J. Indones. Math. Soc.
Vol. 30, No. 02 (2024), pp. 205–217.

# LOAN BENCHMARK INTEREST RATE IN BANKING DUOPOLY MODEL WITH HETEROGENEOUS EXPECTATION

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**Abstract.** A loan benchmark interest rate policy always becomes a challenging problem in the banking industry since it has a role in controlling bank loan expansion, especially when there is competition between two banks. This paper aims to assess the influence of the loan benchmark interest rate on the expansion of loans between two banks. We present a banking duopoly model in the form of two-dimensional difference equations which is constructed from heterogeneous expectation, where one of the banks sets its optimal loan volume based on the other bank's rational expectation. The model's equilibrium is investigated, and its stability is analyzed using the Jury stability condition. Investigation indicates that to ensure the stability of the banking loan equilibrium, it is advisable to establish a loan benchmark interest rate that is lower than the flip bifurcation value. Some numerical simulations, such as the bifurcation diagram, Lyapunov exponent, and chaotic attractor, are presented to confirm the analytical findings.

*Key words and Phrases*: banking duopoly, benchmark rate, bifurcation, chaos, heterogeneous

## 1. INTRODUCTION

The benchmark interest rate for loans set by the central bank, also known as the policy rate or the key interest rate, is a crucial tool used by central banks to affect a nation's overall economic situation and financial stability. It is crucial to monetary policy [1] and acts as the foundation or benchmark for interest rates in the larger economy. To maintain price stability, economic expansion, and full employment, central banks manage a country's money supply. The management of interest rates is one of their key strategies for achieving these goals. The benchmark interest rate on loans set by the central bank is the amount at which it extends credit to businesses and financial organizations. It effectively determines the cost of

*2020 Mathematics Subject Classification*: 39A60, 39A33
Received: 16-09-2023, accepted: 23-08-2024.

borrowing for these banks, which in turn affects the cost of borrowing for consumers, businesses, and the whole economy.

Due to the global low-interest rate environment that has been compounded by economic unpredictability, traditional monetary policies are now much less successful than they once were [2]. Consequently, monetary and interest rate policies instruments have been utilised by central banks in various countries to foster economic expansion [3]. Through the release of quantitative projections for upcoming policy rates, the central bank offers prospective direction, that boosts accountability and openness and may help a central bank better control interest rate expectations [4]. Every change in the benchmark interest rate indicates the likely course of upcoming interest rates: households base their borrowing and lending choices on anticipated borrowing and lending costs, and businesses use central bank statements to reassess their business plan in light of the economy's projected future course [5].

This paper examines banking loan duopoly dynamics with loan benchmark interest rates. One bank has bounded rational expectations on its calculation of the loan disbursement, meanwhile, the other bank determines its optimal loan disbursement based on its opponent. Our model is based on the banking duopoly model introduced by Fanti in [6]. Fanti used the banking loan duopoly model to study the impact of capital regulations. Following Fanti, Brianzoni et al. studied the model by Fanti to assess the role of capital regulations with asymmetric costs and also nonlinear interest rates [7, 8]. The banking duopoly model is also related to the economic duopoly game model, such as [9, 10, 11, 12, 13, 14, 15]. Our proposed model is also inspired by the monopoly banking loan model to study various the central bank's policies, for examples [16, 17, 18, 19, 20, 21, 22, 23]. Our proposed model contributes to literature and the result provides a recommendation to the central bank in determining their benchmark interest rate.

## 2. BANKING DUOPOLY MODEL

A banking duopoly refers to a situation in which two banks hold significant dominance over a market and engage in competition with one another. Suppose, there are two banks, namely with index $i = 1, 2$, and suppose $j \in \{1, 2\}\backslash\{i\}$. Suppose that each bank has a balance sheet that consists of deposit ($D_i$) and equity ($E_i$) on the funding side, and reserve requirement ($R_i$) and loan ($L_i$) on the financing side. The identity of balance sheet gives

$$L_i + R_i = D_i + E_i. \tag{1}$$

The reserve requirement (RR) is a fraction of the deposit that must be placed by the bank in the central bank as a part of monetary or macroprudential policy. Thus, we can write $R_i = \rho D_i$, where $0 < \rho < 1$. This fraction of money of deposits ($\rho D_i$) can not be channelled into loans. Meanwhile, the bank's equity satisfies the capital adequacy ratio (CAR), which is a policy that requires each bank to have an equity-to-risk-weighted assets ratio that is not less than a certain portion set by the regulator. Since the bank's assets only consist of loan and reserve requirements, then we can write $E_i/L_i \geq \kappa$, where $0 < \kappa < 1$. To simplify the model, we assume that $E_i = \kappa L_i$. Thus, eq. (1) becomes

$$D_i = \left(\frac{1-\kappa}{1-\rho}\right) L_i. \tag{2}$$

There is no relationship between the parameter $\kappa$ of the CAR policy and the parameter $\rho$ of the RR policy in banking practice. The eq. (2) can be understood in the following manner. If the condition $\kappa = \rho$ holds, then the equation $L_i = D_i$ implies that the banks will allocate loans in the same quantity as the deposit. If the value of $\kappa$ is less than the value of $\rho$, then the value of $L_i$ is less than the value of $D_i$. When the value of $\kappa$ is greater than $\rho$, it follows that $L_i$ is greater than $D_i$. Can a bank allocate loans that are equal to or greater than the amount of deposits? Affirmative. The bank will utilize a portion of the deposits, specifically $(1-\rho)D_i$, as well as a portion from its capital or equity.

The bank's profit is calculated by

$$\pi_i = r_L L_i - r_D D_i - r_E E_i - C_i, \tag{3}$$

where $r_L$ is loan interest rate, $r_D$ is deposit interest rate, $r_E$ is equity cost, and $C_i$ is the bank's operating cost. It is supposed that the loan interest rate is a linear demand function of the loan, $r_L = a_0 + a_L - b_L(L_i + L_j)$, where $0 < a_0 < 1$ is the benchmark rate, and $0 < a_L, b_L < 1$ is a constant parameter. The parameter $a_0$ is the central topic of this paper. We assume that $0 < r_D, r_E < 1$ are constant, and $C_i = c_D D_i + c_L L_i$, where $0 < c_D, c_L < 1$. Using (2) and the above assumptions, the bank's profit in (3) becomes

$$\pi_i = \left(a_0 + a_L - \left[(r_D + c_D)\left(\frac{1-\kappa}{1-\rho}\right) + r_E\kappa + c_L\right]\right) L_i - b_L L_i^2 - b_L L_i L_j. \tag{4}$$

Following the duopoly version of Monti-Klein model [24, 25, 26], the best strategy of bank $i$ to reply to the bank $j$ is by setting the marginal profit equals to zero, that is

$$\begin{aligned}
\frac{\partial \pi_i}{\partial L_i} &= a_0 + a_L - \left[(r_D + c_D)\left(\frac{1-\kappa}{1-\rho}\right) + r_E\kappa + c_L\right] - 2b_L L_i - b_L L_j = 0 \\
\Leftrightarrow L_i &= \frac{1}{2b_L}\left(a_0 + a_L - \left[(r_D + c_D)\left(\frac{1-\kappa}{1-\rho}\right) + r_E\kappa + c_L\right] - b_L L_j\right)
\end{aligned} \tag{5}$$

Meanwhile, instead of like that, the bank $j$ strategy is based on bounded rational expectation [27, 6] that described by

$$\begin{aligned}
L_{j,t+1} &= L_{j,t} + \alpha_L L_{j,t}\frac{\partial \pi_{j,t}}{\partial L_{j,t}} \\
&= L_{j,t} + \alpha_L L_{j,t}\left(a_0 + a_L - \left[(r_D + c_D)\left(\frac{1-\kappa}{1-\rho}\right) + r_E\kappa + c_L\right] - 2b_L L_{j,t} - b_L L_{i,t}\right),
\end{aligned} \tag{6}$$

where $\alpha_L > 0$. This parameter can be also viewed as the banks' procyclicality. Procyclicality is the banks' behaviour to follow a business cycle, that is they will channel more loans when the economy is in good trends, and will channel fewer loans when the economy is in not good trends.

Since the strategy of bank $i$ and bank $j$ is different, that is why it is called heterogeneous expectation. If we combine the model in (5) and (6), then we have a two-dimensional system of difference equations as follows

$$\begin{cases} L_{i,t+1} &= \frac{1}{2b_L}\left(a_0 + \Lambda - b_L L_{j,t}\right) \\ L_{j,t+1} &= L_{j,t} + \alpha_L L_{j,t}\left(a_0 + \Lambda - 2b_L L_{j,t} - b_L L_{i,t}\right) \end{cases} \tag{7}$$

where $\Lambda = a_L - \left[(r_D + c_D)\left(\frac{1-\kappa}{1-\rho}\right) + r_E\kappa + c_L\right]$. Next, we study the loan benchmark interest rate parameter $a_0$ incorporating with the banking loan stability.

## 3. ANALYSIS

The system (7) has two equilibrium points $P = (L_i^*, L_j^*)$, namely $P_1 = \left(\frac{a_0+\Lambda}{2b_L}, 0\right)$ and $P_2 = \left(\frac{a_0+\Lambda}{3b_L}, \frac{a_0+\Lambda}{3b_L}\right)$. The equilibrium point $P_1$ means that there exists only one bank i.e. the bank $i$. The positivity condition for equilibrium point $P_2$ is $a_0 + \Lambda > 0$, or

$$a_0 + a_L > (r_D + c_D)\left(\frac{1-\kappa}{1-\rho}\right) + r_E\kappa + c_L. \tag{8}$$

First, the Jacobian matrix of system (7) is calculated at point $P = (L_i^*, L_j^*)$ to examine the local stability of $P$.

$$J(P) = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\alpha_L b_L L_j^* & 1 + \alpha_L[a_0 + \Lambda - 4b_L L_j^* - b_L L_i^*] \end{bmatrix} \tag{9}$$

**Theorem 3.1.** *The equilibrium point $P_1$ is a saddle.*

Proof. We have

$$J(P_1) = \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & 1 + \frac{\alpha_L(a_0+\Lambda)}{2} \end{bmatrix}. \tag{10}$$

It has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1 + \frac{\alpha_L(a_0+\Lambda)}{2}$. It is clear that $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Thus, $P_1$ is a saddle point. ■

To investigate $P_2$ stability, we use the Jury stability conditions [28, 29, 30] as follows:

$$\begin{cases} F := 1 + trace(J(P_2)) + det(J(P_2)) > 0 \\ TC := 1 - trace(J(P_2)) + det(J(P_2)) > 0\ , \\ NS := 1 - det(J(P_2)) > 0 \end{cases} \tag{11}$$

If the conditions (11) are met, we can state that $P_2$ is locally asymptotically stable. Moreover, if $F = 0$ then a flip bifurcation happens, or if $TC = 0$ then a transcritical bifurcation happens, or if $NS = 0$ then a Neimark-Sacker bifurcation happens [28, 6].

**Theorem 3.2.** *The equilibrium point* $P_2$ *is locally asymptotically stable if* $a_0 < \frac{4}{\alpha_L} - \Lambda$.

Proof. Note that

$$J(P_2) = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{\alpha_L(a_0+\Lambda)}{3} & 1 - \frac{2\alpha_L(a_0+\Lambda)}{3} \end{bmatrix}, \tag{12}$$

which having trace $T = 1 - \frac{2\alpha_L(a_0+\Lambda)}{3}$ and determinant $D = -\frac{\alpha_L(a_0+\Lambda)}{6}$. Thus, we have

$$F = 1 + T + D = 2 - \frac{\alpha_L(a_0 + \Lambda)}{2} > 0 \text{ if } a_0 < \frac{4}{\alpha_L} - \Lambda,$$
$$TC = 1 - T + D = \frac{\alpha_L(a_0 + \Lambda)}{2} > 0,$$
$$NS = 1 - D = 1 + \frac{\alpha_L(a_0 + \Lambda)}{6} > 0.$$

Thus equilibrium point $P_2$ is locally asymptotically stable if $a_0 < a_0^F := \frac{4}{\alpha_L} - \Lambda$.

The proof of Theorem (3.2) says that only flip bifurcation that happens, with the flip bifurcation value $a_0^F = \frac{4}{\alpha_L} - \Lambda$. It will exist and have economic meaning if it has a value between 0 and 1, or in other words

$$\frac{4}{1+\Lambda} < \alpha_L < \frac{4}{\Lambda}. \tag{13}$$

## 4. NUMERICAL SIMULATIONS

Several simulations are performed using the parameters' value given in Table 1 to confirm the previous analytical results. The value of parameters is chosen for simulation purposes only, but they still satisfy the positivity condition of equilibrium $P_2$ in (8) and the existence condition of flip bifurcation value in (13).

Table 1. The parameters' description and value

| Description | Notation | Value |
|---|---|---|
| Speed of adjustment | $\alpha_L$ | 23.5 |
| Bank's parameter of the loan interest rate | $a_L$ | 0.15 |
| Bank's parameter of the loan interest rate | $b_L$ | 0.05 |
| Bank's interest rate of deposit | $r_D$ | 0.03 |
| Cost of equity | $r_E$ | 0.05 |
| CAR minimum value | $\kappa$ | 0.08 |
| RR minimum value | $\rho$ | 0.12 |
| Marginal cost of deposit | $c_D$ | 0.05 |
| Marginal cost of loan | $c_L$ | 0.05 |

Initially, a bifurcation diagram of the parameter $a_0$ is presented for each bank $i$ and $j$ in Figure 1. The diagram illustrates that a higher value of $a_0$ will make each bank's loan equilibrium higher. But, at some high value of $a_0$, the loan stability is lost. Furthermore, it is evident that bank $j$ exhibits greater loan volatility.

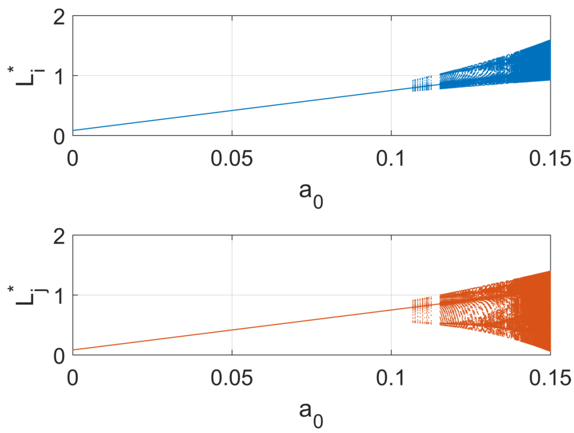



Figure 1. Bifurcation diagram of $a_0$ for each bank.

For another interesting view, we add the loan equilibrium of bank $i$ and bank $j$. We have the depiction of the bifurcation diagram of $a_0$ as in Figure 2a. If we zoom in the area A, we get an appealing picture as shown in Figure 2b. In more detail, we zoom the areas B and C and get very enticing pictures as depicted in Figure 2c and 2d. They show complex dynamics of the loan equilibrium, especially for area C which shows chaotic dynamics.

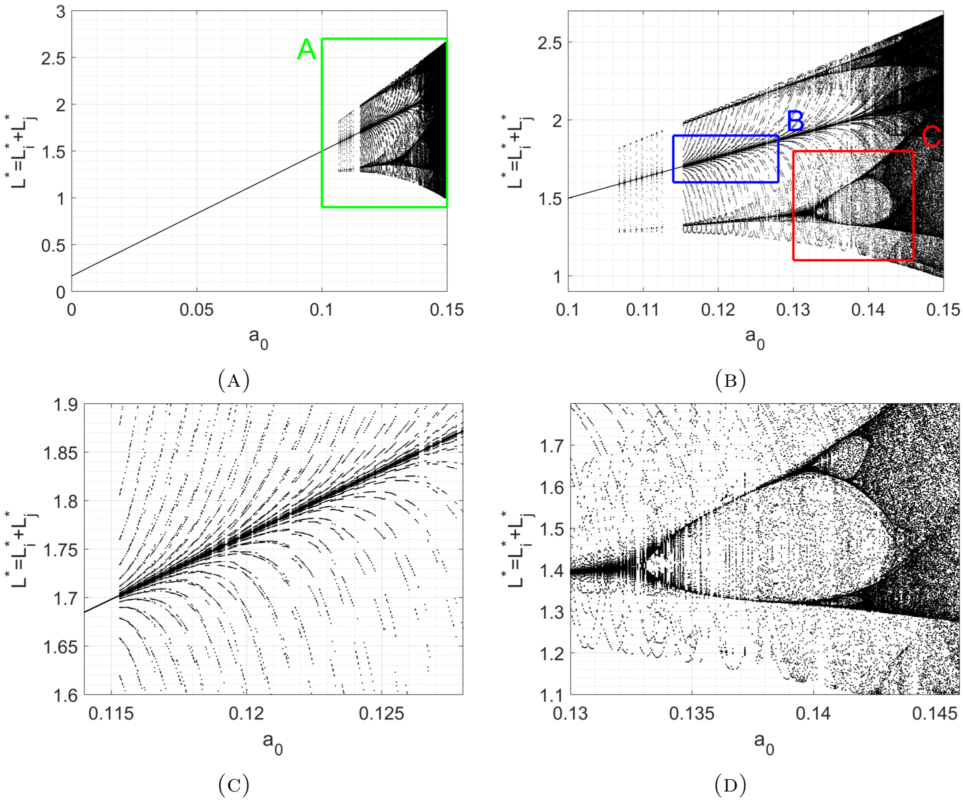
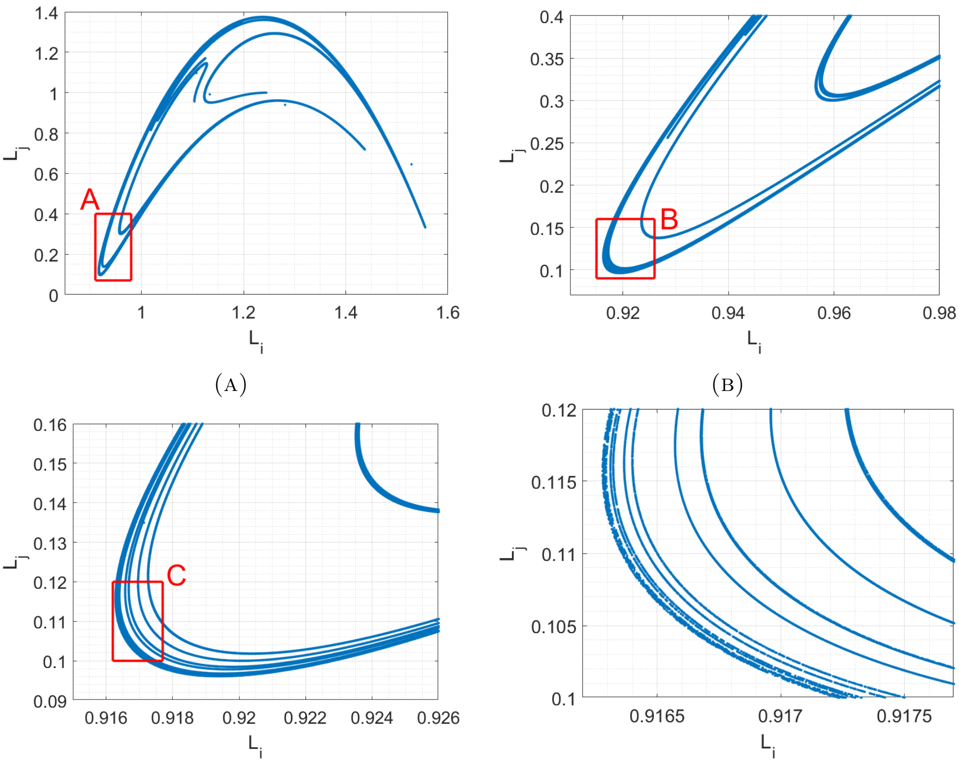



FIGURE 2. (a) Bifurcation diagram of parameter $a_0$ versus $L^* = L_i^* + L_j^*$, (b)-(d) the enlargement of pictures A, B, and C.

The complex dynamics of the chaotic system also can be depicted using its phase portrait. In Figure 3, we plot the phase portrait of points $(L_{i,t}, L_{j,t})$ with $1 \leq t \leq 10^7$ when the system is chaotic. The simulation uses value $a_0 = 0.148$. Since the system (7) is heterogeneous, then in Figure 3a we can see the asymmetrical aspect of the chaotic attractor. The enlargement of areas A, B, and C are displayed in Figures 3b-3d, respectively.




(A) (B) (C) (D)

FIGURE 3. (a) Phase portrait that shows chaotic attractor, and (b)-(d) the enlargement of pictures A, B, C.

## 4.1. **Sensitivity Analysis.**

To ascertain which of the model's parameters exerts the greatest impact on the system's stability, we conduct a sensitivity analysis. Consider the stability condition in Theorem 3.2. The equilibrium $P_2$ is considered stable if $a_0 < \frac{4}{\alpha_L} - (a_L - [(r_D + c_D)(\frac{1-\kappa}{1-\rho}) + r_E\kappa + c_L])$. Define $S := \frac{a_0}{\frac{4}{\alpha_L} - (a_L - [(r_D + c_D)(\frac{1-\kappa}{1-\rho}) + r_E\kappa + c_L])}$. thus, we rewrite the stability condition into the form $S < 1$. This setting is inspired by the basic reproduction number in mathematical epidemiology which is if its value is below 1, then the disease-free equilibrium is stable.

To determine the parameter that has the greatest impact on $S$, we calculate an elasticity index. This index measures the relative changes and provides insight into the relative significance of the parameter's effect on $S$ [31]. This measure is defined by

$$I_x^S = \frac{\partial S}{\partial x} \times \frac{x}{S},$$

where $x \in \{a_0, \alpha_L, a_L, r_D, r_E, \kappa, \rho, c_D, c_L\}$. Using the parameters' value in Table 1, we calculate the elasticity index for each parameter, and the result is presented in Figure 4. Here, we can see that parameter $\alpha_L$ has the most influence on $S$, and it is followed by $a_0$ and $a_L$. The speed of adjustment parameter exerts a significant influence on the stability of the banking loan equilibrium. The stability is also strongly influenced by the loan benchmark and individual bank's loan interest rates.

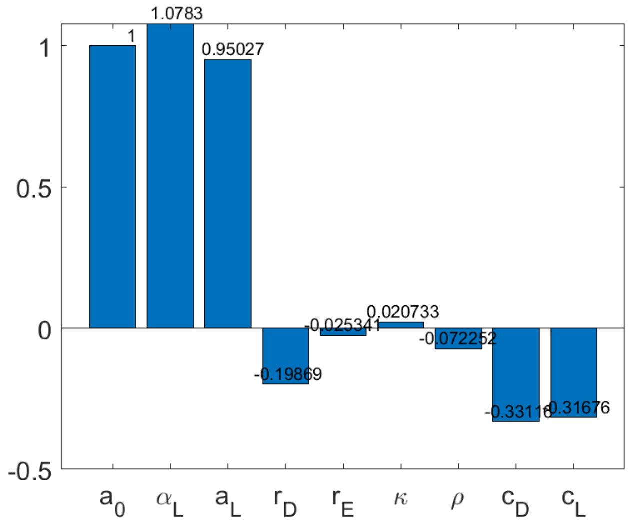



FIGURE 4. Elasticity index to observe which parameter has the most influence on the banking loan stability.

Another way to perform sensitivity analysis is by plotting the contour of $S$, where $S$ is viewed as a function of two parameters. In this case, we assume $S$ as a function of parameter $a_0$ and other parameters. The contour plot is given in Figure 5. By this contour, we can observe when the combination parameters' values produce a value of $S$ below 1, which indicates a stable banking loan. In those figures, we can observe that low $a_0$ will make banking loans stable. The influence of $a_0$ on banking loan stability exceeds the influence of other parameters, except for $\alpha_L$.

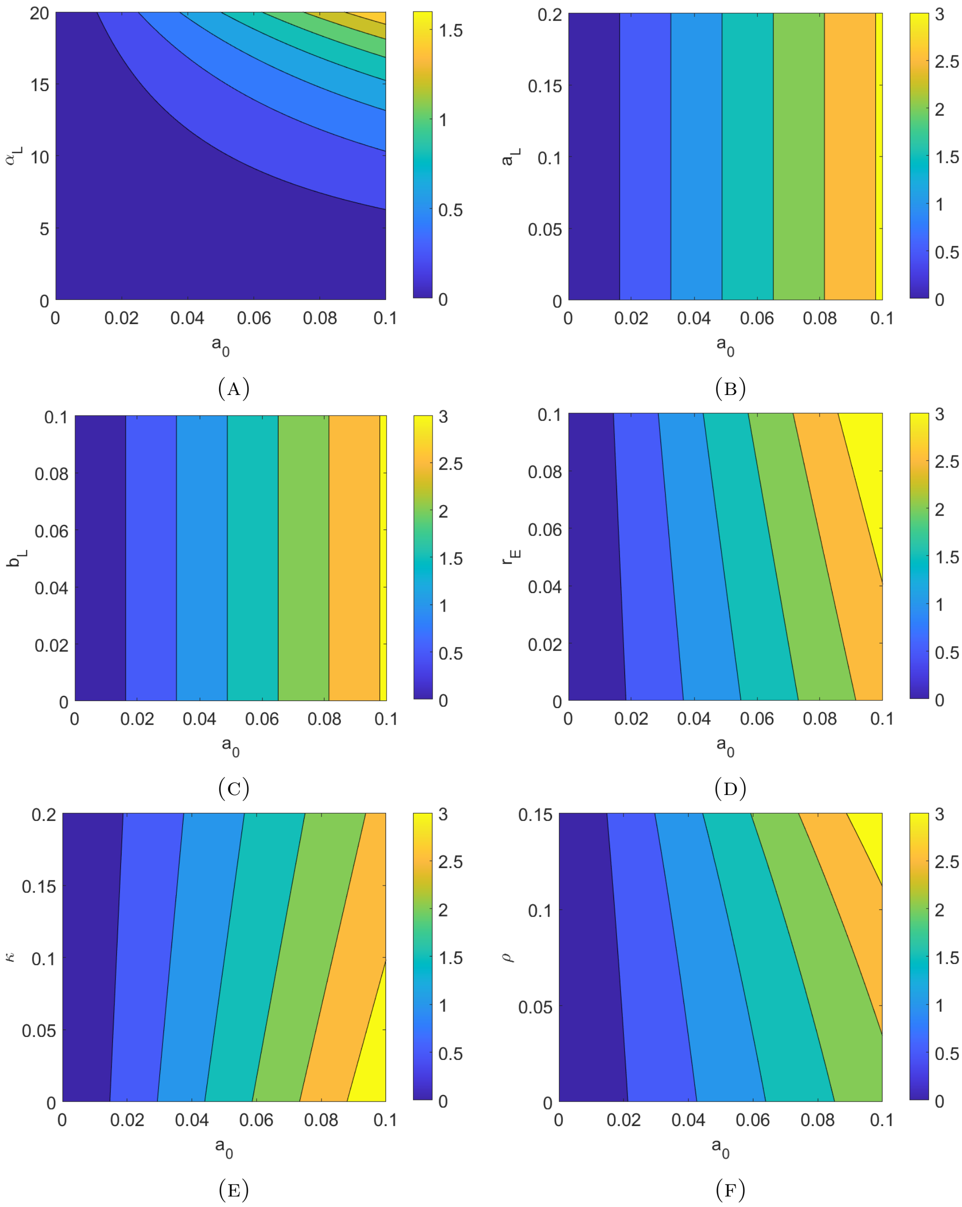



FIGURE 5. Contour plot of $S$ to observe the influence of parameter $a_0$ and other parameters on the banking loan stability.

## 5. CONCLUDING REMARKS

A banking duopoly model is considered with different behaviour of channelling loans. The model is used to examine the influence of the loan benchmark interest rate on the duopoly dynamics. The first bank's optimal channelling loan is based on the other bank, meanwhile, the other bank channels the loan based on bounded rational expectation. One of the equilibriums generated by the model occurs when both banks allocate an equal quantity of loans. If the loan benchmark interest rate falls under certain limits, this equilibrium will be locally asymptotically stable. The regulator can utilize this stability result as a basis for determining the loan benchmark interest rate, ensuring the stability of banking loans.

The model is characterized by the presence of two banks with distinct strategies, making it heterogeneous. The simulations demonstrate that the model's heterogeneity leads to asymmetrical dynamics, which are evident in the bifurcation diagram and chaotic attractor. The bifurcation diagram allows us to observe how changes in the loan benchmark interest rate might affect the stability of banking loans, resulting in a range of outcomes. Therefore, the regulator can utilize this bifurcation diagram to gain insight into the future trajectory of bank loans in the event that they modify the current loan benchmark interest rate. To observe which parameter has the most impact on banking loan stability, a sensitivity analysis is presented. The result shows that the bank's procyclicality behaviour when channelling loans strongly influences the banking loan stability, and it is followed by the loan benchmark interest rate and individual bank's loan interest rate. These three parameters can be considered by the regulator to control the banking loan growth.

**Acknowledgement.** The study is supported by Sumber Dana Selain APBN FSM Undip 2023 (22.D/UN7.F8/PP/II/2023).

## REFERENCES

[1] Q. Duan, Y. Wei, and Z. Chen, "Relationship between the benchmark interest rate and a macroeconomic indicator," *Economic Modelling*, vol. 38, pp. 220–226, 2014.

[2] M. Ulate, "Going negative at the zero lower bound: the effects of negative nominal interest rates," *Am. Econ. Rev.*, vol. 111, pp. 1–40, 2021.

[3] B. Xin and K. Jiang, "Economic uncertainty, central bank digital currency, and negative interest rate policy," *Journal of Management Science and Engineering*, vol. 8, pp. 430–452, 2023.

[4] L. Brubakk, S. ter Ellen, and H. Xu, "Central bank communication through interest rate projections," *Journal of Banking & Finance*, vol. 124, p. 106044, 2021.

[5] A. Rai, R. Seth, and N. White, "Central bank target rates and term structure of interest rates: A study of six asia-pacific countries," *IIMB Management Review*, vol. 31, pp. 223–237, 2019.

[6] L. Fanti, "The dynamics of a banking duopoly with capital regulations," *Economic Modelling*, vol. 37, pp. 340–349, 2014.

[7] S. Brianzoni and G. Campisi, "Dynamical analysis of a banking duopoly model with capital regulation and asymmetric costs," *Discrete and Continuous Dynamical Systems - Series B*, vol. 26, pp. 5807–5825, 2021.

[8] S. Brianzoni, G. Campisi, and A. Colasante, "Nonlinear banking duopoly model with capital regulation: The case of italy," *Chaos, Solitons & Fractals*, vol. 160, p. 112209, 2022.

[9] A. Elsadany, "Dynamics of a delayed duopoly game with bounded rationality," *Mathematical and Computer Modelling*, vol. 52, pp. 1479–1489, 2010.

[10] A. Elsadany, "Dynamics of a cournot duopoly game with bounded rationality based on relative profit maximization," *Applied Mathematics and Computation*, vol. 294, pp. 253–263, 2017.

[11] Y. Peng, Y. Xiao, Q. Lu, X. Wu, and Y. Zhao, "Chaotic dynamics in cournot duopoly model with bounded rationality based on relative profit delegation maximization," *Physica A: Statistical Mechanics and its Applications*, vol. 560, p. 125174, 2020.

[12] X. Zhang, D. Sun, S. Ma, and S. Zhang, "The dynamics of a quantum bertrand duopoly with differentiated products and heterogeneous expectations," *Physica A: Statistical Mechanics and its Applications*, vol. 557, p. 124878, 2020.

[13] S. Askar, "On the dynamics of cournot duopoly game with private firms: Investigations and analysis," *Applied Mathematics and Computation*, vol. 432, p. 127354, 2022.

[14] J. Ren, H. Sun, G. Xu, and D. Hou, "Convergence of output dynamics in duopoly co-opetition model with incomplete information," *Mathematics and Computers in Simulation*, vol. 207, pp. 209–225, 2023.

[15] D. Zhou, H. Yang, J. Pi, and G. Yang, "The dynamics of a quantum cournot duopoly with asymmetric information and heterogeneous players," *Physics Letters A*, vol. 483, p. 129033, 2023.

[16] M. Ansori, N. Sumarti, K. Sidarto, and I. Gunadi, "Analyzing a macroprudential instrument during the covid-19 pandemic using border collision bifurcation," *Rect@: Revista Electrónica de Comunicaciones y Trabajos de ASEPUMA*, vol. 22, pp. 91–103, 2021.

[17] M. Ansori and S. Hariyanto, "Analysis of banking deposit cost in the dynamics of loan: Bifurcation and chaos perspectives," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, pp. 1283–1292, 2022.

[18] M. Ansori and S. Khabibah, "The role of the cost of a loan in banking loan dynamics: Bifurcation and chaos analysis," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, pp. 1031–1038, 2022.

[19] M. Ansori, G. Theotista, and Winson, "Difference equation-based banking loan dynamics with reserve requirement policy," *International Journal of Difference Equations*, vol. 18, pp. 35–48, 2023.

[20] M. Ansori, G. Theotista, and M. Febe, "The influence of the amount of premium and membership of idic on banking loan procyclicality: A mathematical model," *Advances in Dynamical Systems and Applications*, vol. 18, pp. 111–123, 2023.

[21] N. Ashar, M. Ansori, and H. Fata, "The effects of capital policy on banking loan dynamics: A difference equation approach," *International Journal of Difference Equations*, vol. 18, pp. 267–279, 2023.

[22] H. Fata, N. Ashara, and M. Ansori, "Banking loan dynamics with dividend payments," *Advances in Dynamical Systems and Applications (ADSA)*, vol. 18, pp. 87–99, 2023.

[23] M. F. Ansori and F. H. Gümüş, "A difference equation of banking loan with nonlinear deposit interest rate," *Journal of Mathematical Sciences and Modelling*, vol. 7, no. 1, pp. 14–19, 2024.

[24] M. Klein, "A theory of the banking firm," *Journal of Money, Credit, and Banking*, vol. 3, pp. 205–218, 1971.

[25] M. Monti, "Deposit, credit and interest rates determination under alternative objective functions," in *Mathematical methods in investment and finance* (G. P. Szego and K. Shell, eds.), Amsterdam: North-Holland, 1972.

[26] X. Freixas and J. Rochet, *Microeconomics of Banking*. MIT Press, Cambridge, 1997.

[27] A. Dixit, "Comparative statics for oligopoly," *International Economic Review*, vol. 27, pp. 107–122, 1986.

[28] G. Gandolfo, *Economic Dynamics: Methods and Models*. Elsevier Science Publisher BV, Amsterdam, The Netherlands, 2nd ed., 1985.

[29] A. Medio, *Chaotic Dynamics: Theory and Applications to Economics*. Cambridge University Press, Cambridge, UK, 1992.

[30] G.-I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky, *Nonlinear Oligopolies: Stability and Bifurcations*. Springer-Verlag, Berlin, 2010.

[31] P. van den Driessche, "Reproduction numbers of infectious disease models," *Infectious Disease Modelling*, vol. 2, no. 3, pp. 288–303, 2017.