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A FEW MORE PROPERTIES OF SUM AND INTEGRAL SUM GRAPHS

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Abstract. The concepts of sum graph and integral sum graph were introduced by Harary. A *sum graph* is a graph whose vertices can be labeled with distinct positive integers so that the sum of the labels on each pair of adjacent vertices is the label of some other vertex. Integral sum graphs have the same definition except that the labels may be any integers. Harary gave examples of all orders of sum graphs G_n and integral sum graphs $G_{-n,n}, n \in \mathbb{N}$. The family of integral sum graph was extended by Vilfred and in this paper we obtain a few properties of sum and integral sum graphs and two new families of integral sum graphs.

Key words: Supplementary vertices in a sum graph, integral sum labeling, triangular book with a book mark, fan graph with a handle.

Abstrak. Konsep graf penjumlahan dan graf penjumlahan integral diperkenalkan pertama kali oleh Harary. Sebuah *graf penjumlahan* adalah sebuah graf yang titik-titiknya dapat dilabelkan dengan bilangan bulat positif yang berbeda sehingga jumlahan dari label-labelnya pada tiap pasang titik-titik yang bertetangga adalah label dari suatu titik lain. Graf penjumlahan integral memiliki definisi yang sama kecuali dalam hal jumlahan dari label-labelnya adalah suatu bilangan bulat. Harary telah menunjukkan contoh-contoh dari semua orde dari graf penjumlahan G_n dan graf penjumlahan integral $G_{-n,n}, n \in \mathbb{N}$. Keluarga graf penjumlahan integral telah diperluas oleh Vilfred dan dalam paper ini kami mendapatkan beberapa sifat dari graf-graf penjumlahan dan penjumlahan integral dan dua keluarga baru dari graf penjumlahan integral.

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Kata kunci: Titik-titik pengganti, pelabelan graf penjumlahan, buku segitiga, graf kipas dengan pegangan.

1. Introduction

Harary [7],[8] introduced the concepts of sum and integral sum graphs. A graph G is a *sum graph* if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. He extended the concept to allow any integers and called them as *integral sum graphs.* To distinguish between the two types, we call sum graphs that use only positive integers $\mathbb{N} - sum\ graphs$ and those with any integers $\mathbb{Z} - sum\ graphs$ (See [10]). For any non-empty set of integers S, we let $G^+(S)$ denote the integral sum graph on the set S and $-G^+(S) = G^+(-S)$. For integers r and s with $r < s$ we also let $[r, s]$ denote the set of integers $\{r, r+1, \ldots, s\}$. Harary's examples of $\mathbb{N} - sum$ graphs are thus $G^+([1, n]) = G_n$ and his $\mathbb{N} - sum$ graphs are $G^+([-r, r]) = G_{-r,r}$ for $r \in \mathbb{N}$. (Note that his notation is modified and we write $G_{-r,r}$ for what he called $G_{r,r}$. See [10]). The extension of Harary graphs to all intervals of integers was introduced by Vilfred in [14]: for any integers r and s with $r < s$, let $G_{r,s} = G^+([r, s])$. We denote the sum graph $G^+([1, n])$ by G_n^+ when it is labeled and by G_n when it is unlabeled.

In G_n^+, for $n \geq 2, d(u_i) = n - 1 - i$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $d(u_i) = n - i$ if $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n$ where $\lfloor x \rfloor$ denotes the floor of x and u_i is the vertex with label i, $1 \leq i \leq n$. Graphs $G_n^+, (G_n^+)^c$ and $K_n = G_n^+ \cup (G_n^+)^c$ for $n = 3$ to 8 are given in Figures 1 to 6. Different properties of sum and integral sum graphs are studied by several authors [2],[4],[7]-[23]. In [20], for $n \geq 4$, $G_{\Delta n}$ is introduced and defined as an integral sum graph of order n having precisely two vertices each of degree $n - 1$; $G_{-1,1} = K_1 + ((-G_1) + G_1) \cong K_3$ without vertex labels is the only integral sum graph G having more than two vertices, each of degree $|V(G)| - 1$ [20]; it is proved that Fan, Dutch M-Windmill [18], Banana trees and union of stars [19] are integral sum graphs; in [14] maximal integral sum graph is studied; and number of cycles of length 3 [12] and of length 4 [21] in $G_{m,n}$ are obtained. For $m < 0 < n$ and $m, n \in \mathbb{Z}, G_{m,n} = K_1 + ((-G_{-m}) + G_n)$ is an integral sum graph of order $(-m) + n + 1$. Integral sum graphs $G_{0,7}, G_{-1,6}, G_{-2,5}$ and $G_{-3,4}$ are given in Figures 7 and 8. In this paper the *underlying graph* of an integral sum graph is obtained by removing all vertex labels. We obtain new properties of sum and integral sum graphs and two new families of integral sum graphs, triangular book of n pages with book mark and fan graph with a handle. Now let us see the following definitions.

Two distinct vertices of sum graph $G^+(S)$ with n as its maximum value of vertex labels are called *supplementary vertices* if the sum of their vertex labels $=$ $n + 1$. (See [10])

A graph G is an $anti - sum\ graph$ or $anti - \mathbb{N} - sum\ graph$ if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is not a vertex label in G. An

$anti-integral\ sum\ graph$ or $anti-\mathbb{Z}-sum\ graph$ is also defined just as $anti-sum$ graph, the difference being that the labels may be any distinct integers. Clearly, f is an integral sum labeling of graph G if and only if f is an $anti-integral$ sum labeling of G^c.

Two unlabeled graphs are said to be *comparable* if one is a subgraph of the other, while two labeled graphs are *comparable* if one is *a* subgraph of the other with the labels preserved. Clearly, any two Harary graphs G_m^+ and G_n^+ are comparable. In contrast, it is easy to check that labeled graphs G_3^+ and $G_2^+ + G^+(\{3\})$ are not comparable even though as unlabeled graphs G_3 is a (spanning) subgraph of $G_2+G^+(\{3\})$. We also observe that for all r and s with $0 \leq r \leq s, G_r \cup G^+([r+1, s])$ is a (spanning) subgraph of G_s.

A graph G is a *split graph* if its vertices can be partitioned into a clique and a stable set. *A clique* in a graph is a set of pair-wise adjacent vertices and an *independent set* or *stable set* in a graph is a set of pair-wise non-adjacent vertices [3]. G_n and G_n^c are split graphs. Clearly, $[1, m], [1, m+1], [m+1, 2m], [m+2, 2m+1]$ are cliques and $[m+1, 2m], [m+2, 2m+1], [1, m], [1, m+1]$ are stable sets in $G_{2m}, G_{2m+1}, G_{2m}^c, G_{2m+1}^c$, respectively.

A graph H is *decomposable* into the subgraphs $H_1, H_2, \ldots, H_n$ of H, if no H_i has isolated vertices and the edge set of H can be partitioned into the subsets $E(H_1), E(H_2), \ldots, E(H_n), i = 1, 2, \ldots, n$. Graph H is said to be *F-decomposable*, if $H_i \cong F$ for every $i, i = 1, 2, \ldots, n$. If H is F-decomposable, then we say that F *divides* H and we write F/H [6].

When k copies of C_n share a common edge, it will form an $n-gon$ *book of* k *pages*, and it is denoted by $B(n, k)$ or $B_{n,k}$. The common edge is called the *spine* or *base* of the book. *A triangular book* $B(3, k)$ or $B_{3,k}$ consists of k triangles with a common edge and can be described as $B(3, k) = ST(k) + K_1 = P_2 + (k.K_1)$ where $ST(k)$ denotes the star with k leaves. Triangular book $B(3, k)$ with the spine (u, v) is denoted by $TB_k(u, v) = P_2(u, v) + (k.K_1)$. We call $TB_0 = K_2$, *a book without pages as the trivial book* [11].

An $n-gon$ book of k pages $B(n, k)$ with a pendant edge terminating from a vertex of the spine is called an $n-gon$ *book with a book mark*. Triangular book $TB_k(u, v)$ with book mark (u, w) is denoted by $TB_k(u, v)(u, w)$ or $TB_k^*(u, v)$ where w is the pendant vertex adjacent to u. $TB_k^*(u, v)$ is of order $k+3$ and size $2k+2$. $TB_4^*(u_0, v_0)$ is given in Figure 9.

A *fan graph* F_{n-1} is the graph obtained by taking $n-3$ concurrent chords at a vertex in a cycle $C_n, n \geq, 3$. The vertex at which all the $n-3$ chords are concurrent is called the *apex vertex* [23]. The fan graph $F_n = P_n + K_1$ and is an integral sum graph for $n \geq 2$ [18] where P_n is a path on n vertices. Integral sum labeling of F_5 is given in Figure 9.

Fan graph F_n with a pendant edge attached with the apex vertex is called *a fan with a handle* or *a palm fan* and it is denoted by F_n^* [11]. F_5^* , fan graph F_5 with handle u_0v_0 is given in Figure 9.

We consider simple graphs only. For all basic notation and definitions in graph theory, we follow [6] and for additional reading on related graph labeling problems, we refer to [5].

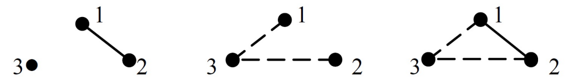

Fig. 1: G_3^+ $(G_3^+)^c$ $K_3 = G_3^+ \cup (G_3^+)^c$

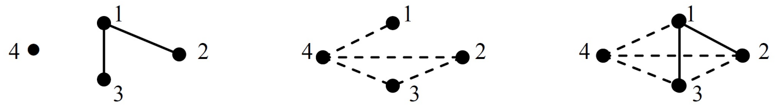

Fig. 2: G_4^+ $(G_4^+)^c$ $K_4 = G_4^+ \cup (G_4^+)^c$

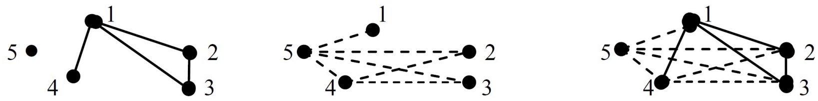

Fig. 3: G_5^+ $(G_5^+)^c$ $K_5 = G_5^+ \cup (G_5^+)^c$

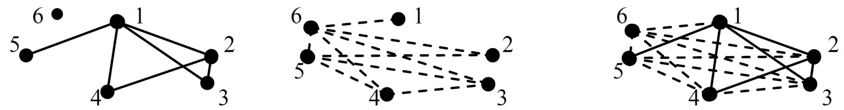

Fig. 4: G_6^+ $(G_6^+)^c$ $K_6 = G_6^+ \cup (G_6^+)^c$

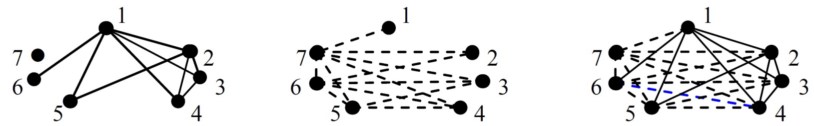

Fig. 5: G_7^+ $(G_7^+)^c$ $K_7 = G_7^+ \cup (G_7^+)^c$

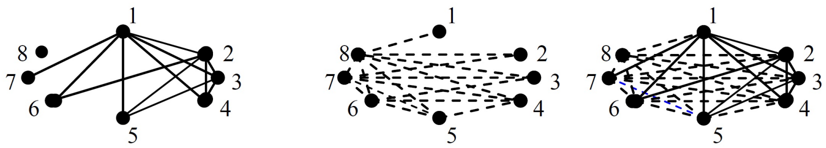

Fig. 6: G_8^+ $(G_8^+)^c$ $K_8 = G_8^+ \cup (G_8^+)^c$

Fig. 7: $G^+_{0,7}$. $G^+_{-1,6}$.

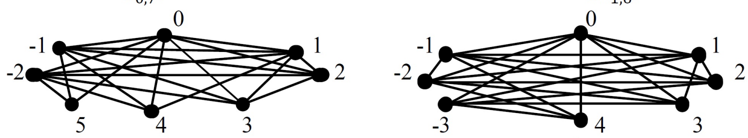

Fig. 8: $G^+_{-2,5}$. $G^+_{-3,4}$.

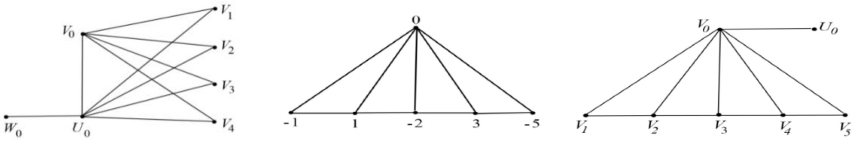

Fig. 9: $TB_4^*(u_0,v_0)$. F_5. F_5^*.

2. A Few Properties of Sum and Integral Sum Graphs

We now turn to some subgraph relationships between graphs. For convenience, if graph F is a subgraph of graph G without the vertex labels, this will often be denoted by $F \subseteq_{wvl} G$. Two graphs are called *non-comparable* if neither is a subgraph of the other. If F is an induced subgraph of G, this will sometimes be denoted by $F \leq G$ and by $F \leq_{wvl} G$. Similarly, if F is congruent to G without the vertex labels, we often write $F \cong_{wvl} G$ [10]. The following theorems from [10], [22] state a few known properties of sum graphs.

Theorem 2.1. [10] If $S \subseteq [r, 2r]$ for some $r \in \mathbb{N}$,then $G^+(S)$ is totally disconnected. □

Theorem 2.2. [10] Let $S = [r + l, r + n]$ with $r \in \mathbb{N}_0, n \geq 3$.

(a) For $n \leq r + 2, G^+(S)$ is totally disconnected.
(b) For $n \geq r + 3, G^+(S) \cong G_{n-r} \cup K_r^c$. □

Corollary 2.3. [10] *If* $0 \leq r \leq n-3$ *and* $S = [r+l, r+n]$,*then* $G_{n-r} \subseteq_{wvl} G^+(S)$. *In particular,* $G_{n-2} \subseteq_{wvl} G^+([2, n]) \subseteq_{wvl} G_{n-1}$. □

Theorem 2.4. [10] Let k and n be such that $2 \leq 2k \leq n$. If k pairs of supplementary vertices are removed from

(i) Harary graph G_n, then the result is isomorphic to G_{n-2k} without the vertex labels and
(ii) the graph G_n^c, then the result is isomorphic to G_{n-2k}^c without the vertex labels. □

Theorem 2.5. [13] For $r \geq 1$ and $n \geq r+2$,

(i) $G_r \cup G^+([r+1,n]) \leq G_n \leq G_1 + G^+([2,n])$ and
(ii) $G_1 \cup (G^+([2,n]))^c \leq G_n^c \leq G_r^c + G^+([r+1,n])^c$. □

Theorem 2.6. [13] Let n and r be positive integers.

(a) For $n \geq 4$ and $n \geq r \geq 2$, the following pairs of labeled graphs
  (i) G_n and $G_r + G^+([r+1,n])$ and
  (ii) G_n^c and $G_r^c \cup (G^+([r+1,n]))^c$ are non-comparable.
(b) For $n \geq 5$, the following pairs of graphs
  (i) G_n and $G_2 + G^+([3,n])$ and
  (ii) G_n^c and $G_2^c \cup (G^+([3,n]))^c$ without the vertex labels are non-comparable.
  □

Theorem 2.7. [10] For $n \geq 2, G_n \cong_{wvl} G_n^c - \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil + 1)\}$. □

Theorem 2.8. [10] For all $n \in \mathbb{N}, G_{0,n} \cong_{wvl} G_{n+1}^c$. □

Theorem 2.9. [10] For all $m,n \in \mathbb{N}_0, G_{-m,n} \cup G_{0,m-1} \cup G_{0,n-1} \cong_{wvl} K_{m+n+1}$. □

We have $|E(G_n)| = \frac{1}{2}(nC_2 - \lfloor \frac{n}{2} \rfloor), |E(G_n^c)| = \frac{1}{2}(nC_2 + \lfloor \frac{n}{2} \rfloor), G_n$ is a disconnected graph with at least one isolated vertex and G_n^c is a connected graph where $\lfloor x \rfloor$ denotes the floor of x. In particular, we have $|E(G_{2n})| = n^2 - n = |E(G_{2n-1}^c)|$ and $|E(G_{2n+1})| = n^2 = |E(G_{2n}^c)|, n \in \mathbb{N}$.

Pythagoreans knew that squared numbers are sums of sequences of odd numbers (That is $1 = 1, 4 = 1+3, 9 = 1+3+5, 16 = 1+3+5+7, \ldots$) [1] and similar to this, we obtain a relation among number of edges of G_{2n+1} and the maximal degree vertices of $G_{2n+1}, G_{2n-1}, \ldots, G_3$. Using Theorem 2.4, the underlying graphs of $G_{2n+1} - \{u_1, u_{2n+1}\}$ and G_{2n-1} are isomorphic and in G_{2n+1}, vertex u_{2n+1} is non-adjacent to all other vertices and vertex u_1 is adjacent to all other vertices, except u_{2n+1} where u_j is the vertex of G_{2n+1} with sum label $j, 1 \leq j \leq 2n+1$. This implies, the underlying graphs of $\{u_{2n+1}\} \cup (\{u_{2n}\} + G_{2n-1})$ and G_{2n+1} are isomorphic where u_j is the vertex of G_{2n+1} with sum label $j, 1 \leq j \leq 2n+1$ and $n \in \mathbb{N}$. Now, using the algorithm, $\{u_{2i+1}\} \cup (\{u_{2i}\} + G_{2i-1}) \cong G_{2i+1}$ and applying the results, $|E(G_{2i+1})| = i^2$ and $|E(G_{2i+1})| = (2i-1) + |E(G_{2i-1})| = d_{G_{2i+1}}(u_1) + |E(G_{2i-1})|$ for $i = n, n-1, \ldots, 3$, successively, we obtain $n^2 = |E(G_{2n+1})| = (2n-1) + |E(G_{2n-1})| = (2n-1) + (2n-3) + |E(G_{2n-3})| = \ldots = (2n-1) + (2n-3) + \ldots + 3 + |E(G_3)| = (2n-1) + (2n-3) + \ldots + 1 = d_{G_{2n+1}}(u_1) + d_{G_{2n-1}}(u_1) + \ldots + d_{G_3}(u_1)$.

3. Decomposition of K_n into Sum and Integral Sum Graphs

In this section we consider decomposition of complete graphs into underlying graphs of sum and integral sum graphs. The first result follows from Theorem 2.7.

Theorem 3.1. For $3 \leq n$, the underlying graphs of $K_n, 2(G_n^+) \cup \lfloor \frac{n}{2} \rfloor .P_2$ and $G_n^c \cup (G_n^c - \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor)\})$ are isomorphic.

Proof. Using Theorem 2.7, the underlying graphs of G_n and $G_n^c - \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor)\}$ are isomorphic. This implies, the underlying graphs of $K_n = G_n \cup G_n^c, 2.G_n \cup \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor)\}, 2(G_n^+) \cup \lfloor \frac{n}{2} \rfloor .P_2$ and $G_n^c \cup (G_n^c - \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor)\})$ are isomorphic. Hence the result. □

Theorem 3.2. The underlying graphs of $G_{0,n}$ and G_{n+1}^c are isomorphic, $n \in \mathbb{N}$.

Proof. Using the definition of anti-sum labeling, we obtain, $G_{n+1}^c = (((G_n^c \cup K_1(n+1)) - \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor)\}) \cup \{(1,n+1),(2,n+1),\ldots,(n,n+1)\}) \cong (G_n^c - \{(1,n),(2,n-1),\ldots,(\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor)\}) \cup K_1(n+1) \cup \{(1,n+1),(2,n+1),\ldots,(n,n+1)\}$ which is isomorphic to the underlying graph of $G_n \cup K_1(n+1) \cup \{(1,n+1),(2,n+1),\ldots,(n,n+1)\} = G_n + K_1(n+1)$ where $K_1(n+1)$ represents a vertex with vertex label $n+1$ in the graph $G_n^c \cup K_1(n+1)$. Thus, the underlying graphs of $G_{n+1}^c, G_n + K_1(n+1)$ and $G_{0,n}$ are isomorphic since the underlying graphs of $G_n + K_1(n+1)$ and $G_{0,n}$ are isomorphic. Hence the result. □

Theorem 3.3. For $m,n \in \mathbb{N}_0$, the underlying graphs of K_{m+n+1} and $G_{-m,n} \cup G_{0,m-1} \cup G_{0,n-1}$ are isomorphic.

Proof. For $m,n \in \mathbb{N}_0, G_{-m,n} = K_1 + ((-G_m) + G_n)$ and hence, $G_{-m,n}^c = K_1(0) \cup \overline{(-G_m) + G_n}^c = K_1(0) \cup (-G_m^c) \cup G_n^c$ which is isomorphic to the underlying graph of $K_1(0) \cup G_{0,m-1} \cup G_{0,n-1}$, using Theorem 3.2. Hence the result. □

Corollary 3.4. For $2 \leq n$, the underlying graphs of $K_{2n}, G_{-(n-1),n} \cup G_{0,n-2} \cup G_{0,n-1}$ and $2.G_n \cup n.P_2$ are isomorphic and that of K_{2n+1} and $G_{-n,n} \cup 2.G_{0,n-1}$ are isomorphic.

Proof. For $2 \leq n$, using Theorem 3.3, we obtain that the underlying graphs of K_{2n} and $G_{-(n-1),n} \cup G_{0,n-2} \cup G_{0,n-1}$ are isomorphic and the underlying graphs of K_{2n+1} and $G_{-n,n} \cup 2.G_{0,n-1}$ are isomorphic and using Theorem 3.1, we obtain that the underlying graphs of K_{2n} and $G_{-(n-1),n} \cup G_{0,n-2} \cup G_{0,n-1} \cong 2.G_n \cup n.P_2$ are isomorphic. Hence the result. □

4. A Few Properties of Integral Sum graphs $G_{0,n}$

In this section we obtain a few structural properties of integral sum graphs $G_{0,n}, n \in \mathbb{N}$.

Theorem 4.1. For $n \geq 3$, the underlying graphs of $G_{0,n} - \{0, n\}$ and $G_{0,n-2}$ are isomorphic.

PROOF. Let $V(G_{0,n-2}) = \{u_0, u_1, \ldots, u_{n-2}\}$ and $V(G_{0,n}) = \{v_0, v_1, \ldots, v_n\}$ where i and j are integral sum labels of u_i and v_j, respectively, $0 \leq i \leq n-2$ and $0 \leq j \leq n$. Define mapping $f : V(G_{0,n-2}) \to V(G_{0,n})$ such that $f(u_i) = v_{i+1}$ for $i = 0, 1, \ldots, n-2$. Now, u_i and u_j are adjacent in $G_{0,n-2}$ if and only if $i \neq j, 0 \leq i, j \leq n-2$ and $0+1 = 1 \leq i+j \leq n-2$ if and only if $i+1 \neq j+1, 1 \leq i+1, j+1 \leq n-1$ and $1+2 = 3 \leq (i+1)+(j+1) \leq n = (n-1)+1$ if and only if v_{i+1} and v_{j+1} are adjacent in $G_{0,n} - \{0, n\}$ if and only if $f(u_i)$ and $f(u_j)$ are adjacent in $G_{0,n} - \{0, n\}$. This implies the mapping f is bijective, preserves adjacency and $f(G_{0,n} - \{0, n\}) = G_{0,n-2}$. $\Box$

Theorem 4.2. For $n \geq 5$, the underlying graphs of $G_{0,n} - (\{0, n, n-1, n-2\} \cup [n] \cup [n-1])$ and $G_{0,n-4}$ are isomorphic.

PROOF. Using the definition of integral sum labeling we obtain isomorphic graphs of the underlying graphs of $G_{0,n} - (\{n, n-1\} \cup [n] \cup [n-1])$ and $G_{0,n-2}$ where $[k]$ in $G^+(S)$ denotes the set of all edges of $G^+(S)$ whose edge sum value is $k, k \in S$ [16]. Using Theorem 4.1, the underlying graphs of $G_{0,n-2} - \{0, n-2\}$ and $G_{0,n-4}$ are isomorphic. Hence the result. $\Box$

Generalizing the above Theorem, we obtain the following result.

Theorem 4.3. For $n \geq 3$, the underlying graphs of $G_{0,n} - \{0, n\}$ and $G_{0,n-2}$ are isomorphic and for $n \geq 2r+3$ and $r \in \mathbb{N}$, the underlying graphs of $G_{0,n} - (\{0, n, n-1, n-2, \ldots, n-2r+1, n-2r\} \cup ([n] \cup [n-1] \cup \ldots \cup [n-2r+1]))$ and $G_{0,n-2r-2}$ are isomorphic. $\Box$

Theorem 4.4. For $5 \leq n$, the underlying graphs of $G_{0,n} - \{0, 1, n-1, n\}$ and $G_{0,n-4}$ are isomorphic where u_j is the vertex of $G_{0,n}$ with integral sum label $j, j = 0, 1, \ldots, n$.

PROOF. Using Theorem 3.2, the underlying graphs of $G_{0,n}$ and G^c_{n+1} are isomorphic and from the structure of these graphs (See graphs G^c_8 in Figures 6 and $G_{0,7}$ in Figure 7.), vertex with integral sum label j in $G_{0,n}$ and vertex with anti-integral sum label $n-j+1$ in G^c_{n+1} are of same degree and thereby the underlying graphs of $G_{0,n} - \{0, 1, n-1, n\}$ and $G^c_{n+1} - \{n+1, n, 2, 1\}$ are isomorphic, $0 \leq j \leq n$. Using Theorem 2.4, the underlying graphs of $G^c_{n+1} - \{1, 2, n, n+1\}$ and G^c_{n-3} are isomorphic and using Theorem 3.2, the underlying graphs of G^c_{n-3} and $G_{0,n-4}$ are isomorphic. This implies, the underlying graphs of $G_{0,n} - \{0, 1, n-1, n\}$ and $G_{0,n-4}$ are isomorphic. Hence the result. $\Box$

Theorem 4.5. For $n \geq 3$, the following pairs of underlying graphs of

(i) $(G_{0,n-1})^c$ and $G_{0,n-2} \cup K_1(n-1)$ and

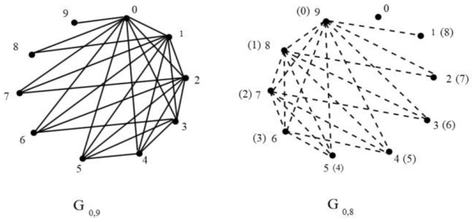

Fig. 10.

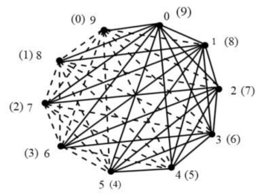

Fig. 11: $K_{10} \cong G_{0,9} \cup (G_{0,8} \cup K_1(9))$.

(ii) K_n and $G_{0,n-1} \cup (G_{0,n-2} \cup K_1(n-1))$ are isomorphic where $K_1(n-1)$ is an isolated vertex with label $n-1$.

Proof. We have the underlying graphs of $(G_{0,n-1})^c$ = the underlying graphs of $(K_1(0) + G_{n-1})^c$ = the underlying graphs of $K_1(0) \cup G^c_{n-1}$ whose underlying graph is isomorphic to the underlying graph of $K_1(0) \cup G_{0,n-2}$ which is isomorphic to the underlying graph of $K_1(n-1) \cup G_{0,n-2}$, using Theorem 3.2 where $K_1(0)$ and $K_1(n-1)$are isolated vertices in the underlying graphs of $K_1(0) \cup G_{0,n-2}$ and $K_1(n-1) \cup G_{0,n-2}$, respectively. This implies the underlying graphs of $(G_{0,n-2} \cup K_1(n-1))^c$ and $G_{0,n-1}$ are isomorphic. This implies the underlying graphs of K_n and $G_{0,n-1} \cup (G_{0,n-2} \cup K_1(n-1))$ are isomorphic where $K_1(n-1)$ is an isolated vertex with label $n-1$. Hence the result. See Figures 10 and 11. □

5. Two New Families of Integral Sum Graphs

In [18] it is proved that fan graph $F_n = P_n + K_1$ is an integral sum graph, $n \geq 2$. In this section we obtain two new families of integral sum graphs, namely, triangular book with a book mark and fan with a handle.

Theorem 5.1. For $n \in \mathbb{N}$,

(i) $TB_n(u_0, v_0)(u_0, w_0)$ and
(ii) F^*_n are integral sum graphs.

Proof.

(i) $TB_n(u_0, v_0)(u_0, w_0)$ is of order $n+3$, size $2n+2$ and (u_0, w_0) is the pendant edge terminating at u_0. Let $V(TB_n(u_0, v_0)(u_0, w_0)) = \{w_0, u_0, v_0, v_1, \ldots, v_n\}$. Define mapping $f : V(TB_n(u_0, v_0)(u_0, w_0)) \to \mathbb{N}_0$ such that $f(u_0) = 0$, $f(v_0) = 2m$, $f(v_i) = 2mi + 1$ for $i = 1, 2, \cdots, n$ and $f(w_0) = 2m(n+1) + 1$, $m \in \mathbb{N}$. Consider the integral sum graph $G^+(S)$ where $S = \{0, 2m, 2m+1, 4m+1, 6m+1, \ldots, 2mn+1, 2m(n+1)+1/m \in \mathbb{N}\} = f(V(TB_n(u_0, v_0)(u_0, w_0)))$. Our aim is to prove that $G^+(S) = TB_n(u_0, v_0)(u_0, w_0)$. $f(u_0) = 0$ implies, $f(u_0) + f(v_i) = f(v_i)$ and $f(u_0) + f(w_0) = f(w_0)$ for $i = 0, 1, 2, \ldots, n$. This implies, u_0 is adjacent to w_0, v_0 and v_i for

$i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n-1$, $f(v_0)+f(v_i) = f(v_{i+1})$, $f(v_0)+f(v_n) = f(w_0)$, $f(v_0) + f(u_0) = f(v_0)$, $f(v_0) + f(w_0) \neq f(u_0), f(v_0), f(w_0), f(v_j)$ for $j = 1, 2, \ldots, n$. This implies v_0 is adjacent to u_0 and v_i and non-adjacent to w_0 for $i = 1, 2, \ldots, n$. Also $f(w_0) + f(u_0) = f(w_0)$ and $f(w_0) + f(v_j) \neq f(w_0), f(u_0), f(v_j)$ for $j = 0, 1, \ldots, n$. This implies, w_0 is a pendant vertex adjacent only to u_0.

For $i, j = 0, 1, 2, \ldots, n$, $f(v_i) + f(w_0) \neq f(u_0), f(v_j)$. Also for $1 \leq i, j, k \leq n$, $f(v_i) + f(v_j) \neq f(v_k)$ since $f(v_i) + f(v_j)$ is an even number and $f(v_k)$ is an odd number. This implies, v_i and v_j are non-adjacent in $TB_n(u_0, v_0)(u_0, w_0)$ when $i \neq j$ and $1 \leq i, j \leq n$. Thus v_j is adjacent only to u_0 and v_0 for $j = 1, 2, \ldots, n$. From all the above conditions integral sum graph $G^+(S) = TB_n(u_0, v_0)(u_0, w_0)$ where $S = \{0, 2m, 2m + 1, 4m + 1, \ldots, 2mn + 1, 2m(n + 1) + 1/m \in \mathbb{N}\}$. Integral sum labeling of TB_7^* is shown in Figure 12.

(ii) Fan graph $F_n = P_n + K_1$ and F_n^* is of order $n+2$ and size $2n$ where P_n is a path on n vertices. Let $V(F_n^*) = \{u_0, v_0, v_1, \ldots, v_n\}$ where u_0 is the pendant vertex, v_0 is the apex vertex and $d(v_0) = n + 1 = \Delta(F_n^*)$. Define mapping $f : V(F_n^*) \to \mathbb{N}$ such that $f(v_0) = 0$, $f(v_1) = p_m$, the m^{th} Fibonacci number, $m \geq 2$, $f(v_i) = p_{m+i-1}$ for $i = 2, \ldots, n$ and $f(u_0) = p_{m+n}$. Here, $f(v_0) = 0 < f(v_1) = p_m < f(v_2) = p_{m+1} < \ldots < f(v_n) = p_{m+n-1} < f(u_0) = p_{m+n}$ and for $i - j \neq 1$ and $1 \leq i, j, k \leq n$, $f(v_i) + f(v_j) \neq f(v_k)$. Also $f(v_i)+f(v_{i+1}) = f(v_{i+2})$ for $i = 1, 2, \ldots, n-2$ and $f(v_{n-1})+f(v_n) = f(u_0)$, $m \geq 2$. Hence the labeling f is an integral sum labeling of graph F_n^*. Integral sum labeling of F_9^* is shown in Figure 12. □

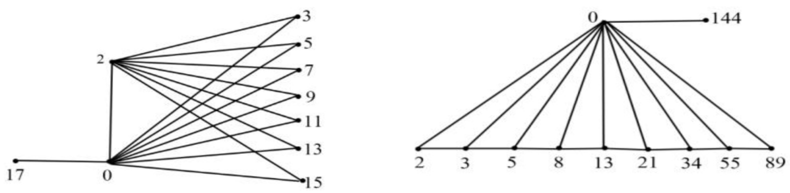

Fig. 12: TB_7^*. F_9^*.

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