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DECOMPOSITIONS OF COMPLETE GRAPHS INTO KAYAK PADDLES

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Abstract. A canoe paddle is a cycle attached to an end-vertex of a path. It was shown by Truszczyński that all canoe paddles are graceful and therefore decompose complete graphs. A kayak paddle is a pair of cycles joined by a path. We prove that the complete graph K_{2n+1} is decomposable into kayak paddles with n edges whenever at least one of its cycles is even.

Key words: Graph decomposition, graceful labeling, rosy labeling.

Abstrak. Sebuah dayung kano adalah lingkaran yang dikaitkan ke sebuah titik ujung dari lintasan. Truszczyński telah menunjukkan bahwa semua dayung kano adalah graceful dan oleh karenanya mendekomposisi graf lengkap. Sebuah dayung kayak adalah sepasang lingkaran yang dikaitkan dengan lintasan. Kami membuktikan bahwa graf lengkap K_{2n+1} dapat didekomposisi menjadi dayung-dayung kayak dengan n sisi jika sekurang-kurangnya satu lingkarannya adalah lingkaran genap.

Kata kunci: Dekomposisi graf, pelabelan graceful, pelabelan rosy.

1. Introduction

Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* (or that it is *G-decomposable*) if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i.

In 1967 Rosa [4] introduced some important types of vertex labelings serving as tools for finding decompositions of complete graphs. Graceful labeling (called β-valuation by AR) and rosy labeling (called ρ-valuation by Rosa) are useful tools

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for decompositions of complete graphs K_{2n+1} into graphs with n edges. A *labeling* of a graph G with n edges is an injection ρ from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \ldots, 2n\}$ of elements of the additive group Z_{2n+1}. The *length* of an edge $e = xy$ with endvertices x and y is defined as $\ell(xy) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. Notice that the subtraction is performed in Z_{2n+1} and hence $1 \leq \ell(e) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \ldots, n\}$, then ρ is a *rosy labeling*; if $S \subseteq \{0, 1, \ldots, n\}$, then ρ is a *graceful labeling*. A graceful labeling α is said to be an *α-labeling* if there exists a number α_0 with the property that for every edge e in G with endvertices x and y and with $\alpha(x) < \alpha(y)$ it holds that $\alpha(x) \leq \alpha_0 < \alpha(y)$. Obviously, G must be bipartite to allow an α-labeling. For an exhaustive survey of graph labelings, see [3] by Gallian.

Rosa observed that if a graph G with n edges has a graceful or rosy labeling, then K_{2n+1} can be cyclically decomposed into $2n + 1$ copies of G. It is so because K_{2n+1} has exactly $2n+1$ edges of length i for every $i = 1, 2, \ldots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G, say G_0, and then adding a non-zero element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \ldots, 2n$.

A *canoe paddle* (also called a *kite* or *dragon*) is a cycle attached to an endvertex of a path. It was shown by Truszczyński [6] that all canoe paddles are graceful. A *kayak paddle* (or a *double kite* or *double dragon*) is a pair of cycles joined by a path. In particular, $\mathrm{KP}(r, s, l)$ stands for cycles of lengths r and s joined by a path of length l.

In this paper we introduce an auxiliary tool, called gap graceful labeling, and use it to find rosy labelings of kayak paddles with at least one even cycle. We will prove that every kayak paddle $\mathrm{KP}(r, s, l)$ with $n = r + s + l$ edges and r even has a rosy labeling and hence K_{2n+1} is $\mathrm{KP}(r, s, l)$-decomposable.

2. **Basic notions and auxiliary results**

To deal with graph decomposition problems for graphs that are not graceful, Rosa [4] introduced a modification of graceful labelings, called *$\hat{\rho}$-labeling* (Frucht [1] used the term *nearly graceful labeling*). In a nearly graceful labeling the vertices are labeled by elements of the set $\{0, 1, 2, \ldots, n, n+1\}$ while the edge lengths are either $\{1, 2, \ldots, n\}$ or $\{1, 2, \ldots, n-1, n+1\}$. To avoid confusion, we note that the edge length was in these papers defined as $\hat{\ell}(xy) = \hat{\rho}(y) - \hat{\rho}(x)$ for $0 \leq \hat{\rho}(x) < \hat{\rho}(y) \leq n+1$. We observe that for an edge with $\hat{\ell}(xy) = n+1$ the vertices would be labeled 0 and $n+1$, respectively. Using our definition of the edge length, we would conclude that $\ell(xy) = \min\{\rho(y) - \rho(x), \rho(x) - \rho(y)\} = n$, since the subtraction is carried out in Z_{2n+1}. We will use this observation later in our constructions.

Seoud and Elsakhawi [5] have shown that all cycles are nearly graceful, and Barrientos [2] proved that C_n is nearly graceful with edge lengths $\{1, 2, \ldots, n-1, n+1\}$ if and only if $n \equiv 1$ or $2 \pmod 4$.

We generalize this notion further by defining a *gap graceful labeling*, which we denote $\ddot{\beta}$. Here the vertices are again labeled by elements of the set $\{0,1,2,\dots,n, n+1\}$ while the set of the edge lengths is an n-element subset of $\{1,2,\dots,n,n+1\}$. If we want to point out the missing length p, we will call the labeling p-gap graceful labeling and denote it by $\ddot{\beta}_p$. We will call this labeling a *p-gap α-labeling* and denote by $\ddot{\alpha}$ if it moreover has the alpha-like property that there exists a number α_0 such that for every edge $e = xy$ in G with $\ddot{\alpha}(x) < \ddot{\alpha}(y)$ it holds that $\ddot{\alpha}(x) \le \alpha_0 < \ddot{\alpha}(y)$. Again, a p-gap α-labeling will be denoted by $\ddot{\alpha}_p$. One can notice that in our terminology, the nearly graceful labeling is an n-gap graceful labeling.

Now we prove some lemmas that will be useful in our constructions. We will often describe labelings as follows. An edge $e = xy$ along with the endvertices x, y whose labels are i and j, respectively, and $i > j$, will be denoted as $\dots(i)[i-j](j)\dots$.

Lemma 2.1. *The cycle C_{4k+2} has a p-gap α-labeling $\ddot{\alpha}_p$ for $p = 2$ and $p = 4k+2$ for any $k \ge 1$.*

Proof. For $p = 2$ the labeling is

$$(0)[4k+3](4k+3)[4k+2](1)[4k+1](4k+2)[4k](2)\dots$$
$$\dots(k)[2k+3](3k+3)[2k+2](k+1)[2k](3k+1)[2k-1](k+2)\dots$$
$$\dots(2k-1)[4](2k+3)[3](2k)[1](2k+1)[2k+1](0).$$

For $p = 4k+2$ the labeling is

$$(0)[4k+3](4k+3)[4k+1](2)[4k](4k+2)[4k-1](3)\dots$$
$$\dots(k)[2k+4](3k+4)[2k+3](k+1)[2k+1](3k+2)[2k](k+2)\dots$$
$$\dots(2k)[3](2k+3)[2](2k+1)[1](2k+2)[2k+2](0).$$

Lemma 2.2. *The cycle C_{4k+1} has a 1-gap α-labeling $\ddot{\beta}_1$ for any $k \ge 1$.*

Proof. For $p = 2$ the labeling is

$$(0)[4k+2](4k+2)[4k+1](1)[4k](4k+1)[4k-1](2)\dots$$
$$\dots(k)[2k+2](3k+2)[2k+1](k+1)[2k-1](3k)[2k-2](k+2)\dots$$
$$\dots(2k-1)[3](2k+2)[2](2k)[2k](0).$$

3. **Case** KP$(4k, s, l)$

This well known construction was proved by many authors. One of the first ones was Truszczyński [6].

Lemma 3.1. *Let G be a graceful graph with t edges, $\beta(x_0) = 0, \beta(x_1) = t$ and H a graph with an α-labeling with $\alpha(y_0) = \alpha_0, \alpha(y_1) = \alpha_0 + 1$. For $i \in \{0,1\}$ denote by $F_i = (G \odot H)_i$ the graph with $V(F_i) = V(G) \cup V(H)$ where the vertex x_i is identified with y_i. Then each of F_0 and F_1 has a graceful labeling.*

The following then holds.

Observation 3.2. *The graphs* $\mathrm{KP}(4k, 4m, l)$ *and* $\mathrm{KP}(4k, 4m-1, l)$ *are graceful for any* $k \geq 1, m \geq 1, l \geq 1$.

PROOF. Because all cycles C_{4m} and C_{4m-1} are graceful and all paths P_{l+1} have α-labelings, by applying Lemma 3.1 we observe that both $C_{4m} \odot P_{l+1}$ and $C_{4m-1} \odot P_{l+1}$ are graceful. Because C_{4k} has an α-labeling, both $\mathrm{KP}(4k, 4m, l) = (C_{4m} \odot P_{l+1}) \odot C_{4k}$ and $\mathrm{KP}(4k, 4m-1, l) = (C_{4m-1} \odot P_{l+1}) \odot C_{4k}$ are graceful.

The following Lemma is a straightforward generalization of Lemma 3.1 and therefore the proof can be left to the reader.

Lemma 3.3. *Let* G *be a* p*-gap graceful graph with* t *edges,* $\ddot{\beta}_p(x_0) = 0, \ddot{\beta}_p(x_1) = t+1$ *and* H *a graph with an* α*-labeling with* $\alpha(y_0) = \alpha_0, \alpha(y_1) = \alpha_0 + 1$. *For* $i \in \{0, 1\}$ *denote by* $F_i = (G \odot H)_i$ *the graph with* $V(F_i) = V(G) \cup V(H)$ *where the vertex* x_i *is identified with* y_i. *Then each of* F_0 *and* F_1 *has a* p*-gap graceful labeling.*

Notice that if we have $G = C_t$ and $H = P_l$, we can always choose the α-labeling of P_l so that one of the vertices of degree one receives label $l-1$ while the other one receives label α_0 (when l is even) or $\alpha_0 + 1$ (when l is odd). Hence, we use $(C_t \odot P_l)_0$ when l is even and $(C_t \odot P_l)_1$ when l is odd. In both cases the resulting graph will have the vertex of degree one labeled $t + l$.

Theorem 3.4. *The kayak paddles* $\mathrm{KP}(4k, 4m+1, l)$ *and* $\mathrm{KP}(4k, 4m+2, l)$ *have rosy labeling for any* $k \geq 1, m \geq 1, l \geq 1$.

PROOF. By Lemma 2.2, C_{4m+1} has a 1-gap graceful labeling $\ddot{\beta}_1$ for any $m \geq 1$. For $l = 1$, we use the missing edge length and attach the single edge of P_2 to vertex $4m+2$ of C_{4m+1}. The other end-vertex is then $4m+3 = 4m+l+2$.

For $l > 1$ we use Lemma 3.3 and find a 1-gap graceful labeling of $C_{4m+1} \odot P_l$ with the vertex of degree one labeled $4m+l+1$. As in the previous case, we join the vertices $4m+l+1$ and $4m+l+2$ by the edge with unused length 1. In both cases, we have so far used edge lengths $1, 2, \ldots, 4m+l+1$ and the vertex of degree one is labeled $4m+l+2$.

Now because C_{4k} has an α-labeling, we can use it and increase the vertex labels in the "lower" partite set by $4m+l+2$ and the labels in the "upper" partite set by $(4m+l+2)+(4m+l+1) = 8m+2l+3$ to obtain labels $4m+l+2, 4m+l+3, \ldots, 4m+2k+l+1$ and $8m+2k+2l+3, 8m+2k+2l+4, \ldots, 8m+4k+2l+3$. The edge lengths in C_{4k} are $4m+l+2, 4m+l+3, \ldots, 4m+4k+l+1$. Hence, we have used precisely the lengths $1, 2, \ldots, n$, where $n = 4m+4k+l+1$ is the number of edges in $\mathrm{KP}(4k, 4m+1, l)$. Because $k > 0$, we have $8m+4k+2l+3 < 8m+8k+2l+2 = 2n$ and the labeling is rosy.

For C_{4k+2} the construction is essentially the same. The cycle has a 2-gap α-labeling $\ddot{\alpha}_2$. Therefore, the only difference is that the edge we add to $C_{4m+2} \odot P_l$ (or to C_{4m+2} if $l = 1$) is of length two and joins vertices $4m+l+2$ and $4m+l+4$. The cycle C_{4k} then receives labels $4m+l+4, 4m+l+5, \ldots, 4m+2k+l+3$ and $8m+2k+2l+6, 8m+2k+2l+7, \ldots, 8m+4k+2l+7$ and the edge lengths are $4m + l + 3, 4m + l + 4, \ldots, 4m + 4k + l + 2$. All lengths $1, 2, \ldots, n$, where $n = 4m + 4k + l + 2$ are used and the highest label satisfies $8m + 4k + 2l + 5 < 8m + 8k + 2l + 4 = 2n$, since $k > 0$. This completes the proof.

4. **Case** KP$(4k+2, s, l)$

Theorem 4.1. *The kayak paddle* KP$(4k+2, 4m+1, l)$ *has a rosy labeling for any* $k \geq 1, m \geq 1, l \geq 1$.

Proof. This proof is similar to the proof of Theorem 3.4. First we construct the labeling of $C_{4m+1} \odot P_{l+1}$ with the vertex of degree one labeled $4m + l + 2$ and edge lengths $1, 2, \ldots, 4m + l + 1$. Then we use the $(4k + 2)$-gap α-labeling of C_{4k+2} and increase the vertex labels in the lower and upper partite sets by $4m + l + 2$ and $(4m + l + 2) + (4m + l + 1) = 8m + 2l + 3$, respectively. The edge lengths become $4m + l + 2, 4m + l + 3, \ldots, 4m + 4k + l + 2, 4m + 4k + l + 3$. Notice that while there was the $(4k + 2)$-gap between the longest edges of lengths $(4k + 1)$ and $(4k + 3)$, it disappears when the cycle is stretched. The longest edge is now between the edges $4m + l + 2$ and $8m + 4k + 2l + 6$ and is equal to $(4m + l + 2) - (8m + 4k + 2l + 6) = -(4m + 4k + l + 4) = 4m + 4k + l + 3$, because the labels are elements of the group Z_{2n+1} and $n = 4m + 4k + l + 3$.

It remains to check that the largest label satisfies $8m + 4k + 2l + 6 \leq 8m + 8k + 2l + 6 = 2n$, which is obvious. This concludes the proof.

Theorem 4.2. *The kayak paddle* KP$(4k+2, 4m+2, l)$ *has a rosy labeling for any* $k \geq 1, m \geq 1, l \geq 1$.

Proof. Here we combine the ideas of the second part of the proof of Theorem 3.4 with the previous proof. First we construct the labeling of $C_{4m+2} \odot P_{l+1}$ with the vertex of degree one labeled $4m+l+4$ and edge lengths $1, 2, \ldots, 4m+l+2$. Then we use the $(4k+2)$-gap α-labeling of C_{4k+2} and increase the vertex labels in the lower and upper partite sets by $4m+l+4$ and $(4m+l+4)+(4m+l+2) = 8m+2l+6$, respectively. The edge lengths become $4m + l + 3, 4m + l + 4, \ldots, 4m + 4k + l + 3, 4m + 4k + l + 4$. The original $(4k + 2)$-gap between the longest edges of C_{4k+2} disappears for the same reasons as in the previous proof. Hence we have used all lengths $1, 2, \ldots, 4m + 4k + l + 4$. We verify that the largest label satisfies $8m + 4k + 2l + 9 \leq 8m + 8k + 2l + 8 = 2n$, which is true because $k > 0$.

Theorem 4.3. *The kayak paddle* KP$(4k+2, 4m+3, l)$ *has a rosy labeling for any* $k \geq 1, m \geq 0, l \geq 1$.

Proof. We know that each of C_{4m+3} and $C_{4m+3} \odot P_l$ for $l \geq 2$ has a graceful labeling. Moreover, the labeling of $C_{4m+3} \odot P_l$ can be chosen so that the single vertex of degree one receives label 0.

Because C_{4k+2} has a 2-gap α-labeling with $\alpha_0 = 2k$, we observe that by increasing all labels in the upper partite set by $4m + l + 2$ the edges will have lengths $4m + l + 3, 4m + l + 5, 4m + l + 6, \ldots, 4m + 4k + l + 5$. We place the edge of length $4m + l + 4$ which was not used yet between vertices 0 and $4m + l + 4$ to complete the graph $C_{4m+3} \odot P_{l+1}$. Finally, we increase each label in C_{4k+2} by $4m+l+4$ to obtain $\mathrm{KP}(4k+2, 4m+3, l)$. The highest used label is $8m+4k+2l+9$. Because $2n = 8m + 8k + 2l + 10$, we have a rosy labeling and the proof is complete.

We summarize the results as follows.

Corollary 4.4. *The kayak paddle* $\mathrm{KP}(r, s, l)$ *has a rosy labeling for any* r *even,* $r \geq 4, s \geq 3, l \geq 1$.

Our main result then follows immediately.

Theorem 4.5. *Let* r *be even,* $r \geq 4, s \geq 3, l \geq 1$, *and* $n = r + s + l$. *Then the complete graph* K_{2n+1} *is decomposable into kayak paddles* $\mathrm{KP}(r, s, l)$.

5. Concluding Remarks

We believe that similar although more sophisticated methods can be used to solve the remaining case, that is, the decomposition into kayak paddles with both cycles odd.

On the other hand, finding graceful labelings of kayak paddles seems to be much more complicated (expect when $r \equiv 0 \pmod 4$ and $s \equiv 0, 3 \pmod 4$—see Observation 3.2). Hence, we conclude the paper by posing the following open problem.

Problem 5.1. *Characterize all graceful kayak paddles* $\mathrm{KP}(r, s, l)$.

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