorem presents our main result in this paper, namely the F-index of the zero divisor graph for the ring $\mathbb{Z}_{2\rho^n}$.

Theorem 4.1. *The F-index of* $\Gamma(\mathbb{Z}_{2\rho^n})$,

$$\begin{aligned}
FT(\Gamma(\mathbb{Z}_{2\rho^n})) = \left(\rho^n - \rho^{n-1}\right)\Bigg[&\frac{9\rho^2\left(\rho^{2(n-1)} - 1\right)}{\rho^2 - 1} - \frac{3\rho\left(\rho^{n-1} - 1\right)}{\rho - 1} - \frac{2\left(\rho^{n-1} - 1\right)}{\rho^{n-1}(\rho - 1)} \\
&- \frac{12\rho^{\left\lfloor \frac{n+1}{2}\right\rfloor}\left(\rho^{\left\lceil \frac{n-1}{2}\right\rceil} - 1\right)}{\rho - 1} - \frac{12\rho\left(\rho^{n-1} - 1\right)}{\rho - 1} + 18\left\lceil \frac{n-1}{2}\right\rceil + 1 \\
&+ 9(n-1) - \frac{7\left(\rho^{n-1} - \rho^{\left\lfloor \frac{n-1}{2}\right\rfloor}\right)}{\rho^{\left\lfloor \frac{3n-1}{2}\right\rfloor}(\rho - 1)}\Bigg] + (\rho^n - 1)^3.
\end{aligned}$$

Proof. By applying Definition 1.4 and using Proposition 3.1 to Proposition 3.6, we obtain

$$\begin{aligned}
FT\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right) &= \sum_{a\in V\left(\Gamma\left(\mathbb{Z}_{2\rho^n}\right)\right)} \left(\delta\left(a\right)\right)^3 \\
&= \left(\rho^n - \rho^{n-1}\right) + \sum_{i=1}^{n-1}\left(\rho^{n-i} - \rho^{n-(i+1)}\right)\left(\rho^i - 1\right)^3 \\
&+ \sum_{i=1+n-1}^{n} \rho^{n-i}\left(v^i - 1\right)^3 + \sum_{i=1}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\left(\rho^{n-i} - \rho^{n-(i+1)}\right)\left(2\rho^i - 1\right)^3 \\
&+2\sum_{i=1+\left\lfloor \frac{n-1}{2}\right\rfloor}^{n-1}\left(\rho^{n-i} - \rho^{n-(i+1)}\right)\left(\rho^i - 1\right)^3 \\
&= \left(\rho^n - \rho^{n-1}\right)\left[\frac{9\rho^2\left(\rho^{2(n-1)} - 1\right)}{\rho^2 - 1} - \frac{3\rho\left(\rho^{n-1} - 1\right)}{\rho - 1} - \frac{2\left(\rho^{n-1} - 1\right)}{\rho^{n-1}\left(\rho - 1\right)}\right. \\
&- \frac{12\rho^{\left\lfloor \frac{n+1}{2}\right\rfloor}\left(\rho^{\left\lceil \frac{n-1}{2}\right\rceil} - 1\right)}{\rho - 1} - \frac{12\rho\left(\rho^{n-1} - 1\right)}{\rho - 1} + 18\left\lceil \frac{n-1}{2}\right\rceil + 1 \\
&\left. +9\left(n-1\right) - \frac{7\left(\rho^{n-1} - \rho^{\left\lfloor \frac{n-1}{2}\right\rfloor}\right)}{\rho^{\left\lfloor \frac{3n-1}{2}\right\rfloor}\left(\rho - 1\right)}\right] + \left(\rho^n - 1\right)^3.
\end{aligned}$$

□

An example of $\mathbb{Z}_{50}$ is provided in the following in computing its set of vertices, number of edges, and F-index using the definitions, propositions, and theorem.

Example 4.2. Using Definition 1.2, the vertices of $\Gamma\left(\mathbb{Z}_{50}\right)$ can be found which are $2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 25, 26, 28, 30, 32, 34, 35, 36, 38, 40, 42, 44, 45,$ 46 and 48, as shown in Figure 1. The edges are also found using Definition 1.2. From Figure 1, $\left|V\left(\Gamma\left(\mathbb{Z}_{50}\right)\right)\right| = 29$ and $\left|E\left(\Gamma\left(\mathbb{Z}_{50}\right)\right)\right| = 46$.

By Proposition 2.2 with $\rho = 5$ and $n = 2$, the number of zero divisors for $\Gamma\left(\mathbb{Z}_{50}\right)$ is $5^2 + 5^{2-1} - 1 = 29$. While by Theorem 3.7,

$$\begin{aligned}
\left|E\left(\Gamma\left(\mathbb{Z}_{50}\right)\right)\right| &= \frac{1}{2}\left[\left(5^2 - 5^{2-1}\right)\left(3(2) - 2 - \frac{2\left(5^{2-1} - 1\right)}{5^{2-1}\left(5 - 1\right)} - \frac{5^{2-1} - 5^{\left\lfloor \frac{2-1}{2}\right\rfloor}}{5^{\left\lfloor \frac{3(2-1)}{2}\right\rfloor}\left(5 - 1\right)}\right)\right. \\
&\left. + 5^2 - 1\right] \\
&= 46.
\end{aligned}$$

By Definition 1.4, the F-index can be computed as follows:

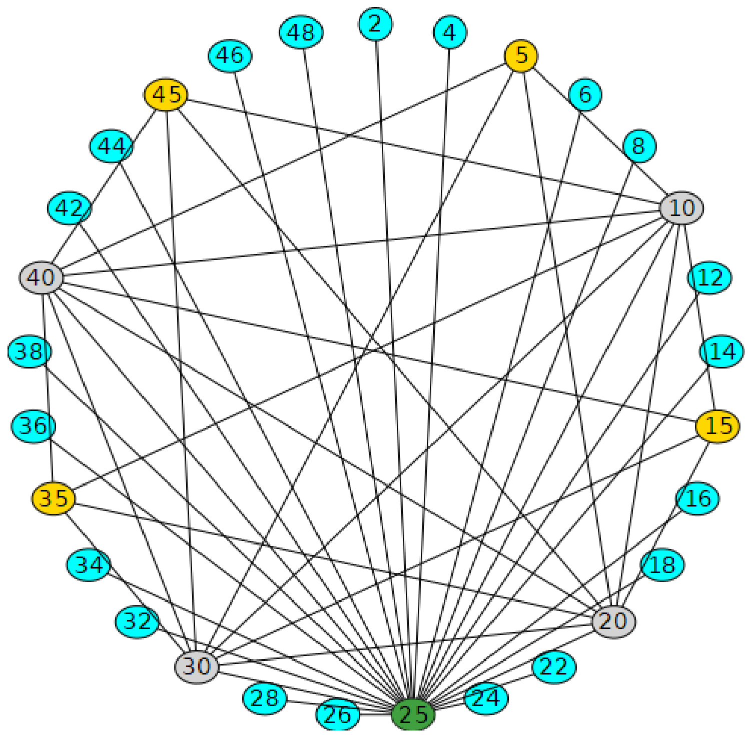

FIGURE 1. The zero divisor graph for $\mathbb{Z}_{50}$, $\Gamma(\mathbb{Z}_{50})$

$$\begin{aligned}
FT(\Gamma(\mathbb{Z}_{50})) &= \sum_{a\in V(\Gamma(\mathbb{Z}_{50}))} \delta(a)^3 \\
&= (\delta(2))^3 + (\delta(4))^3 + (\delta(5))^3 + (\delta(6))^3 + (\delta(8))^3 \\
&\quad + (\delta(10))^3 + (\delta(12))^3 + (\delta(14))^3 + (\delta(15))^3 + (\delta(16))^3 \\
&\quad + (\delta(18))^3 + (\delta(20))^3 + (\delta(22))^3 + (\delta(24))^3 + (\delta(25))^3 \\
&\quad + (\delta(26))^3 + (\delta(28))^3 + (\delta(30))^3 + (\delta(32))^3 + (\delta(34))^3 \\
&\quad + (\delta(35))^3 + (\delta(36))^3 + (\delta(38))^3 + (\delta(40))^3 + (\delta(42))^3 \\
&\quad + (\delta(44))^3 + (\delta(45))^3 + (\delta(46))^3 + (\delta(48))^3 \\
&= 16148.
\end{aligned}$$

On the other hand, using Theorem 4.1,

$$FT(\Gamma(\mathbb{Z}_{50})) = \left(5^2 - 5^{2-1}\right)\left[\frac{9\left(5^2\right)\left(5^{2(2-1)}-1\right)}{5^2-1} - \frac{3(5)\left(5^{2-1}-1\right)}{5-1} - \frac{2\left(5^{2-1}-1\right)}{5^{2-1}(5-1)}\right.$$

$$
\begin{aligned}
&- \frac{12\left(5^{\left\lfloor \frac{2+1}{2} \right\rfloor}\right)\left(5^{\left\lceil \frac{2-1}{2} \right\rceil}-1\right)}{5-1} - \frac{12(5)\left(5^{2-1}-1\right)}{5-1} + 18\left\lceil \frac{2-1}{2} \right\rceil + 1 \\
&+ 9\left(2-1\right) - \left.\frac{7\left(5^{2-1}-5^{\left\lfloor \frac{2-1}{2} \right\rfloor}\right)}{5^{\left\lfloor \frac{3(2-1)}{2} \right\rfloor}\left(5-1\right)}\right] + \left(5^2-1\right)^3 \\
&= 16148.
\end{aligned}
$$

The answers derived from the example above have shown that the results are identical using both methods.

5. CONCLUDING REMARKS

In this article, the general expression of the F-index of the zero divisors for the commutative ring $\mathbb{Z}_{2\rho^n}$ has been established. We also provided an example for $FT\left(\Gamma\left(\mathbb{Z}_{50}\right)\right)$ to illustrate the main findings.

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REFERENCES

[1] K. Elahi, A. Ahmad, and R. Hasni, "Construction algorithm for zero divisor graphs of finite commutative rings and their vertex-based eccentric topological indices," *Mathematics*, vol. 6, no. 12, p. 301, 2018. `https://doi.org/10.3390/math6120301`.

[2] J. B. Fraleigh, *A first course in abstract algebra*. Pearson Education India, 2003.

[3] D. Anderson and P. Livingston, "The zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, pp. 434–447, 1999. `https://doi.org/10.1006/jabr.1998.7840`.

[4] D. S. Gunderson and K. H. Rosen, *Handbook of mathematical induction: Theory and applications*. CRC Press, 2014.

[5] D. F. Anderson and D. Weber, "The zero-divisor graph of a commutative ring without identity," *International Electronic Journal of Algebra*, vol. 23, no. 23, pp. 176–202, 2018.

[6] A. Cherrabi, H. Essannouni, E. Jabbouri, and A. Ouadfel, "On a new extension of the zero-divisor graph," in *Algebra Colloquium*, vol. 27, pp. 469–476, World Scientific, 2020. `https://doi.org/10.1142/S1005386720000383`.

[7] S. Bajaj and P. Panigrahi, "On the adjacency spectrum of zero divisor graph of ring $\mathbb{Z}_n$," *Journal of Algebra and Its Applications*, vol. 21, no. 10, p. 2250197, 2022. `https://doi.org/10.1142/S0219498822501973`.

[8] M. F. Hanif, H. Mahmood, and S. Ahmad, "On degree-based entropy measure for zero-divisor graphs," *Discrete Mathematics, Algorithms and Applications*, vol. 16, no. 08, p. 2350104, 2024. `https://doi.org/10.1142/S1793830923501045`.

[9] S. Mondal, N. De, and A. Pal, "Topological indices of some chemical structures applied for the treatment of covid-19 patients," *Polycyclic Aromatic Compounds*, vol. 42, no. 4, pp. 1220–1234, 2022. `https://doi.org/10.1080/10406638.2020.1770306`.

[10] B. Furtula and I. Gutman, "A forgotten topological index," *Journal of Mathematical Chemistry*, vol. 53, pp. 1184–1190, 2015. `https://doi.org/10.1007/s10910-015-0480-z`.

[11] G. E. Mehak and A. A. Bhatti, "Forgotten topological index of line graphs of some chemical structures in drugs," *Acta Chemica Iasi*, 2018.

[12] M. Cancan, M. Imran, S. Akhter, M. K. Siddiqui, and M. F. Hanif, "Computing forgotten topological index of extremal cactus chains," *Applied Mathematics and Nonlinear Sciences*, vol. 6, no. 1, pp. 439–446, 2021. `https://doi.org/10.2478/amns.2020.2.00075`.

[13] A. Gursoy, N. K. Gursoy, and A. Ulker, "Computing forgotten topological index of zero-divisor graphs of commutative rings," *Turkish Journal of Mathematics*, vol. 46, no. 5, pp. 1845–1863, 2022. `https://doi.org/10.55730/1300-0098.3236`.