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ON THE DETOUR AND VERTEX DETOUR HULL NUMBERS OF A GRAPH

A.P. Santhakumaran[1] and S.V. Ullas Chandran[2]

[1]**Department of Mathematics Hindustan University Hindustan Institute of Technology and Science, Padur, Chennai - 603 103, India**
apskumar1953@yahoo.co.in
[2]**Department of Mathematics, Mahatma Gandhi College, Kesavdasapuram, Pattom P.O., Thiruvanathapuram - 695 004, India**
svuc.math@gmail.com

Abstract. For vertices x and y in a connected graph G, the detour distance $D(x, y)$ is the length of a longest $x - y$ path in G. An $x - y$ path of length $D(x, y)$ is an $x - y$ detour. The closed detour interval $I_D[x, y]$ consists of x, y, and all vertices lying on some $x - y$ detour of G; while for $S \subseteq V(G)$, $I_D[S] = \bigcup_{x,y\in S} I_D[x, y]$. A set S of vertices is a detour convex set if $I_D[S] = S$. The detour convex hull $[S]_D$ is the smallest detour convex set containing S. The detour hull number $dh(G)$ is the minimum cardinality among subsets S of $V(G)$ with $[S]_D = V(G)$. Let x be any vertex in a connected graph G. For a vertex y in G, denote by $I_G[y]^x$, the set of all vertices distinct from x that lie on some $x - y$ detour of G; while for $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y\in S} I_D[y]^x$. For $x \notin S$, S is an x-detour set of G if $I_D[S]^x = V(G) - \{x\}$ and an x-detour set of minimum cardinality is the x-detour number $d_x(G)$ of G. For $x \notin S$, S is an x-detour convex set if $I_D[S]^x = S$. The x-detour convex hull of S, $[S]^x_D$ is the smallest x-detour convex set containing S. The x-detour hull number $dh_x(G)$ is the minimum cardinality among the subsets S of $V(G) - \{x\}$ with $[S]^x_D = V(G) - \{x\}$. In this paper, we investigate how the detour hull number and the vertex detour hull number of a connected graph are affected by adding a pendant edge.

Key words and Phrases:d Detour, detour number, detour hull number, x-detour number, x-detour hull number.

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Abstrak. Misalkan x dan y berada di graf terhubung G, jarak detour $D(x, y)$ adalah panjang dari lintasan $x - y$ yang terpanjang di G. Lintasan $x - y$ dengan panjang $D(x, y)$ adalah suatu detour $x-y$. Interval detour tertutup $I_D[x, y]$ memuat x, y dan semua titik yang berada dalam suatu detour $x - y$ dari G; sedangkan untuk $S \subseteq V(G)$, $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$. Himpunan titik S adalah suatu himpunan konveks detour jika $I_D[S] = S$. Konveks hull detour $[S]_D$ adalah himpunan konveks detour terkecil yang memuat S. Bilangan hull detour $dh(G)$ adalah kardinalitas minimum diantara sub-subhimpunan S dari $V(G)$ dengan $[S]_D = V(G)$. Misalkan x adalah suatu titik di graf terhubung G. Untuk suatu titik y di G, dinotasikan dengan $I_G[y]^x$, himpunan dari semua titik berbeda dari x yang terletak pada suatu detour $x - y$ dari G; sedangkan untuk $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. Untuk $x \notin S$, S adalah suatu himpuan detour-x dari G jika $I_D[S]^x = V(G) - \{x\}$ dan suatu himpuan detour-x dengan kardinalitas minimum adalah bilangan detour-x $d_x(G)$ dari G. Untuk $x \notin S$, S adalh suatu himpunan detour-x konveks jika $I_D[S]^x = S$. Konveks hull detour-x dari S, $[S]^x_D$ adalah himpunan konveks detour-x yang memuat S. Bilangan hull detour-x $dh_x(G)$ adalah kardinalitas minimum diantara sub-subhimpunan S dari $V(G) - \{x\}$ dengan $[S]^x_D = V(G) - \{x\}$. Pada paper ini, kami memeriksa pengaruh penambahan sisi anting dari suatu graf terhubung terhadap bilangan hull detour dan bilangan hull detour titik.

Kata kunci: Detour, bilangan detour, bilangan hull detour, bilangan detour-x, bilangan hull detour-x.

1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies, we refer to [1, 6]. For vertices x and y in a nontrivial connected graph G, the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G. An $x - y$ path of length $D(x, y)$ is an $x - y$ *detour*. It is known that the detour distance is a metric on the vertex set $V(G)$. The *detour eccentricity* of a vertex u is $e_D(u) = \max\{D(u, v) : v \in V(G)\}$. The *detour radius*, $rad_D(G)$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G. The detour distance and the detour center of a graph were studied in [2]. The *closed detour interval* $I_D[x, y]$ consists of x, y, and all vertices lying on some $x - y$ detour of G; while for $S \subseteq V(G)$, $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$; S is a *detour set* if $I_D[S] = V(G)$ and a detour set of minimum cardinality is the *detour number* $dn(G)$ of G. Any detour set of cardinality $dn(G)$ is the *minimum detour set* or *dn-set* of G. A vertex x in G is a *detour extreme vertex* if it is an initial or terminal vertex of any detour containing x. The detour number of a graph was introduced in [3] and further studied in [4, 8]. These concepts have interesting applications in Channel Assignment Problem in radio technologies [5, 7].

A set S of vertices of a graph G is a *detour convex set* if $I_D[S] = S$. The *detour convex hull* $[S]_D$ of S is the smallest detour convex set containing S. The detour convex hull of S can also be formed from the sequence $\{I_D^k[S], k \geq 0\}$, where $I_D^0[S] = S, I_D^1[S] = I_D[S]$ and $I_D^k = I_D[I_D^{k-1}[S]]$. From some term on, this sequence must be constant. Let p be the smallest number such that $I_D^p[S] = I_D^{p+1}[S]$. Then $I_D^p[S]$ is the *detour convex hull* $[S]_D$ and we call p as the *detour iteration number* $din(S)$ of S. A set S of vertices of G is a *detour hull set* if $[S]_D = V(G)$ and a detour hull set of minimum cardinality is the *detour hull number* $dh(G)$. The detour hull number of a graph was introduced and studied in [11].

For the graph G given in Figure 1, and $S = \{v_1, v_6\}$, $I_D[S] = V - \{v_7\}$ and $I_D^2[S] = V$. Thus S is a minimum detour hull set of G and so $d_h(G) = 2$. Since S is not a detour set and $S \cup \{v_7\}$ is a detour set of G, it follows from Theorem 1.2 that $dn(G) = 3$. Hence the detour number and detour hull number of a graph are different. Note that the sets $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_2, v_3, v_4, v_5, v_7\}$ are detour convex sets in G. Let x be any vertex of G. For a vertex y in G, $I_G[y]^x$ denotes

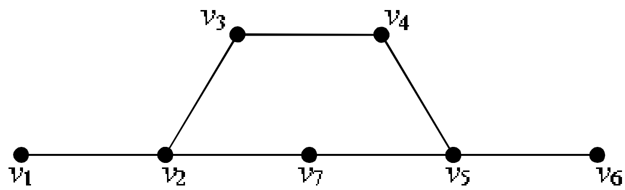

FIGURE 1. Graph G with $d_h(G) = 2$ and $dn(G) = 3$

the set of all vertices distinct from x that lie on some $x - y$ detour of G; while for $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. It is clear that $I_D[x]^x = \phi$. For $x \notin S$, S is an *x-detour set* if $I_D[S]^x = V(G) - \{x\}$ and an x-detour set of minimum cardinality is the *x-detour number* $d_x(G)$ of G. Any x-detour set of cardinality $d_x(G)$ is the *minimum x-detour set* or *d_x-set* of G. The vertex detour number of a graph was introduced and studied in [9].

Let G be a connected graph and x a vertex in G. Let S be a set of vertices in G such that $x \notin S$. Then S is an *x-detour convex set* if $I_D[S]^x = S$. The *x-detour convex hull* of S, $[S]_D^x$ is the smallest x-detour convex set containing S. The x-detour convex set can also formed from the sequence $\{I_D^k[S]^x, k \geq 0\}$, where $I_D^0[S]^x = S, I_D^1[S]^x = I_D[S]^x$ and $I_D^k[S]^x = I_D[I_D^{k-1}[S]^x]^x$. From some term on, this sequence must be constant. Let p_x be the smallest number such that $I_D^{p_x}[S]^x = I_D^{p_x+1}[S]^x$. Then $I_D^{p_x}[S]^x$ is the x-detour convex hull $[S]_D^x$ of S and we call p_x as the *x-detour iteration number* $din_x(S)$ of S. The set S is an *x-detour hull set* if $[S]_D^x = V(G) - \{x\}$ and an x-detour hull set of minimum cardinality is the *x-detour hull number* $dh_x(G)$ of G. Any x-detour hull set of cardinality $dh_x(G)$ is the *minimum x-detour hull set* or *d_x-hull set* of G.

For the graph G in Figure 2, the minimum vertex detour hull numbers and vertex detour numbers are given in Table 1. Table 1 shows that, for a vertex x, the x-detour number and the x-detour hull number of a graph are different.

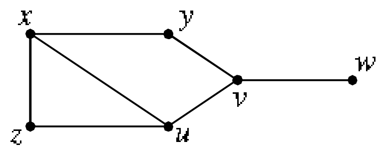

FIGURE 2. G

Table 1. x-detour numbers and x-detour hull numbers of G in Figure 2

Vertex	Minimum vertex detour sets	Minimum vertex detour hull sets	Vertex detour number	Vertex detour hull number
x	$\{y,w\},\{z,w\},\{u,w\}$	$\{w\}$	2	1
y	$\{w\}$	$\{w\}$	1	1
z	$\{w\}$	$\{w\}$	1	1
u	$\{w\}$	$\{w\}$	1	1
v	$\{y,w\},\{z,w\},\{u,w\}$	$\{x,w\},\{y,w\},\{z,w\},\{u,w\}$	2	2
w	$\{y\},\{z\},\{u\}$	$\{x\},\{y\},\{z\},\{u\}$	1	1

It is clear that every minimum x-detour hull set of a connected graph G of order n contains at least one vertex and at most $n-1$ vertices. Also, since every x-detour set is a x-detour hull set, we have the following proposition. Throughout this paper G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1. [11] *Let G be a connected graph. Then*
(i) Each detour extreme vertex of G belongs to every detour hull set of G.
(ii) No cut vertex of G belongs to any minimum detour hull set of G.

Theorem 1.2. [9] *Each end vertex of G other than x (whether x is an end vertex or not) belongs to every minimum x-detour set of G.*

Theorem 1.3. [10] *Let x be a vertex of a connected graph G. Let S be any x-detour hull set of G. Then*
(i) Each x-detour extreme vertex of G belongs to S.
(ii) If v is a cut vertex of G and C a component of $G-v$ such that $x \notin V(C)$, then $S \cap V(C) \neq \emptyset$.
(iii) No cut-vertex of G belongs to any minimum x-detour hull set of G.

Theorem 1.4. [10] *For any vertex x in a connected graph G of order n, $dh_x(G) \leq n - e_D(x)$.*

2. GRAPHS OF ORDER n WITH VERTEX DETOUR HULL NUMBER $n-1$, $n-2$ AND $n-3$

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$. Then $dh_x(G) = n-1$ for every vertex x in G if and only if $G = K_2$.*

Proof. Suppose that $G = K_2$. Then $dh_x(G) = 1 = n - 1$. The converse follows from Theorem 1.4. □

Theorem 2.2. *Let G be a connected graph of order $n \geq 3$. Then $dh_x(G) = n - 2$ for every vertex x in G if and only if $G = K_3$.*

Proof. Suppose that $G = K_3$. Then it is clear that $dh_x(G) = 1 = n - 2$ for every vertex x in G. Conversely, suppose that $dh_x(G) = n - 2$ for every vertex x in G. Then by Theorem 1.4, $e_D(x) \leq 2$ for every vertex x in G. It follows from Theorem 2.1 that $e_D(x) \neq 1$ for every vertex x in G. Thus $e_D(x) = 2$ for every vertex x in G; or the vertex set can be partitioned into V_1 and V_2 such that $e_D(x) = 1$ for $x \in V_1$ and $e_D(x) = 2$ for $x \in V_2$. Thus either $rad_D(G) = diam_D(G) = 2$; or we have $rad_D(G) = 1$ and $diam_D(G) = 2$. This implies that either $G = K_3$ or $G = K_{1,n-1}$. If $G = K_{1,n-1}$, then by Theorem 1.3, $dh_x(G) = n - 1$ for the cut vertex x and $dh_y(G) = n - 2$ for any end vertex y in G, which is a contradiction to the hypothesis. Hence $G = K_3$. □

Theorem 2.3. *Let G be a connected graph of order $n \geq 2$. Then $G = K_{1,n-1}$ if and only if the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $dh_x(G) = n - 1$ for $x \in V_1$ and $dh_y(G) = n - 2$ for $y \in V_2$.*

Proof. Suppose that $G = K_{1,n-1}$. Then $dh_x(G) = n - 1$ for the cut vertex x in G and $dh_y(G) = n - 2$ for any end vertex y in G. Conversely, suppose that the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $dh_x(G) = n - 1$ for $x \in V_1$; and we have $dh_y(G) = n - 2$ for $y \in V_2$. Then by Theorem 1.4, $e_D(x) = 1$ for each $x \in V_1$ and $e_D(y) = 1$ or $e_D(y) = 2$ for each $y \in V_2$. It follows from Theorem 2.1 that $e_D(y) = 2$ for some $y \in V_2$. Hence $rad_D(G) = 1$ and $diam_D(G) = 2$. Thus $G = K_{1,n-1}$. □

Theorem 2.4. *Let G be a connected graph of order $n \geq 5$. Then G is a double star or $G = K_{1,n-1} + e$ if and only if the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $dh_x(G) = n - 2$ for $x \in V_1$ and $dh_y(G) = n - 3$ for $y \in V_2$.*

Proof. Suppose that G is a double star or $G = K_{1,n-1} + e$. Then it follows from Theorem 1.3 that $dh_x(G) = n - 2$ or $dh_x(G) = n - 3$ according to whether x is a cut vertex of G or not. Conversely, suppose that $dh_x(G) = n - 2$ for $x \in V_1$ and $dh_x(G) = n - 3$ for $x \in V_2$. Then by Theorem 1.4, $e_D(x) \leq 3$ for every x and so $diam_D(G) \leq 3$. It follows from Theorem 2.1 that $G \neq K_2$ and so $diam_D(G) \geq 2$. If $diam_D(G) = 2$, then G is the star $K_{1,n-1}$ and by Theorem 2.3, $dh_x(G) = n - 1$ or $dh_x(G) = n - 2$ for every vertex x. This is a contradiction to the hypothesis. Now, suppose that $diam_D(G) = 3$. If G is a tree, then G is a double star. If G is not a tree, then it is clear that $3 \leq cir(G) \leq 4$, where $cir(G)$ denotes the length of a longest cycle in G. We prove that $cir(G) = 3$. Suppose that $cir(G) = 4$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G. Since $n \geq 5$ and G is connected, there is a vertex x not on C_4 such that x is adjacent to some vertex say, v_1 of G. Then x, v_1, v_2, v_4, v_4 is a path of length 4 in G and so $diam_D(G) \geq 4$, which is a contradiction. Thus $cir(G) = 3$. Also, if G contains two or more cycles, then it follows that $diam_D(G) \geq 4$. Hence G contains a unique triangle, say $C_3 : v_1, v_2, v_3, v_1$. Since $n \geq 5$, at least one vertex of C_3 has degree at least 3. If there are two or more vertices of C_3 having degree at least 3, then $diam_D(G) \geq 4$, which is a contradiction. Thus exactly one vertex of C_3 has degree at least 3 and it follows that $G = K_{1,n-1} + e$. This completes the proof. □

3. Detour and Vertex Detour Hull Numbers and Addition of A Pendant Edge

In this section we discuss how the detour hull number and the vertex detour hull number of a connected graph are affected by adding a pendant edge to G. Let G' be a graph obtained from a connected graph G by adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G.

Theorem 3.1. *If G' is a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G, then $d_h(G) \leq d_h(G') \leq d_h(G) + 1$.*

Proof. Let S be a minimum detour hull set of G and let $S' = S \cup \{u\}$. We show that S' is a detour hull set of G'. Let $x \in V(G')$. If $x = u$, then $x \in S'$. So, assume that $x \in V(G)$. Then $x \in I_D^k[S]_G$ for some $k \geq 0$. Since $I_D^n[S]_G = I_D^n[S]_{G'}$ for all $n \geq 0$, we have $x \in I_D^k[S]_{G'}$. Also, since $S \subseteq S'$, we see that $I_D^n[S]_{G'} \subseteq I_D^n[S']_{G'}$ for all $n \geq 0$. Hence $x \in I_D^k[S']_{G'}$. This implies that S' is a detour hull set of G' so that $d_h(G') \leq |S'| = |S| + 1 = d_h(G) + 1$. For the lower bound, let S' be a minimum detour hull set of G'. Then by Theorem 1.1, $u \in S'$ and $v \notin S'$. Let $S = (S' - \{u\}) \cup \{v\}$. We prove that S is a detour hull set of G. For this, first we claim that $I_D^k[S']_{G'} - \{u\} \subseteq I_D^k[S]_G$ for all $k \geq 0$. We use induction on k. Since $S' - \{u\} \subseteq S$, the result is true for $k = 0$. Let $k = 1$ and let $x \in I_D[S']_{G'} - \{u\}$. Then $x \neq u$. If $x \in S'$, then $x \in S \subseteq I_D[S]_G$. If $x \notin S'$, then there exist $y, z \in S'$ such that $x \in I_D[y, z]_{G'}$ with $x \neq y, z$. If $y \neq u$ and $z \neq u$, then $y, z \in S$ and so $I_D[y, z]_G = I_D[y, z]_{G'}$. Thus $x \in I_D[S]_G$. Now, let $y = u$ or $z = u$, say $z = u$. Since v is a cut vertex of G', it follows that $x \in I_D[y, v]_{G'} = I_D[y, v]_G$ and hence $x \in I_D[S]_G$. Assume that the result is true for $k = l$. Then $I_D^l[S']_{G'} - \{u\} \subseteq I_D^l[S]_G$. Now, let $x \in I_D^{l+1}[S']_{G'} - \{u\}$. If $x \in I_D^l[S']_{G'}$, then by induction hypothesis, we have $x \in I_D^l[S]_G \subseteq I_D^{l+1}[S]_G$. If $x \notin I_D^l[S']_{G'}$, then there exist $y, z \in I_D^l[S']_{G'}$ such that $x \in I_D[y, z]_{G'}$ with $x \neq y, z$. If $y \neq u$ and $z \neq u$, then it follows from induction hypothesis that $y, z \in I_D^l[S]_G$. Also, since $I_D[y, z]_{G'} = I_D[y, z]_G$, we have $x \in I_D^{l+1}[S]_G$. Let $y = u$ or $z = u$, say $z = u$. Then $y \neq u$ and so by induction hypothesis, $y \in I_D^l[S]_G$. Since v is a cut vertex of G', it follows that $x \in I_D[y, v]_{G'} = I_D[y, v]_G$. Also, since $v \in S \subseteq I_D^l[S]_G$, it follows that $x \in I_D^{l+1}[S]_G$. Hence the proof of the claim is complete by induction. Now, since S' is a minimum detour hull set of G', there is an integer $r \geq 0$ such that $I_D^r[S']_{G'} = V(G')$ and it follows from the above claim that $I_D^r[S]_G = V(G)$. Thus S is a detour hull set of G so that $d_h(G) \leq |S| = |S'| = d_h(G')$. This completes the proof. □

Remark 3.2. The bounds for $d_h(G')$ in Theorem 3.1 are sharp. Let G' be the graph obtained from the graph G in Figure 3, by adding a pendant edge at one of its end vertices. Then $d_h(G') = d_h(G) = 2$. If G' is obtained from G by adding a pendant edge at one of its cut vertices, then $d_h(G') = d_h(G) + 1$.

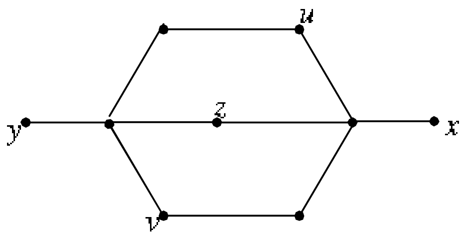

FIGURE 3. Graph G with $d_h(G') = d_h(G) + 1$

Theorem 3.3. *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. Then $d_h(G) = d_h(G')$ if and only if v is a vertex of some minimum detour hull set of G.*

Proof. First, assume that there is a minimum detour hull set S of G such that $v \in S$. Let $S' = (S - \{v\}) \cup \{u\}$. Then $|S'| = |S|$. We show that S' is a detour hull set of G'. First, we claim that $I_D^k[S]_G \subseteq I_D^{k+1}[S']_{G'}$ for all $k \geq 0$. We prove this by using induction on k. Let $k = 0$. Let $x \in S$. If $x \neq v$, then $x \in S' \subseteq I_D[S']_{G'}$. If $x = v$, then $x \in I_D[y, u]_{G'} \subseteq I_D[S']_{G'}$, where $y \in S$ such that $y \neq v$. Thus $S \subseteq I_D[S']_{G'}$. Assume the result for $k = l$. Then $I_D^l[S]_G \subseteq I_D^{l+1}[S']_{G'}$. Let $x \in I_D^{l+1}[S]_G$. If $x \in I_D^l[S]_G$, then by induction hypothesis, $x \in I_D^{l+1}[S']_{G'} \subseteq I_D^{l+2}[S']_{G'}$. If $x \notin I_D^l[S]_G$, then there exist $y, z \in I_D^l[S]_G$ such that $x \in I_D[y, z]_G = I_D[y, z]_{G'}$. By induction hypothesis, we have $y, z \in I_D^{l+1}[S']_{G'}$ and so $x \in I_D^{l+2}[S']_{G'}$.Hence by induction $I_D^k[S]_G \subseteq I_D^{k+1}[S']_{G'}$ for all $k \geq 0$. Now, since S is a detour hull set of G, there exists an integer $r \geq 0$ such that $I_D^r[S]_G = V(G)$ and it follows from the above claim that $I_D^{r+1}[S']_{G'} = V(G')$. Thus S' is a detour hull set of G so that $d_h(G') \leq |S'| = |S| = d_h(G)$. The other inequality follows from Theorem 3.1.

Conversely, let $d_h(G) = d_h(G')$. Let S' be a minimum detour hull set of G'. Then by Theorem 1.3, $u \in S'$ and $v \notin S'$. Let $S = (S' - \{u\}) \cup \{v\}$. Then, as in the proof of Theorem 3.1, we can prove that S is a detour hull set of G. Since $|S| = |S'| = d_h(G') = d_h(G)$, we see that S is a minimum detour hull set of G and $v \in S$. This completes the proof. □

Theorem 3.4. *Let G be a connected graph and let x be any vertex in G. If G' is a graph obtained from G by adding a pendant edge xu, then $dh_x(G') = dh_x(G) + 1$.*

Proof. Let S be a minimum x-detour hull set of G and let $S' = S \cup \{u\}$. Then, as in Theorem 3.1, it is straight forward to verify that $I_D^n[S]_G^x \subseteq I_D^n[S']_{G'}^x$ for all $n \geq 0$. Since S is an x-detour hull set of G, there is an integer $r \geq 0$ such that

$I_D^r[S]_G^x = V(G) - \{x\}$ and it is clear that $I_D^r[S']_{G'}^x = V(G') - \{x\}$. Hence S' is an x-detour hull set of G' so that $dh_x(G') \leq |S'| = dh_x(G) + 1$. Now, suppose that $dh_x(G') < dh_x(G) + 1$. Let S' be a minimum x-detour hull set of G'. Then, by Theorem 1.3, $u \in S'$. Let $S = S' - \{u\}$. Then, as in Theorem 3.1, it is straight forward to prove that $I_D^n[S']_{G'}^x - \{u\} \subseteq I_D^n[S]_G^x$ for all $n \geq 0$. Since S' is an x-detour hull set of G', there is an integer $r \geq 0$ such that $I_D^r[S']_{G'}^x = V(G') - \{x\}$. Hence $I_D^r[S]_G^x = V(G) - \{x\}$. Thus S is an x-detour hull set of G so that $dh_x(G) \leq |S| = dh_x(G') - 1$, which is a contradiction to $dh_x(G') < dh_x(G) + 1$. Hence the result follows. □

Theorem 3.5. *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. Then $dh_u(G') = dh_v(G)$.*

Proof. Let S be a minimum v-detour hull set of G. Then $v \notin S$. As in Theorem 3.1, it is straight forward to prove that $I_D^n[S]_G^v \subseteq I_D^n[S]_{G'}^u$ for all $n \geq 0$. Since S is a v-detour hull set of G, there is an integer $r \geq 0$ such that $I_D^r[S]_G^v = V(G) - \{v\}$. Now, since $v \in I_D[z]_G^u$ for any $z \in S$, it follows that $I_D^r[S]_G^u = V(G') - \{u\}$. Hence S is a u-detour hull set of G' so that $dh_u(G') \leq |S| = dh_v(G)$. For the other inequality, let T be a minimum u-detour hull set of G'. Then $u \notin T$ and by Theorem 1.3(iii), $v \notin T$. As in Theorem 3.1, it is straight forward to prove that $I_D^n[T]_{G'}^u - \{v\} \subseteq I_D^n[T]_G^v$ for all $n \geq 0$. Since T is a u-detour hull set of G', there is an integer $r \geq 0$ such that $I_D^r[T]_{G'}^u = V(G') - \{u\}$. Hence it follows that $I_D^r[T]_G^v = V(G) - \{v\}$ and T is a v-detour hull set of G. Thus $dh_v(G) \leq |T| = dh_u(G')$. This completes the proof. □

Theorem 3.6. *Let G be a connected graph and x any vertex of G. Let G' be a graph obtained from G by adding a pendant edge uv at a vertex $v \neq x$ of G. Then $dh_x(G) \leq dh_x(G') \leq dh_x(G) + 1$.*

Proof. The proof is similar to Theorem 3.1. □

Theorem 3.7. *Let G be a connected graph and x any vertex of G. Let G' be a graph obtained from G by adding a pendant edge uv at a vertex $v \neq x$ of G. Then $dh_x(G) = dh_x(G')$ if and only if v belongs to some minimum x-detour hull set of G.*

Proof. The proof is similar to Theorem 3.3. □

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