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WEAK LOCAL RESIDUALS IN AN ADAPTIVE FINITE VOLUME METHOD FOR ONE-DIMENSIONAL SHALLOW WATER EQUATIONS

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Abstract. Weak local residual (WLR) detects the smoothness of numerical solutions to conservation laws. In this paper we consider balance laws with a source term, the shallow water equations (SWE). WLR is used as the refinement indicator in an adaptive finite volume method for solving SWE. This is the first study in implementing WLR into an adaptive finite volume method used to solve the SWE, where the adaptivity is with respect to its mesh or computational grids. We limit our presentation to one-dimensional domain. Numerical simulations show the effectiveness of WLR as the refinement indicator in the adaptive method.

Key words and Phrases: Finite volume methods, weak local residuals, refinement indicator, adaptive mesh refinement, shallow water equations.

Abstrak. "Residu lokal lemah" (RLL) dapat mendeteksi kehalusan solusi numerik dari hukum kekekalan. Pada makalah ini kami membahas hukum kesetimbangan dengan sebuah suku masukan, yakni persamaan air dangkal (PAD). Hal ini merupakan yang pertama dalam implementasi RLL untuk sebuah metode volume hingga yang adaptif untuk penyelesaian PAD, di mana adaptivitas dikaitkan dengan mesh atau grid komputasi. Bahasan dibatasi hanya untuk domain satu dimensi. Simulasi numerik memperlihatkan keefektifan RLL sebagai indikator penghalusan dalam metode adaptif.

Kata kunci: Metode volume hingga, residu lokal lemah, indikator penghalusan, penghalusan mesh adaptif, persamaan air dangkal.

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1. Introduction

Weak local residuals as smoothness indicators for conservation laws were proposed by Karni, Kurganov, and Petrova [4, 5]. In their work the weak local residual (WLR) was called the weak local truncation error. In the present paper we follow the term "weak local residual" used by Kurganov et al. [1, 6], as this is more appropriate by definition.

Even though the corresponding theory is so far available only for scalar conservation laws, WLR as a smoothness indicator was also valid for systems of conservation laws [4, 5] and even for systems of balance laws [7]. The order of accuracy of WLR in a rough region is lower than the order of accuracy in a smooth region [2, 4, 5]. Therefore WLR is able to detect the smoothness of solutions.

The ability of detecting the smoothness of solutions makes the WLR a good candidate as the indicator in an adaptive numerical method to solve balance laws. So far, WLR has been implemented in adaptive numerical methods for gas dynamics [2, 4, 5, 6]. WLR has also been implemented in an adaptive numerical method for the shallow water equations (SWE) [1], but the adaptivity was with respect to artificial viscosity.

In this paper we implement WLR as the refinement indicator in an adaptive finite volume method used to solve the SWE. This work is the first study in implementing WLR into the adaptive finite volume method used to solve the SWE, where the adaptivity is with respect to its mesh or computational grids. We consider one-dimensional problems of water flows. We follow the frame work presented by Constantin and Kurganov [2] to compute the WLR, as it gives a simple and cheap computation of WLR. Note that an approach different from WLR is available for solving SWE adaptively, such as using the Winslow's monitor function [3] instead of WLR. However, in this paper we shall limit our presentation to WLR to solve the SWE adaptively.

This paper is organised as follows. Section 2 recalls WLR formulations for conservation laws. The WLR as the refinement indicator is implemented in an adaptive method in Section 3. Finally Section 4 draws some concluding remarks.

2. Weak Local Residuals of Conservation Laws

This section recalls the formulation of WLR based on the work of Constantin and Kurganov [2]. We use the following conventions for our notations: x is a one-dimesional space variable, t is the time variable, q is a conserved quantity, f is a flux function, and q_0 is an arbitrary function defined for an initial condition.

Consider the scalar conservation laws

$$q_t + f(q)_x = 0 \tag{1}$$

on $-\infty < x < \infty$, with the initial condition $q(x,t) = q_0(x)$ at $t = 0$. The weak form of this initial value problem is

$$\int_0^\infty \int_{-\infty}^\infty [q(x,t)T_t(x,t) + f(q(x,t))T_x(x,t)]\, dx\, dt + \int_{-\infty}^\infty q_0(x)T(x,0)\, dx = 0, \quad (2)$$

where $T(x,t)$ is an arbitrary test function.

Now we recall the strategy of Karni and Kurganov [4] in defining the WLR. First we consider uniform grids. Uniform grids are taken as $(x_j := j\Delta x\,, t^n := n\Delta t)$ and let q_j^n be approximate values of $q(x_j, t^n)$ computed by a conservative method. We denote by $q^\Delta(x,t)$ the corresponding piecewise constant approximation,

$$q^\Delta(x,t) := q_j^n \quad \text{if} \quad (x,t) \in \left[x_{j-1/2}, x_{j+1/2}\right] \times \left[t^{n-1/2}, t^{n+1/2}\right] \quad (3)$$

where $x_{j\pm 1/2} := x_j \pm \Delta x/2$ and $t^{n\pm 1/2} := t^n \pm \Delta t/2$. We construct a test function $T_j^n(x,t) := B_j(x)B^n(t)$, where $B_j(x)$ and $B^n(t)$ are centered at $x = x_{j+1/2}$ and $t = t^{n-1/2}$ with support of size $2\Delta x$ and $2\Delta t$. That is,

$$B_{j+1/2}(x) = \begin{cases} \frac{x - x_{j-1/2}}{\Delta x} & \text{if } x_{j-1/2} \le x \le x_{j+1/2}\,, \\ \frac{x_{j+3/2} - x}{\Delta x} & \text{if } x_{j+1/2} \le x \le x_{j+3/2}\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad (4)$$

and

$$B^{n-1/2}(t) = \begin{cases} \frac{t - t^{n-3/2}}{\Delta t} & \text{if } t^{n-3/2} \le t \le t^{n-1/2}\,, \\ \frac{t^{n+1/2} - t}{\Delta t} & \text{if } t^{n-1/2} \le t \le t^{n+1/2}\,, \\ 0 & \text{otherwise}\,. \end{cases} \quad (5)$$

This results in a cheap computation of the WLR

$$E_{j+1/2}^{n-1/2} = -\int_{t^{n-3/2}}^{t^{n+1/2}} \int_{x_{j-1/2}}^{x_{j+3/2}} \left[q^\Delta(x,t)\left(T_{j+1/2}^{n-1/2}\right)_t + f(q^\Delta(x,t))\left(T_{j+1/2}^{n-1/2}\right)_x\right] dx\, dt, \quad (6)$$

which can be expressed after a straightforward calculation as

$$\begin{aligned} E_{j+1/2}^{n-1/2} &= \frac{\Delta x}{2}\left[q_j^n - q_j^{n-1} + q_{j+1}^n - q_{j+1}^{n-1}\right] \\ &\quad + \frac{\Delta t}{2}\left[f\left(q_{j+1}^{n-1}\right) - f\left(q_j^{n-1}\right) + f\left(q_{j+1}^n\right) - f\left(q_j^n\right)\right]. \end{aligned} \quad (7)$$

This taking linear B-splines in constructing the test function adapts from the work of Constantin and Kurganov [2] on conservation laws.

3. Weak Local Residual in An Adaptive Method

In this section we implement the WLR as the refinement indicator in an adaptive finite volume method used to solve the one-dimensional SWE. We present some numerical results of the implementation. The WLR, also known as local truncation errors, as smoothness indicators for conservation laws were proposed by Kurganov et al. [5]. A conservation law is homogeneous, so its source term is zero.

In this article we extend the implementation of WLR as the refinement indicator for balance laws with a nonzero source term. In particular we consider the one-dimensional shallow water equations. These equations are

$$h_t + (hu)_x = 0\,, \tag{8}$$

$$(hu)_t + \left(hu^2 + \tfrac{1}{2}gh^2\right)_x = -ghz_x. \tag{9}$$

Here, x represents the coordinate in one-dimensional space, t represents the time variable, $u = u(x,t)$ denotes the water velocity, $h = h(x,t)$ denotes the water height, $z = z(x)$ is the bed topography, and g is the acceleration due to gravity. We define stage $w(x,t)$ as $w = h + z$.

Note that the WLR presented in Section 2 is defined for conservation laws. Therefore the best way to compute the WLR of the SWE is considering the mass equation (8) with the quantity $q = h$ and flux $f(q) = hu$. The WLR based on the mass equation for the SWE is then

$$\begin{aligned} E_{i+1/2}^{n-1/2} \;&=\; (1/2)\left\{\Delta x\left[h_i^n - h_i^{n-1} + h_{i+1}^n - h_{i+1}^{n-1}\right]\right. \\ &\quad \left. +\Delta t\left[h_{i+1}^{n-1}u_{i+1}^{n-1} - h_i^{n-1}u_i^{n-1} + h_{i+1}^n u_{i+1}^n - h_i^n u_i^n\right]\right\}\,. \end{aligned} \tag{10}$$

We do not consider the WLR based on the momentum equation (9). That is because the computation would require well-balanced technique, as the source term is nonzero for the momentum equation (9). The well-balanced technique for the computation of WLR is beyond the scope of this paper.

Formulation (10) is defined at each vertex and for uniform grids. To define the CK indicator at the centroid of each cell, we choose one of two available indicators at its cell vertices having the largest magnitude and divide it by the local cell width. That is, the CK indicator at the centroid of the ith cell is

$$E_i^{h,n-1/2} = \frac{1}{\Delta x_i} \times \begin{cases} E_{i-1/2}^{h,n-1/2} & \text{if} \quad \left|E_{i-1/2}^{h,n-1/2}\right| \geq \left|E_{i+1/2}^{h,n-1/2}\right|\,, \\ E_{i+1/2}^{h,n-1/2} & \text{otherwise}\,. \end{cases} \tag{11}$$

We denote by CK (Constantin-Kurganov) indicator the WLR (11) [8]. Formulation (11) can be used to compute the WLR at centroids of non-uniform cells, hence, adaptive grids. When the numerical method has a formal order r, the order of $\left|E_i^{h,n-1/2}\right|$ is $O(1)$ near discontinuities and $O\left(\Delta^{\min\{3,r+1\}}\right)$ in smooth regions [2]. The order difference between discontinuous and smooth regions makes $\left|E_i^{h,n-1/2}\right|$ able to detect the smoothness of the numerical solution.

As a test case, we recall the problem with topography considered by Felcman and Kadrnka [3]. In this test, all quantities are measured in Systeme International (SI) units and we omit the writing of the units. We consider a channel of length 25 with topography

$$z(x) = \begin{cases} 0.2 - 0.05\,(x-10)^2 & \text{if} \quad 8 \leq x \leq 12\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{12}$$

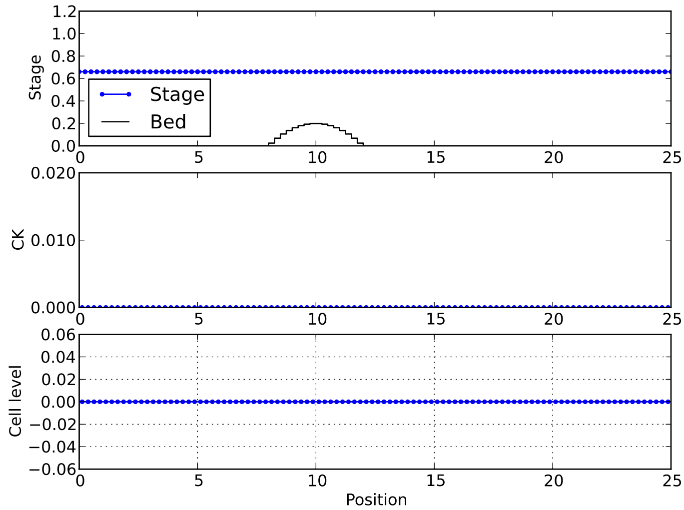


FIGURE 1. Initial setting for the adaptive method: water is at rest at $t = 0$.

and an initial condition

$$u(x, 0) = 0\,, \quad w(x, 0) = 0.66 \tag{13}$$

together with the Dirichlet boundary conditions

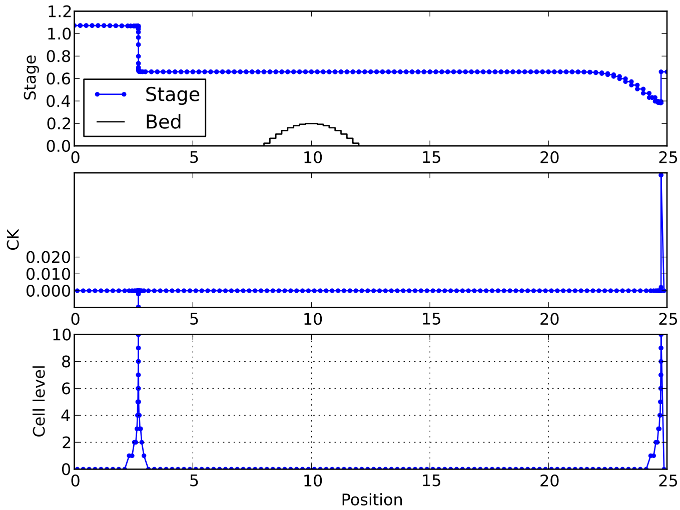
$$h(25, t) = 0.66\,, \quad hu(0, t) = hu(25, t) = 1.53\,. \tag{14}$$

The numerical settings are as follows. First order finite volume scheme [9] is used. The spatial domain is discretised into 100 cells initially. The tolerance of the CK indicator is

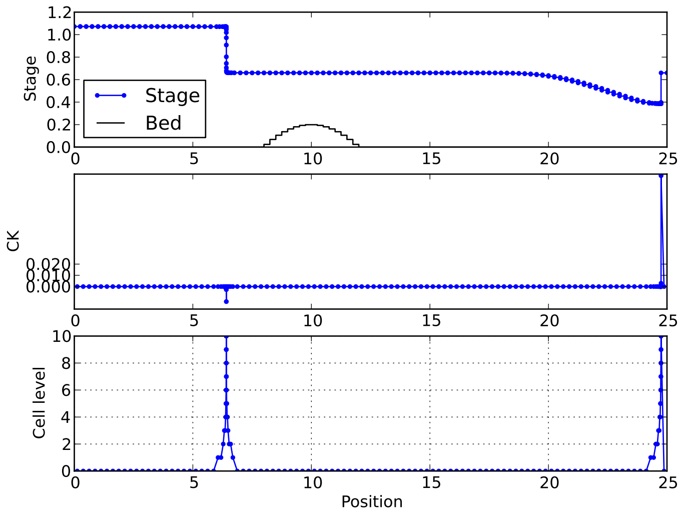
$$\mathrm{CK_{tol}} = 0.05 \max |\mathrm{CK}|\ , \tag{15}$$

where CK is defined as in (11). Cells with $|\mathrm{CK}| > \mathrm{CK_{tol}}$ are refined binary, that is, a "parent" cell is refined into two "children" cells with equal width. Cells with $|\mathrm{CK}| \le 0.1\,\mathrm{CK_{tol}}$ are coarsened. Two neighbouring cells are coarsened as long as they are at the same level and have the same parent. The maximum level of binary refinement is 10 and the width of the coarsest cell allowed is 0.25.

Our results of the test case are as follows. The initial condition is a river at rest with a parabolic bump at the bottom, as shown in Figure 1. For time $t > 0$ there is a constant inflow from the left-end and constant outflow at the right-end of the domain. The inflow results in shock waves coming into the domain. The numerical results for time $t > 0$ are represented by Figures 2(A), 2(B) and 3(A), which show the adaptive results at time $t = 0.67$, 1.67 and 2.67 respectively. Shock waves are rough, so our adaptive method takes action by refining the grids around the shocks. Therefore, we get accurate results. Note that if the standard uniform

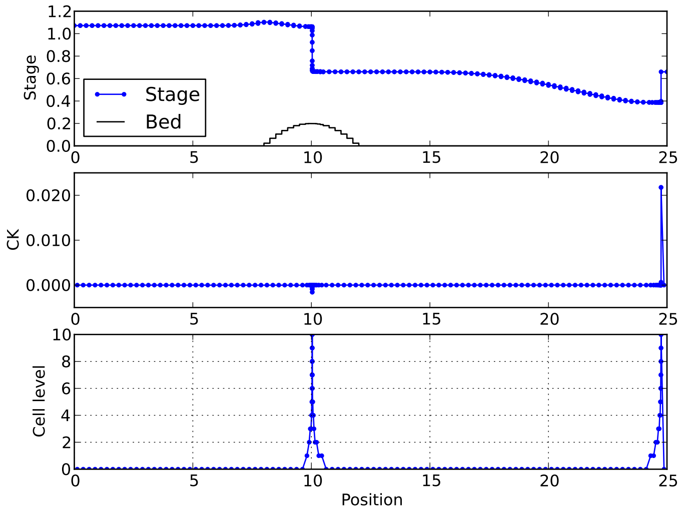


(A)

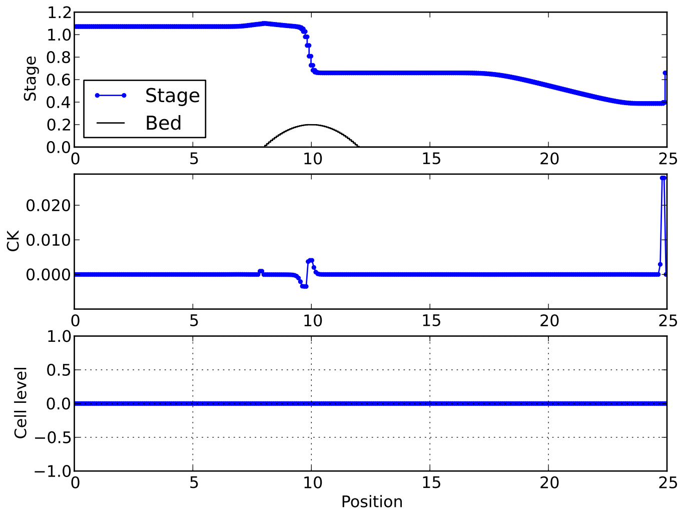


(B)

FIGURE 2. Results produced by the adaptive method for time: (A) $t = 0.67$ and (B) $t = 1.67$. The flow is unsteady and moving to the right.



(A)



(B)

FIGURE 3. Result comparison for time $t = 2.67$ between: (A) adaptive method involving 154 cells and (B) standard method involving 308 cells. The shock over the bump is unsteady and moving to the right.

grid were used, the shocks could not be sharply resolved. For comparison, this result at time $t = 2.67$ of the standard method is shown in Figure 3(b). We separate these figures from $t = 0$ until $t = 2.67$ in order that we can clearly see the evolution of the flow together with its mesh adaptivity and its residuals.

The adaptive strategy presented in this paper can be used to solve the one-dimensional SWE in general. The example on unsteady flow over a bump that we have discussed is a representative of one-dimensional shallow water flows.

4. Concluding Remarks

We have implemented weak local residuals as the refinement indicator in an adaptive finite volume method used to solve the shallow water equations in one dimension. The adaptivity is based on the magnitude of the residuals at each time step. Numerical results show that the coarsening and refinement can be done successfully using these weak local residuals. Possible future work is extending this technique to solve the shallow water equations in higher dimensions.

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References

[1] Chen, Y., Kurganov, A., Lei, M., and Liu, Y., "An adaptive artificial viscosity method for the Saint-Venant system", *Notes on Numerical Fluid Mechanics and Multidisciplinary Design*, **120** (2012), 125–141.

[2] Constantin, L.A., and Kurganov, A., "Adaptive central-upwind schemes for hyperbolic systems of conservation laws", in *Hyperbolic problems: Theory, numerics, and applications*, Volume 1 (F. Asakura et al., editors), Yokohama Publishers, Yokohama, 2006, pages 95–103.

[3] Felcman, J., and Kadrnka, L., "Adaptive finite volume approximation of the shallow water equations", *Applied Mathematics and Computation*, **219** (2012), 3354–3366.

[4] Karni, S., and Kurganov, A., "Local error analysis for approximate solutions of hyperbolic conservation laws", *Advances in Computational Mathematics*, **22** (2005), 79–99.

[5] Karni, S., Kurganov, A., and Petrova, G., "A smoothness indicator for adaptive algorithms for hyperbolic systems", *Journal of Computational Physics*, **178** (2002), 323–341.

[6] Kurganov, A., and Liu, Y., "New adaptive artificial viscosity method for hyperbolic systems of conservation laws", *Journal of Computational Physics*, **231** (2012), 8114–8132.

[7] Mungkasi, S., Li, Z., and Roberts, S. G., "Weak local residuals as smoothness indicators for the shallow water equations", *Applied Mathematics Letters*, **30** (2014), 51–55.

[8] Mungkasi, S., and Roberts, S. G., "Numerical entropy production for shallow water flows", *ANZIAM Journal*, **52** (2011), C1–C17.

[9] Mungkasi, S., and Roberts, S. G., "On the best quantity reconstructions for a well balanced finite volume method used to solve the shallow water wave equations with a wet/dry interface", *ANZIAM Journal*, **51** (2010), C48–C65.