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Multi-Decomposition of Product Graphs into Kites and Stars on Four Edges

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Abstract. A decomposition of a graph G is a set of edge-disjoint subgraphs $H_1, H_2, ..., H_r$ of G such that every edge of G belongs to exactly one H_i. If all the subgraphs in the decomposition of G are isomorphic to a graph H then we say that G is H-decomposable. The graph G has an $\{H_1^{\alpha}, H_2^{\beta}\}$-decomposition, if α copies of H_1 and β copies of H_2 decompose G, where α and β are non-negative integers. In this paper, we have obtained the decomposition of $K_m \times K_n$ into α kites and β stars on four edges for some of the admissible pairs (α, β), whenever $mn(m-1)(n-1) \equiv 0(mod\ 8)$, for $m \geq 3$ and $n \geq 4$. Also, we have obtained the decomposition of $K_m \otimes \overline{K_n}$ into α kites and β stars on four edges for some of the admissible pairs (α, β), whenever $m(m-1)n^2 \equiv 0(mod\ 8)$, for $m \geq 3$ and $n \geq 4$. Here $K_m \times K_n$ and $K_m \otimes \overline{K_n}$ respectively denotes the tensor and wreath product of complete graphs.

Keywords: Kite, star, tensor product, wreath product, multi-decomposition.

1. INTRODUCTION

All graphs considered here are finite, simple, and undirected. Let $S_4(= K_{1,4})$ denote a *star* on 4 edges. A *Kite* $\mathcal{K}$ is a graph in which one edge is attached to a vertex of the triangle C_3. Let $K_{n,n} - I$ denote a complete bipartite graph minus one factor. For two graphs G_1 and G_2, their *tensor product* $G_1 \times G_2$ and *wreath product* $G_1 \otimes G_2$ have the same vertex set $V(G_1) \times V(G_2)$, and the corresponding edge sets are defined as follows: $E(G_1 \times G_2) = \{(g_1, g_2)(g_1', g_2') | g_1g_1' \in E(G_1)\ and\ g_2g_2' \in E(G_2)\}$ and $E(G_1 \otimes G_2) = \{(g_1, g_2)(g_1', g_2') | g_1g_1' \in E(G_1)\ or\ g_1 = g_1',\ g_2g_2' \in E(G_2)\}$. In general, if G is the product of two graphs G_1 and G_2 of order m and n, respectively, then G can be considered a multi-partite graph with mn vertices where $V(G) = \{v_i^j | i \in \{1, 2, ..., m\}\ and\ j \in \{1, 2, ..., n\}\}$ and the edge set of G is defined according to the product considered. For a graph G, if $E(G)$ can be

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partitioned into subsets $E_1, \cdots, E_k$ such that the subgraph of G induced by E_i is H_i for each $1 \le i \le k$, we write $G = H_1 \oplus \cdots \oplus H_k$. For $1 \le i \le k$, if $H_i \cong H$, we say that G has an *H-decomposition.* If G can be decomposed into α copies of H_1 and β copies of H_2, then we say that G has an *$\{H_1^\alpha, H_2^\beta\}$-decomposition* or an *$(H_1, H_2)$-multi-decomposition.* If the necessary conditions are satisfied for the existence of a $\{H_1^\alpha, H_2^\beta\}$-decomposition of G by pairs (α, β), then we say that (α, β) is an admissible pair. Note that if G_1 and G_2 have an (H_1, H_2)-multi-decomposition, then the (vertex or edge) disjoint union of G_1 and G_2 also has a decomposition of the same type.

The study of $\{H_1^\alpha, H_2^\beta\}$-decomposition has been introduced by Abueida and Daven [1]. Moreover, Abueida and O'Neil [2] have settled the existence of $\{H_1^\alpha, H_2^\beta\}$ - decomposition of $K_m(\lambda)$ when $\{H_1, H_2\} = \{S_n, C_n\}$ for $n = 3, 4, 5$. In [3], Yizhe Gao and Dan Roberts obtained the multi-designs for the graph pair formed by the 6-cycle and 3-prism. In [4], Pauline Ezhilarasi et al. proved the existence of multi-decomposition in the tensor product of complete graphs into cycles and stars with four edges. In [5], Pauline Ezhilarasi and Muthusamy have obtained the multi-decomposition of some product graphs into paths and stars with three edges. In [6], Jeevadoss and Muthusamy proved the existence of multi-decomposition of some product graphs into paths and cycles of length four. In [7], Ilayaraja, Sowndhariya, et al. studied the multi-decomposition of some product graphs into paths and stars on five vertices. All these studies motivated us to study the multi-decomposition of simple graphs into kites and stars on four edges. In [8], we have obtained the necessary and sufficient conditions for the existence of a multi-decomposition of complete graphs into kites and stars on four edges. Also, the multi-decomposition of the cartesian product of complete graphs into kites and stars has been studied in [9]. In this paper, we have obtained a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m \times K_n$ and $K_m \otimes \overline{K_n}$ for some admissible pairs (α, β), whenever $m \ge 3$ and $n \ge 4$.

Notations:

(1) In a multi-partite graph G, we define F as a *1-factor* of distance p if $F = F_1 \cup F_2$, where $F_1 = \bigcup_{i=1}^{m} v_{i+p}^j v_i^j$ and $F_2 = \bigcup_{j=1}^{n} v_i^{j+p} v_i^j$ and the additions in the subscripts and the superscripts are taken modulo m and n respectively, with residues $0, 1, 2, ..., m-1$ and $0, 1, 2, ..., n-1$.

(2) A kite in a multi-partite graph G is of the form $(v_i^j, v_k^l, v_p^q; v_x^y)$, where v_i^j, v_k^l, v_p^q is a triangle C_3 and $v_p^q v_x^y$ is the pendant edge e attached to the vertex v_p^q of C_3. It is denoted by $\mathcal{K}$.

(3) A star S_4 in a multi-partite graph G is of the form $(v_p^q; v_i^j, v_k^l, v_m^n, v_x^y)$, where v_p^q is the center vertex(of degree 4) and $v_i^j, v_k^l, v_m^n, v_x^y$ are the end vertices(pendant vertices).

(4) The n stars with the same end vertices $v_a^u, v_b^w, v_c^x, v_d^y$ and different center vertices $a_i^1, a_m^2, \cdots, a_p^n$ are denoted by $(a_i^1, a_m^2, \cdots, a_p^n; v_a^u, v_b^w, v_c^x, v_d^y)$.

1.1. **Preliminary results.**

In this section, we will present some of the known lemmas and theorems which will be used in proving our main results.

Lemma 1.1. [8] *The graph K_8 admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ - decomposition if and only if $\alpha+\beta=7$, where $\alpha,\beta\geq 0$.*

Lemma 1.2. [8] *The graph K_9 admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ - decomposition if and only if $\alpha+\beta=9$, where $\alpha,\beta\geq 0$.*

Theorem 1.3. [8] *For all even $t\geq 4$, if $n=4t$ or $n=4t+1$, then K_n admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$ - decomposition for the pairs (α,β) satifying $0\leq\alpha\leq\frac{9t-4}{2}$, $\beta=\frac{1}{4}\binom{n}{2}-\alpha$.*

Theorem 1.4. [10] *Let k, m and $n\in Z^+$ with $m\leq n$. Then there exits an S_k-decomposition of $K_{m,n}$ if and only if one of the follwing holds:*
(i) $k\leq m$ and $mn\equiv 0(mod\ k)$
(ii) $m<k\leq n$ and $n\equiv 0(mod\ k)$.

Observation 1.1. *If $G=K_{n,n}-I$, where $n\equiv 1(mod\ 4)$, then the degree of every vertex in G is a multiple of 4 and hence G admits an S_4-decomposition in this case, by Theorem 1.4.*

Observation 1.2. *The graph $P_3\times K_n$ admits an S_4-decomposition when n is odd.*

2. $\{\mathcal{K}^\alpha, S_4^\beta\}$-DECOMPOSITION OF $K_m\times K_n$

In this section, the necessary and sufficient conditions for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in the tensor product of complete graphs have been discussed.

Theorem 2.1. *Let $m\geq 3$ and $n\geq 4$ be the given integers. Then the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m\times K_n$ is $mn(m-1)(n-1)\equiv 0(mod\ 8)$.*

Proof. Obvious from the edge divisibility condition required for the desired decomposition in $K_m\times K_n$. □

The following lemmas are useful in proving our main result on the sufficient condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m\times K_n$ for $m\geq 3$ and $n\geq 4$.

Lemma 2.2. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_3\times K_4$ for all admissible pairs (α,β) such that $\alpha+\beta=9$.*

Proof. The decomposition of $K_3\times K_4$ into α kites(i.e, the set K) and β stars(i.e, the set S) on 4 edges for all admissible pairs (α,β) such that $\alpha+\beta=9$ is discussed in the following cases:
Case 1: $(\alpha,\beta)=(9,0)$.

K=$\{(v_2^2,v_3^3,v_1^1;v_3^4), (v_1^2,v_3^4,v_2^3;v_1^1), (v_3^1,v_1^3,v_2^2;v_1^4), (v_1^2,v_2^1,v_3^3;v_1^4), (v_1^3,v_3^2,v_2^4;v_3^3), (v_2^1,v_1^4, v_3^2;v_1^1), (v_2^1,v_1^3,v_3^4;v_2^2), (v_1^2,v_3^1,v_2^4;v_1^1), (v_3^1,v_1^4,v_2^3;v_3^2)\}$.
Case 2: $(\alpha,\beta)=(8,1)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_3^2), (v_1^2,v_3^4,v_2^3;v_1^1), (v_3^1,v_1^3,v_2^2;v_3^4), (v_2^1,v_1^2,v_3^3;v_2^4), (v_1^3,v_3^2,v_2^4;v_1^1), (v_2^1,v_1^4, v_3^2;v_2^3), (v_2^1,v_1^3,v_3^4;v_1^1), (v_1^2,v_2^4,v_3^1;v_2^3)\}$ and S=$\{(v_1^4;v_2^2,v_3^1,v_2^3,v_3^3)\}$.
Case 3: $(\alpha,\beta)=(7,2)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_2^4), (v_2^3,v_3^4,v_1^2;v_2^4), (v_2^2,v_1^3,v_3^1;v_2^3), (v_2^1,v_3^3,v_1^2;v_3^1), (v_1^3,v_3^2,v_2^4;v_3^1), (v_2^1,v_1^3, v_3^4;v_2^2), (v_2^1,v_1^4,v_3^2;v_2^3)\}$ and S=$\{(v_1^1;v_3^2,v_3^4,v_2^3,v_2^4), (v_1^4;v_2^2,v_3^1,v_2^3,v_3^3)\}$.
Case 4: $(\alpha,\beta)=(6,3)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_2^4), (v_1^2,v_3^4,v_2^3;v_3^1), (v_1^3,v_3^1,v_2^2;v_3^4), (v_3^2,v_1^4,v_2^3;v_1^1), (v_2^1,v_3^3,v_1^2;v_3^1), (v_1^3,v_2^1, v_3^4;v_1^1)\}$ and S=$\{(v_1^4;v_2^1,v_2^2,v_3^1,v_3^3), (v_2^4;v_1^3,v_1^2,v_1^1,v_3^1), (v_3^2;v_2^1,v_1^1,v_1^3,v_2^4)\}$.
Case 5: $(\alpha,\beta)=(5,4)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_2^3), (v_2^3,v_3^4,v_1^2;v_3^1), (v_2^2,v_3^1,v_1^3;v_2^1), (v_1^4,v_3^2,v_2^3;v_3^1), (v_1^2,v_2^1,v_3^3;v_2^4)\}$ and S= $\{(v_1^4;v_2^1,v_2^2,v_3^1,v_3^3), (v_2^4;v_1^3,v_1^2,v_1^1,v_3^1), (v_3^2;v_1^1,v_2^1,v_1^3,v_2^4), (v_3^4;v_1^3,v_1^1,v_2^2,v_2^1)\}$.
Case 6: $(\alpha,\beta)=(4,5)$.
K=$\{(v_1^1,v_2^4,v_3^2;v_2^3), (v_2^4,v_3^3,v_1^2;v_2^3), (v_2^2,v_1^4,v_3^3;v_2^1), (v_1^3,v_3^2,v_2^1;v_1^2)\}$ and S=$\{(v_1^1;v_2^3,v_2^2,v_3^3, v_3^4), (v_1^3;v_2^2,v_2^4,v_3^1,v_3^4), (v_1^4;v_2^1,v_2^3,v_3^1,v_3^2), (v_3^1;v_1^2,v_2^2,v_2^3,v_2^4), (v_3^4;v_2^1,v_2^2,v_2^3,v_1^2)\}$.
Case 7: $(\alpha,\beta)=(3,6)$.
K=$\{(v_1^1,v_2^4,v_3^2;v_2^3), (v_1^3,v_3^2,v_2^1;v_3^3), (v_1^4,v_2^2,v_3^3;v_2^4)\}$ and S=$\{(v_1^1;v_2^3,v_2^2,v_3^3,v_3^4), (v_1^3;v_2^2,v_2^4, v_3^1,v_3^4), (v_1^4;v_2^1, v_2^3,v_3^1,v_3^2), (v_3^1;v_1^2,v_2^2,v_2^3,v_2^4), (v_3^4;v_2^1,v_2^2,v_2^3,v_1^2), (v_1^2;v_2^1,v_2^3,v_2^4,v_3^3)\}$.
Case 8: $(\alpha,\beta)=(2,7)$.
K=$\{(v_1^1,v_2^4,v_3^2;v_1^4), (v_2^1,v_1^3,v_3^2;v_2^3)\}$ and S=$\{(v_1^1;v_2^3,v_2^2,v_3^3,v_3^4), (v_1^3;v_2^2,v_2^4,v_3^1,v_3^4), (v_3^1;v_1^2,v_2^2, v_2^3,v_2^4), (v_3^4;v_2^1,v_2^2,v_2^3,v_1^2), (v_1^2;v_2^1,v_2^3,v_2^4,v_3^3), (v_1^4;v_2^1,v_2^2,v_3^1,v_3^2), (v_3^3;v_2^1,v_2^2,v_1^4,v_2^4)\}$.
Case 9: $(\alpha,\beta)=(1,8)$.
K=$\{(v_1^1,v_3^2,v_2^4;v_1^3)\}$ and S=$\{(v_1^1;v_2^3,v_2^2,v_3^3,v_3^4), (v_1^3;v_2^1,v_2^2,v_3^1,v_3^4), (v_3^1;v_1^2,v_2^2,v_2^3,v_2^4), (v_3^4;v_2^1, v_2^2,v_2^3,v_1^2), (v_1^2;v_2^1,v_2^3,v_2^4,v_3^3), (v_1^4;v_2^1,v_2^2,v_2^3,v_3^1), (v_3^3;v_2^1,v_2^2,v_1^4,v_2^4). (v_3^2;v_2^1,v_2^3,v_1^3,v_1^4)\}$.
Case 10: $(\alpha,\beta)=(0,9)$.
S=$\{(v_1^1;v_2^3,v_2^4,v_3^3,v_3^4), (v_3^2;v_1^1,v_1^3,v_2^3,v_1^4), (v_2^3;v_1^2,v_1^4,v_3^1,v_3^4), (v_2^1;v_1^2,v_1^3,v_1^4,v_3^2), (v_3^1;v_1^2,v_1^3,v_1^4, v_2^4), (v_2^4;v_1^2,v_1^3,v_3^2,v_3^3), (v_3^4;v_1^2,v_1^3,v_2^1,v_2^2), (v_3^3;v_2^1,v_2^2,v_1^2,v_1^4), (v_2^2;v_1^1,v_1^3,v_1^4,v_3^1)\}$. □

Lemma 2.3. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_3\times K_5$ for all admissible pairs (α,β) such that $\alpha+\beta=15$.*

Proof. The decomposition $K\cup S$ of $K_3\times K_5$ into α kites and β stars on 4 edges for all admissible pairs (α,β) such that $\alpha+\beta=15$ is discussed in the following cases:
Case 1: $(\alpha,\beta)=(15,0)$.
K=$\{(v_1^5,v_3^3,v_2^4;v_1^1), (v_2^3,v_3^2,v_1^4;v_2^2), (v_1^3,v_2^2,v_3^1;v_1^2), (v_2^1,v_1^2,v_3^5;v_1^1), (v_3^4,v_2^5,v_1^1;v_3^2), (v_2^4,v_1^3,v_3^5; v_1^4), (v_1^2,v_2^3,v_3^4;v_1^3), (v_2^2,v_3^3,v_1^1;v_2^3), (v_3^2,v_1^5,v_2^1;v_3^4), (v_1^4,v_2^5,v_3^1;v_2^4), (v_1^5,v_3^1,v_2^3;v_3^5), (v_1^5,v_3^4,v_2^2; v_3^5), (v_2^5,v_1^3,v_3^2;v_2^4), (v_2^5,v_3^3,v_1^2;v_2^4), (v_1^4,v_3^3,v_2^1;v_1^3).\}$
Case 2: $(\alpha,\beta)=(14,1)$.
K=$\{(v_1^5,v_2^4,v_3^3;v_2^2), (v_2^3,v_3^2,v_1^4;v_2^2), (v_1^3,v_2^2,v_3^1;v_1^2), (v_2^1,v_1^2,v_3^5;v_1^1), (v_3^4,v_2^5,v_1^1;v_3^2), (v_2^4,v_1^3,v_3^5; v_1^4), (v_1^2,v_2^3,v_3^4;v_1^3), (v_3^2,v_1^5,v_2^1;v_3^4), (v_1^4,v_2^5,v_3^1;v_2^4), (v_1^5,v_3^1,v_2^3;v_3^5), (v_1^5,v_3^4,v_2^2;v_3^5), (v_2^5,v_1^3,v_3^2; v_2^4), (v_2^5,v_3^3,v_1^2;v_2^4), (v_1^4,v_3^3,v_2^1;v_1^3)\}$ and S=$\{(v_1^1;v_2^2,v_3^3,v_2^3,v_2^4)\}$.
Case 3: $(\alpha,\beta)=(13,2)$.
K=$\{(v_1^5,v_2^4,v_3^3;v_2^2), (v_2^3,v_3^2,v_1^4;v_2^2), (v_1^3,v_2^2,v_3^1;v_1^2), (v_3^4,v_2^5,v_1^1;v_3^2), (v_3^2,v_1^5,v_2^1;v_3^4), (v_1^4,v_2^5,v_3^1;$

$v_2^4), (v_1^5,v_3^1,v_2^3;v_3^5), (v_1^5,v_3^4,v_2^2;v_3^5), (v_2^5,v_1^3,v_3^2;v_2^4), (v_2^5,v_3^3,v_1^2;v_2^4), (v_1^4,v_3^3,v_2^1;v_1^3), (v_2^3,v_3^4,v_1^2;v_2^1), (v_2^4,v_3^5,v_1^3;v_3^4)\}$ and S=$\{(v_1^1;v_2^2,v_3^3,v_2^3,v_2^4), (v_3^5;v_1^1,v_1^4,v_1^2,v_2^1)\}$.

Case 4: $(\alpha,\beta)=(12,3)$.

K=$\{(v_1^5,v_2^4,v_3^3;v_2^2), (v_2^3,v_3^2,v_1^4;v_2^5), (v_1^3,v_3^1,v_2^2;v_1^4), (v_3^4,v_2^5,v_1^1;v_3^2), (v_3^2,v_1^5,v_2^1;v_3^4), (v_1^5,v_3^1,v_2^3;v_3^5), (v_1^5,v_3^4,v_2^2;v_3^5), (v_2^5,v_1^3,v_3^2;v_2^4), (v_2^5,v_3^3,v_1^2;v_2^4), (v_1^4,v_3^3,v_2^1;v_1^3), (v_2^3,v_3^4,v_1^2;v_2^1), (v_2^4,v_3^5,v_1^3;v_3^4)\}$ and S=$\{(v_1^1;v_2^2,v_3^3,v_2^3,v_2^4), (v_3^5;v_1^1,v_1^4,v_1^2,v_2^1), (v_3^1;v_2^5,v_1^4,v_2^4,v_1^2)\}$.

Case 5: $(\alpha,\beta)=(11,4)$.

K=$\{(v_1^5,v_2^4,v_3^3;v_1^4), (v_1^4,v_2^3,v_3^2;v_1^1), (v_1^3,v_3^1,v_2^2;v_3^4), (v_2^4,v_1^3,v_3^5;v_1^2), (v_3^4,v_1^2,v_2^3;v_3^5), (v_3^3,v_2^2,v_1^1;v_3^5), (v_3^5,v_1^4,v_2^2;v_1^5), (v_2^5,v_1^3,v_3^2;v_2^4), (v_1^2,v_3^1,v_2^4;v_3^2), (v_3^4,v_1^3,v_2^1;v_3^5), (v_3^3,v_1^2,v_2^1;v_1^4)\}$ and S=$\{(v_1^5;v_3^4,v_3^2,v_2^3,v_2^1), (v_2^5;v_1^4,v_1^2,v_3^3,v_3^4), (v_1^1;v_2^5,v_2^4,v_3^5,v_3^4), (v_3^1;v_2^5,v_1^5,v_2^3,v_1^4)\}$.

Case 6: $(\alpha,\beta)=(10,5)$.

K=$\{(v_1^5,v_3^3,v_2^4;v_1^2), (v_1^4,v_2^3,v_3^2;v_1^5), (v_1^3,v_2^2,v_3^1;v_1^5), (v_2^4,v_1^3,v_3^5;v_2^2), (v_2^3,v_1^2,v_3^4;v_2^2), (v_3^3,v_2^2,v_1^1;v_2^4), (v_1^4,v_3^3,v_2^5;v_1^2), (v_1^3,v_3^4,v_2^5;v_3^2), (v_1^2,v_3^3,v_2^1;v_3^4), (v_2^1,v_1^3,v_3^2;v_2^4)\}$ and S=$\{(v_1^5;v_3^4,v_2^3,v_2^2,v_2^1), (v_3^5;v_1^2,v_1^1, v_2^3,v_2^1), (v_3^1;v_1^2,v_2^3,v_2^4,v_2^5), (v_1^1;v_2^5,v_2^3,v_3^4,v_3^2), (v_1^4;v_3^5,v_3^1,v_2^2,v_2^1)\}$.

Case 7: $(\alpha,\beta)=(9,6)$.

K=$\{(v_2^4,v_3^4,v_1^5;v_3^1), (v_1^4,v_2^3,v_3^2;v_2^1), (v_2^2,v_3^1,v_1^3;v_2^1), (v_2^4,v_1^3,v_3^5;v_2^2), (v_2^3,v_1^2,v_3^4;v_2^2), (v_3^3,v_2^2,v_1^1;v_2^4), (v_1^4,v_3^3,v_2^5;v_1^2), (v_2^5,v_1^3,v_3^4;v_2^1), (v_3^3,v_2^1,v_1^2;v_2^4)\}$ and S=$\{(v_1^5;v_3^4,v_2^3,v_2^2,v_2^1), (v_3^5;v_1^2,v_1^1,v_2^3,v_2^1), (v_3^1;v_1^2, v_2^3,v_2^4,v_2^5), (v_1^1;v_2^5,v_2^3,v_3^4,v_3^2), (v_1^4;v_3^5,v_3^1,v_2^2,v_2^1), (v_3^2;v_2^5,v_2^4,v_1^5,v_1^3)\}$.

Case 8: $(\alpha,\beta)=(8,7)$.

K=$\{(v_2^4,v_3^3,v_1^5;v_3^1), (v_2^4,v_2^3,v_3^2;v_2^1), (v_2^2,v_3^1,v_1^3;v_2^5), (v_2^4,v_1^3,v_3^5;v_2^2), (v_2^3,v_3^4,v_1^2;v_2^4), (v_3^3,v_2^2,v_1^1;v_2^4), (v_1^4,v_3^3,v_2^5;v_1^2), (v_1^2,v_3^3,v_2^1;v_1^3)\}$ and S=$\{(v_1^5;v_3^4,v_2^3,v_2^2,v_2^1), (v_3^5;v_1^2,v_1^1,v_2^3,v_2^1), (v_3^1;v_1^2,v_2^3,v_2^4,v_2^5), (v_1^1; v_2^5,v_2^3,v_3^4,v_3^2), (v_1^4;v_3^5,v_3^1,v_2^2,v_2^1), (v_3^2;v_2^5,v_2^4,v_1^5,v_1^3), (v_3^4;v_2^5,v_1^3,v_2^2,v_2^1)\}$.

Case 9: $(\alpha,\beta)=(7,8)$.

K=$\{(v_1^5,v_2^4,v_3^3;v_2^1), (v_1^4,v_2^3,v_3^2;v_2^1), (v_1^3,v_2^2,v_3^1;v_1^5), (v_2^4,v_1^3,v_3^5;v_2^2), (v_3^4,v_2^3,v_1^2;v_3^3), (v_3^3,v_2^2,v_1^1;v_2^4), (v_1^4,v_3^3,v_2^5;v_1^3)\}$ and S=$\{(v_1^5;v_3^4,v_2^3,v_2^2,v_2^1), (v_3^5;v_1^2,v_1^1,v_2^3,v_2^1), (v_3^1;v_1^2,v_2^3,v_2^4,v_2^5), (v_1^1;v_2^5,v_2^3,v_3^4,v_3^2), (v_1^4;v_3^5,v_3^1,v_2^2,v_2^1), (v_3^2;v_2^5,v_2^4,v_1^5,v_1^3), (v_3^4;v_2^5,v_1^3,v_2^2,v_2^1), (v_1^2;v_2^5,v_2^4,v_3^3,v_2^1)\}$.

Case 10: $(\alpha,\beta)=(6,9)$.

K=$\{(v_1^5,v_2^4,v_3^3;v_1^2), (v_1^4,v_3^2,v_2^3;v_3^1), (v_1^3,v_2^2,v_3^1;v_1^2), (v_2^4,v_1^3,v_3^5;v_2^2), (v_2^3,v_1^2,v_3^4;v_1^1), (v_3^3,v_2^2,v_1^1;v_2^3)\}$ and S=$\{(v_1^5;v_2^3,v_2^2,v_2^1,v_3^1), (v_1^4;v_3^3,v_3^1,v_2^2,v_2^1), (v_2^5;v_1^4,v_1^3,v_1^2,v_1^1), (v_2^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_2^4;v_1^2,v_1^1,v_3^1,$
$v_3^2), (v_2^1;v_1^2,v_1^3,v_3^5,v_3^3), (v_3^5;v_1^4,v_1^2,v_2^3,v_1^1), (v_3^4;v_1^5,v_1^3,v_2^2,v_2^1), (v_3^2;v_1^5,v_2^4,v_1^3,v_2^1)\}$.

Case 11: $(\alpha,\beta)=(5,10)$.

K=$\{(v_1^5,v_2^4,v_3^3;v_2^2), (v_1^4,v_3^2,v_2^3;v_3^1), (v_1^3,v_2^2,v_3^1;v_1^2), (v_2^4,v_1^3,v_3^5;v_2^2), (v_3^4,v_2^3,v_1^2;v_3^3)\}$ and S=$\{(v_1^5;v_2^3,v_2^2,v_2^1, v_3^1), (v_1^4;v_3^3,v_3^1,v_2^2,v_2^1), (v_2^5;v_1^4,v_1^3,v_1^2,v_1^1), (v_2^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_2^4;v_1^2,v_1^1,v_3^1,v_3^2), (v_2^1;v_1^2,v_1^3,v_3^5, v_3^3), (v_3^5;v_1^4,v_1^2,v_2^3,v_1^1), (v_3^4;v_1^5,v_1^3,v_2^2,v_2^1), (v_3^2;v_1^5,v_2^4,v_1^3,v_2^1), (v_1^1;v_2^3,v_2^2,v_3^4,v_3^3)\}$.

Case 12: $(\alpha,\beta)=(4,11)$.

K=$\{(v_1^5,v_2^4,v_3^3;v_1^2), (v_1^4,v_3^3,v_2^3;v_3^1), (v_3^5,v_2^4,v_1^3;v_3^1), (v_3^4,v_2^3,v_1^2;v_3^1)\}$ and S=$\{(v_1^5;v_2^3,v_2^2,v_2^1,v_3^1), (v_1^4;v_3^3,v_3^1,v_2^2,v_2^1), (v_2^5;v_1^4,v_1^3,v_1^2,v_1^1), (v_2^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_2^4;v_1^2,v_1^1,v_3^1,v_3^2), (v_2^1;v_1^2,v_1^3,v_3^5,v_3^3), (v_3^5;v_1^4,v_1^2,v_2^3,v_1^1), (v_3^4;v_1^5,v_1^3,v_2^2,v_2^1), (v_3^2;v_1^5,v_2^4,v_1^3,v_2^1), (v_1^1;v_2^3,v_2^2,v_3^4,v_3^3), (v_2^2;v_1^3,v_3^1,v_3^5,v_3^3)\}$.

Case 13: $(\alpha,\beta)=(3,12)$.

K=$\{(v_3^3,v_2^4,v_1^5;v_3^4)\ (v_1^4,v_3^2,v_2^3;v_3^1), (v_3^5,v_2^4,v_1^3;v_3^1)\}$ and S=$\{(v_1^5;v_2^3,v_2^2,v_2^1,v_3^1), (v_1^4;v_3^3,v_3^1,v_2^2,v_2^1), (v_2^5;v_1^4,v_1^3,v_1^2,v_1^1), (v_2^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_2^4;v_1^2,v_1^1,v_3^1,v_3^2), (v_2^1;v_1^2,v_1^3,v_3^5,v_3^3), (v_3^5;v_1^4,v_1^2,v_2^3,v_1^1), (v_3^4;v_2^3,v_1^3,v_2^2,v_2^1), (v_3^2;v_1^5,v_2^4,v_1^3,v_2^1), (v_1^1;v_2^3,v_2^2,v_3^4,v_3^3), (v_2^2;v_1^3,v_3^1,v_3^5,v_3^3), (v_1^2;v_3^4,v_2^3,v_3^3,v_3^1)\}$.

Case 14: $(\alpha,\beta)=(2,13)$.
K=$\{(v_1^5,v_2^4,v_3^3;v_1^4), (v_3^5,v_2^4,v_1^3;v_3^1)\}$ and S=$\{(v_1^5;v_2^3,v_2^2,v_2^1,v_3^1), (v_1^4;v_3^2,v_3^1,v_2^2,v_2^1), (v_2^5;v_1^4,v_1^3,v_1^2,v_1^1), (v_2^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_2^4;v_1^2,v_1^1,v_3^1,v_3^2), (v_2^1;v_1^2,v_1^3,v_3^5,v_3^3), (v_3^5;v_1^4,v_1^2,v_2^3,v_1^1), (v_3^4;v_1^5,v_1^3,v_2^2,v_2^1), (v_3^2;v_1^5,v_2^4,v_1^3,v_2^1), (v_1^1;v_2^3,v_2^2,v_3^4,v_3^3), (v_2^2;v_1^3,v_3^1,v_3^5,v_3^3), (v_1^2;v_3^4,v_2^3,v_3^3,v_3^1), (v_2^3;v_1^4,v_3^4,v_3^2,v_3^1)\}$.

Case 15: $(\alpha,\beta)=(1,14)$.
K=$\{(v_3^5,v_2^4,v_1^3;v_3^1)\}$ and S=$\{(v_1^5;v_2^2,v_2^3,v_3^1,v_2^4), (v_1^4;v_2^1,v_2^2,v_3^1,v_3^2), (v_1^2;v_3^4,v_2^3,v_3^1,v_2^1), (v_1^1;v_2^3,v_2^2,v_3^4,v_3^3), (v_2^1;v_1^3,v_1^5,v_3^5,v_3^3), (v_3^2;v_2^1,v_1^1,v_1^3,v_1^5), (v_3^3;v_2^4,v_1^5,v_1^4,v_1^2), (v_3^4;v_1^5,v_1^3,v_2^2,v_2^1), (v_3^5;v_1^4,v_1^2,v_2^3,v_1^1), (v_2^5;v_1^4,v_1^3,v_1^2,v_1^1), (v_2^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_2^4;v_1^2,v_1^1,v_3^1,v_3^2), (v_2^3;v_1^4,v_3^4,v_3^2,v_3^1), (v_2^2;v_1^3,v_3^5,v_3^3,v_3^1)\}$.

Case 16: $(\alpha,\beta)=(0,15)$.
S=$\{(v_1^5,v_3^5;v_2^4,v_2^3,v_2^2,v_2^1), (v_1^4,v_3^4;v_2^5,v_2^3,v_2^2,v_2^1), (v_1^3,v_3^3;v_2^5,v_2^4,v_2^2,v_2^1), (v_1^2,v_3^2;v_2^5,v_2^4,v_2^3,v_2^1), (v_1^1,v_3^1;v_2^5,v_2^4,v_2^3,v_2^2), (v_1^5;v_3^4,v_3^3,v_3^2,v_3^1), (v_1^4;v_3^5,v_3^3,v_3^2,v_3^1), (v_1^3;v_3^5,v_3^4,v_3^2,v_3^1), (v_1^2;v_3^5,v_3^4,v_3^3,v_3^1), (v_1^1;v_3^5,v_3^4,v_3^3,v_3^2)\}$. □

Lemma 2.4. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_4\times K_4$ for all admissible pairs (α,β) such that $\alpha+\beta=18$.*

Proof. The decomposition $K\cup S$ of $K_4\times K_4$ into α kites and β stars on 4 edges for all admissible pairs (α,β) such that $\alpha+\beta=18$ is discussed in the following cases:

Case 1: $(\alpha,\beta)=(18,0)$.
K=$\{(v_2^4,v_3^1,v_4^2;v_1^1), (v_1^2,v_2^1,v_3^4;v_4^1), (v_1^4,v_3^2,v_2^3;v_4^2), (v_1^1,v_3^3,v_2^4;v_4^3), (v_1^2,v_3^3,v_4^4;v_2^1), (v_4^1,v_2^2,v_1^3;v_3^1), (v_1^2,v_4^3,v_3^1;v_2^2), (v_1^3,v_2^1,v_3^2;v_4^3), (v_2^2,v_3^3,v_1^4;v_4^1), (v_2^3,v_3^4,v_1^1;v_4^3), (v_1^2,v_2^3,v_4^1;v_3^3), (v_4^2,v_3^4,v_1^3;v_2^4), (v_4^1,v_3^2,v_2^4;v_1^2), (v_1^1,v_3^2,v_4^4;v_2^2), (v_2^1,v_3^3,v_4^2;v_1^4), (v_2^1,v_4^3,v_1^4;v_3^1), (v_3^1,v_2^3,v_4^4;v_1^3), (v_4^3,v_3^4,v_2^2;v_1^1)\}$.

Case 2: $(\alpha,\beta)=(17,1)$.
K=$\{(v_2^2,v_4^3,v_3^4;v_2^4), (v_3^1,v_4^2,v_2^4;v_3^3), (v_1^2,v_2^1,v_3^4;v_4^1), (v_1^4,v_3^2,v_2^3;v_4^2), (v_1^2,v_3^3,v_4^4;v_2^1), (v_4^1,v_2^2,v_1^3;v_3^1), (v_1^2,v_4^3,v_3^1;v_2^2), (v_1^3,v_2^1,v_3^2;v_4^3), (v_2^2,v_3^3,v_1^4;v_4^1), (v_2^3,v_3^4,v_1^1;v_4^3), (v_1^2,v_2^3,v_4^1;v_3^3), (v_4^2,v_3^4,v_1^3;v_2^4), (v_4^1,v_3^2,v_2^4;v_1^2), (v_1^1,v_3^2,v_4^4;v_2^2), (v_2^1,v_3^3,v_4^2;v_1^4), (v_2^1,v_4^3,v_1^4;v_3^1), (v_3^1,v_2^3,v_4^4;v_1^3)\}$ and S=$\{(v_1^1;v_4^2,v_2^2,v_3^3,v_2^4)\}$.

Case 3: $(\alpha,\beta)=(16,2)$.
K=$\{(v_1^4,v_2^1,v_4^3;v_2^4), (v_3^4,v_4^3,v_2^2;v_3^3), (v_3^1,v_4^2,v_2^4;v_3^3), (v_1^2,v_2^1,v_3^4;v_4^1), (v_1^4,v_3^2,v_2^3;v_4^2), (v_1^2,v_3^3,v_4^4;v_2^1), (v_4^1,v_2^2,v_1^3;v_3^1), (v_1^2,v_4^3,v_3^1;v_2^3), (v_1^3,v_2^1,v_3^2;v_4^3), (v_1^2,v_2^3,v_4^1;v_3^3), (v_4^2,v_3^4,v_1^3;v_2^4), (v_4^1,v_3^2,v_2^4;v_1^2), (v_1^1,v_3^2,v_4^4;v_2^2), (v_2^1,v_3^3,v_4^2;v_1^4), (v_3^1,v_2^3,v_4^4;v_1^3), (v_2^3,v_3^4,v_1^1;v_4^3)\}$ and S=$\{(v_1^1;v_4^2,v_2^2,v_3^3,v_2^4), (v_1^4;v_2^2,v_3^1,v_4^1,v_3^3)\}$.

Case 4: $(\alpha,\beta)=(15,3)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_4^3), (v_2^4,v_3^2,v_1^1;v_4^4), (v_3^4,v_4^2,v_1^1;v_2^3), (v_2^4,v_3^1,v_1^2;v_2^1), (v_1^2,v_4^1,v_3^3;v_2^4), (v_1^2,v_2^3,v_3^4;v_2^1), (v_2^1,v_4^4,v_1^3;v_3^2), (v_2^2,v_3^1,v_1^3;v_4^2), (v_3^4,v_4^1,v_1^3;v_2^4), (v_2^3,v_3^2,v_1^4;v_4^1), (v_2^1,v_3^3,v_4^2;v_2^4), (v_3^2,v_4^3,v_2^1;v_1^4), (v_3^4,v_4^3,v_2^2;v_1^4), (v_3^1,v_4^4,v_2^3;v_4^2), (v_3^1,v_4^2,v_1^4;v_3^3)\}$ and S=$\{(v_4^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_4^3;v_1^2,v_1^4,v_2^4,v_3^1), (v_4^4;v_1^2,v_2^2,v_3^2,v_3^3)\}$.

Case 5: $(\alpha,\beta)=(14,4)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_4^3), (v_2^4,v_3^2,v_1^1;v_4^4), (v_3^4,v_4^2,v_1^1;v_2^3), (v_2^4,v_3^1,v_1^2;v_2^1), (v_1^2,v_4^1,v_3^3;v_2^4), (v_1^2,v_2^3,v_3^4;v_2^1), (v_2^1,v_4^4,v_1^3;v_3^2), (v_2^2,v_3^1,v_1^3;v_4^2), (v_3^4,v_4^1,v_1^3;v_2^4), (v_2^3,v_3^2,v_1^4;v_4^1), (v_2^1,v_4^2,v_3^3;v_1^4), (v_3^2,v_4^3,v_2^1;v_1^4), (v_3^4,v_4^3,v_2^2;v_1^4), (v_2^3,v_4^4,v_3^1;v_1^4)\}$ and S=$\{(v_4^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_4^3;v_1^2,v_1^4,v_2^4,v_3^1), (v_4^4;v_1^2,v_2^2,v_3^2,v_3^3), (v_4^2;v_1^4,v_2^3,v_2^4,v_3^1)\}$.

Case 6: $(\alpha,\beta)=(13,5)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_4^3), (v_2^4,v_3^2,v_1^1;v_4^4), (v_3^4,v_4^2,v_1^1;v_2^3), (v_2^4,v_3^1,v_1^2;v_2^1), (v_1^2,v_4^1,v_3^3;v_2^4), (v_1^2,v_2^3,v_3^4;v_2^1), (v_2^1,v_4^4,v_1^3;v_3^2), (v_2^2,v_3^1,v_1^3;v_4^2), (v_3^4,v_4^1,v_1^3;v_2^4), (v_2^3,v_3^2,v_1^4;v_4^1), (v_2^1,v_4^2,v_3^3;v_1^4), (v_2^1,v_4^3,v_3^2;v_2^3), (v_3^4,v_4^3,v_2^2;v_1^4), (v_2^3,v_4^4,v_3^1;v_1^4)\}$ and S=$\{(v_4^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_4^3;v_1^2,v_1^4,v_2^4,v_3^1), (v_4^4;v_1^2,v_2^2,v_3^2,v_3^3), (v_4^2;v_1^4,v_2^3,v_2^4,v_3^1), (v_1^4;v_2^1,v_2^3,v_3^2,v_4^1)\}$.
Case 7: $(\alpha,\beta)=(12,6)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_4^3), (v_2^4,v_3^2,v_1^1;v_4^4), (v_3^4,v_4^2,v_1^1;v_2^3), (v_2^4,v_3^1,v_1^2;v_2^1), (v_1^2,v_4^1,v_3^3;v_2^4), (v_1^2,v_2^3,v_3^4;v_2^1), (v_2^1,v_4^4,v_1^3;v_3^2), (v_1^3,v_3^4,v_4^1;v_1^4), (v_2^3,v_3^2,v_1^4;v_4^1), (v_2^1,v_4^2,v_3^3;v_1^4), (v_2^1,v_4^3,v_3^2;v_2^3), (v_3^4,v_4^3,v_2^2;v_3^1), (v_2^3,v_4^4,v_3^1;v_1^4)\}$ and S=$\{(v_4^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_4^3;v_1^2,v_1^4,v_2^4,v_3^1), (v_4^4;v_1^2,v_2^2,v_3^2,v_3^3), (v_4^2;v_1^4,v_2^3,v_2^4,v_3^1), (v_1^4;v_2^1,v_2^2,v_2^3,v_3^2), (v_1^3;v_2^2,v_3^1,v_4^2,v_2^4)\}$.
Case 8: $(\alpha,\beta)=(11,7)$.
K=$\{(v_2^2,v_3^3,v_1^1;v_4^2), (v_2^1,v_3^3,v_1^2;v_4^3), (v_2^3,v_3^4,v_1^2;v_2^4), (v_2^2,v_3^1,v_1^3;v_4^1), (v_2^4,v_3^2,v_1^3;v_3^4), (v_1^4,v_3^3,v_4^1;v_2^4), (v_2^3,v_3^2,v_1^4;v_4^3), (v_3^2,v_4^3,v_2^1;v_4^4), (v_3^4,v_4^3,v_2^2;v_1^4), (v_3^3,v_4^2,v_2^4;v_4^3), (v_3^1,v_4^2,v_2^3;v_4^1)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3)\}$.
Case 9: $(\alpha,\beta)=(10,8)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_4^1), (v_2^1,v_3^3,v_1^2;v_2^4), (v_1^2,v_2^3,v_3^4;v_1^3), (v_2^2,v_3^1,v_1^3;v_4^1), (v_1^3,v_3^2,v_2^4;v_4^1), (v_2^3,v_3^2,v_1^4;v_4^1), (v_3^2,v_4^3,v_2^1;v_4^4), (v_3^4,v_4^3,v_2^2;v_1^4), (v_3^1,v_4^2,v_2^3;v_4^1), (v_2^4,v_4^2,v_3^3;v_1^4)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3), (v_4^3;v_1^1,v_1^2,v_1^4,v_2^4)\}$.
Case 10: $(\alpha,\beta)=(9,9)$.
K=$\{(v_1^1,v_3^3,v_2^2;v_1^4), (v_1^2,v_2^1,v_3^3;v_4^1), (v_1^2,v_3^4,v_2^3;v_4^2), (v_1^3,v_2^2,v_3^1;v_2^3), (v_1^3,v_3^2,v_2^4;v_1^2), (v_2^3,v_3^2,v_1^4;v_3^3), (v_3^2,v_4^3,v_2^1;v_4^4), (v_2^2,v_4^3,v_3^4;v_1^3), (v_2^4,v_3^3,v_4^2;v_3^1)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3), (v_4^3;v_1^1,v_1^2,v_1^4,v_2^4), (v_4^1;v_1^3,v_1^4,v_2^3,v_2^4)\}$.
Case 11: $(\alpha,\beta)=(8,10)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_4^1), (v_1^2,v_2^1,v_3^3;v_1^4), (v_2^3,v_3^4,v_1^2;v_2^4), (v_1^3,v_2^2,v_3^1;v_2^3), (v_1^3,v_3^2,v_2^4;v_3^3), (v_2^3,v_3^2,v_1^4;v_2^2), (v_3^2,v_4^3,v_2^1;v_4^4), (v_2^2,v_4^3,v_3^4;v_1^3)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3), (v_4^3;v_1^1,v_1^2,v_1^4,v_2^4), (v_4^1;v_1^3,v_1^4,v_2^3,v_2^4), (v_4^2;v_2^3,v_2^4,v_3^1,v_3^3)\}$.
Case 12: $(\alpha,\beta)=(7,11)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_4^1), (v_1^2,v_3^3,v_2^1;v_4^4), (v_2^3,v_3^4,v_1^2;v_2^4), (v_1^3,v_2^2,v_3^1;v_2^3), (v_1^3,v_3^2,v_2^4;v_3^3), (v_2^1,v_4^3,v_3^2;v_2^3), (v_2^2,v_4^3,v_3^4;v_1^3)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3), (v_4^3;v_1^1,v_1^2,v_1^4,v_2^4), (v_4^1;v_1^3,v_1^4,v_2^3,v_2^4), (v_4^2;v_2^3,v_2^4,v_3^1,v_3^3), (v_1^4;v_2^2,v_2^3,v_3^2,v_3^3)\}$.
Case 13: $(\alpha,\beta)=(6,12)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_4^1), (v_2^1,v_3^3,v_1^2;v_2^4), (v_2^2,v_3^1,v_1^3;v_3^4), (v_1^3,v_3^2,v_2^4;v_3^3), (v_3^2,v_4^3,v_2^1;v_4^4), (v_2^2,v_4^3,v_3^4;v_1^2)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3), (v_4^3;v_1^1,v_1^2,v_1^4,v_2^4), (v_4^1;v_1^3,v_1^4,v_2^3,v_2^4), (v_4^2;v_2^3,v_2^4,v_3^1,v_3^3), (v_1^4;v_2^2,v_2^3,v_3^2,v_3^3), (v_2^3;v_1^2,v_3^1,v_3^2,v_3^4)\}$.
Case 14: $(\alpha,\beta)=(5,13)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_4^1), (v_2^1,v_3^3,v_1^2;v_2^4), (v_1^3,v_3^2,v_2^4;v_3^3), (v_3^2,v_4^3,v_2^1;v_4^4), (v_1^3,v_3^1,v_2^2;v_4^3)\}$ and S= $\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2), (v_2^1;v_1^3,v_1^4,v_3^4,v_4^2), (v_3^1;v_1^2,v_1^4,v_2^4,v_4^3), (v_4^1;v_1^2,v_2^2,v_3^2,v_3^4), (v_4^2;v_1^1,v_1^3,v_1^4,v_3^4), (v_4^4;v_2^2,v_3^1,v_3^2,v_3^3), (v_4^4;v_1^1,v_1^2,v_1^3,v_2^3), (v_4^3;v_1^1,v_1^2,v_1^4,v_2^4), (v_4^1;v_1^3,v_1^4,v_2^3,v_2^4), (v_4^2;v_2^3,v_2^4,v_3^1,v_3^3), (v_1^4;v_2^2,v_2^3,v_3^2,v_3^3), (v_2^3;v_1^2,v_3^1,v_3^2,v_3^4), (v_3^4;v_1^2,v_1^3,v_2^2,v_4^3)\}$.

Case 15: $(\alpha,\beta)=(4,14)$.
K=$\{(v_1^1,v_2^2,v_3^3;v_4^1)$, $(v_1^2,v_3^3,v_2^1;v_4^4)$, $(v_1^3,v_3^1,v_2^2;v_4^3)$, $(v_2^1,v_4^3,v_3^2;v_1^3)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2)$, $(v_2^1;v_1^3,v_1^4,v_3^4\ v_4^2)$, $(v_3^1;v_1^2,v_1^4,v_2^4,v_4^3)$, $(v_4^1;v_1^2,v_2^2,v_3^2,v_3^4)$, $(v_4^2;v_1^1,v_1^3,v_1^4,v_3^4)$, $(v_4^4;v_2^2,v_3^1,v_3^2,v_3^3)$, $(v_4^4;v_1^1,v_1^2,v_1^3,v_2^3)$, $(v_4^3;v_1^1,v_1^2,v_1^4,v_2^4)$, $(v_4^1;v_1^3,v_1^4,v_2^3,v_2^4)$, $(v_4^2;v_2^3,v_2^4,v_3^1,v_3^3)$, $(v_1^4;v_2^2,v_2^3,v_3^2,v_3^3)$, $(v_2^3;v_1^2,v_3^1,v_3^2,v_3^4)$, $(v_3^4;v_1^2,v_1^3,v_2^2,v_4^3)$, $(v_2^4;v_1^2,v_1^3,v_3^2,v_3^3)\}$.
Case 16: $(\alpha,\beta)=(3,15)$.
K=$\{(v_1^1,v_3^3,v_2^2;v_3^1)$, $(v_1^2,v_2^1,v_3^3;v_4^1)$, $(v_2^1,v_3^2,v_4^3;v_2^2)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2)$, $(v_2^1;v_1^4,v_3^4,v_4^2,v_4^3)$, $(v_3^1;v_1^2,v_1^4,v_2^4,v_4^3)$, $(v_4^1;v_1^2,v_2^2,v_3^2,v_3^4)$, $(v_4^2;v_1^1,v_1^3,v_1^4,v_3^4)$, $(v_4^4;v_2^2,v_3^1,v_3^2,v_3^3)$, $(v_4^4;v_1^1,v_1^2,v_1^3,v_2^3)$, $(v_4^3;v_1^1,v_1^2,v_1^4,v_2^4)$, $(v_4^1;v_1^3,v_1^4,v_2^3,v_2^4)$, $(v_4^2;v_2^3,v_2^4,v_3^1,v_3^3)$, $(v_1^4;v_2^2,v_2^3,v_3^2,v_3^3)$, $(v_2^3;v_1^2,v_3^1,v_3^2,v_3^4)$, $(v_3^4;v_1^2,v_1^3,v_2^2,v_4^3)$, $(v_2^4;v_1^2,v_1^3,v_3^2,v_3^3)$, $(v_1^3;v_2^1,v_2^2,v_3^1,v_3^2)\}$.
Case 17: $(\alpha,\beta)=(2,16)$.
K=$\{(v_1^2,v_2^1,v_3^3;v_1^1)$, $(v_2^1,v_4^3,v_3^2;v_4^1)\}$ and S=$\{(v_1^1;v_2^2,v_2^3,v_2^4,v_3^2)$, $(v_2^1;v_1^4,v_3^4,v_4^2,v_4^3)$, $(v_3^1;v_1^2,v_1^4,v_2^4,v_4^3)$, $(v_4^1;v_1^2,v_2^2,v_3^3,v_3^4)$, $(v_4^2;v_1^1,v_1^3,v_1^4,v_3^4)$, $(v_4^4;v_2^2,v_3^1,v_3^2,v_3^3)$, $(v_4^4;v_1^1,v_1^2,v_1^3,v_2^3)$, $(v_4^3;v_1^1,v_1^2,v_1^4,v_2^4)$, $(v_4^1;v_1^3,v_1^4,v_2^3,v_2^4)$, $(v_4^2;v_2^3,v_2^4,v_3^1,v_3^3)$, $(v_1^4;v_2^2,v_2^3,v_3^2,v_3^3)$, $(v_2^3;v_1^2,v_3^1,v_3^2,v_3^4)$, $(v_3^4;v_1^2,v_1^3,v_2^2,v_4^3)$, $(v_2^4;v_1^2,v_1^3,v_3^2,v_3^3)$, $(v_1^3;v_2^1,v_2^2,v_3^1,v_3^2)$, $(v_2^2;v_1^1,v_3^1,v_3^3,v_4^3)\}$.
Case 18: $(\alpha,\beta)=(1,17)$.
K=$\{(v_2^2,v_4^1,v_1^3;v_2^4)\}$ and S=$\{(v_1^1,v_3^1;v_2^2,v_2^4,v_2^3,v_4^3)$, $(v_1^2,v_3^2;v_2^1,v_2^3,v_2^4,v_4^4)$, $(v_1^4,v_3^4;v_2^1,v_2^2,v_2^3,v_4^2)$, $(v_3^3;v_2^1,v_2^2,v_2^4,v_4^2)$, $(v_4^1;v_2^3,v_2^4,v_3^3,v_3^4)$, $(v_4^2;v_1^3,v_2^1,v_2^3,v_2^4)$, $(v_4^3;v_2^1,v_2^2,v_2^4,v_3^4)$, $(v_4^4;v_2^1,v_2^2,v_2^3,v_3^3)$, $(v_1^1;v_3^3,v_3^4,v_4^2,v_4^4)$, $(v_1^3;v_3^3,v_3^4,v_4^1,v_4^3)$, $(v_1^4;v_2^1,v_3^1,v_3^4,v_4^4)$, $(v_1^5;v_4^1,v_3^2,v_3^3,v_4^3)$, $(v_3^1;v_1^3,v_1^5,v_4^2,v_4^4)$, $(v_3^2;v_1^1,v_1^3,v_4^1,v_4^3)\}$.
Case 19: $(\alpha,\beta)=(0,18)$.
S=$\{(v_4^1;v_2^2,v_2^3,v_2^4,v_3^4)$, $(v_4^2;v_2^1,v_2^3,v_2^4,v_3^3)$, $(v_4^3;v_2^2,v_2^1,v_2^4,v_3^4)$, $(v_4^4;v_2^2,v_2^3,v_2^1,v_3^3)$, $(v_1^1;v_3^3,v_3^4,v_4^2,v_4^4)$, $(v_1^2;v_3^3,v_3^4,v_4^1,v_4^3)$, $(v_1^3;v_3^1,v_3^4,v_4^2,v_4^4)$, $(v_1^4;v_3^3,v_3^2,v_4^1,v_4^3)$, $(v_3^1;v_1^2,v_1^4,v_4^2,v_4^4)$, $(v_3^2;v_1^1,v_1^3,v_4^1,v_4^3)$, $(v_1^1,v_3^1;v_2^2,v_2^4,v_2^3,v_4^3)$, $(v_1^2,v_3^2;v_2^1,v_2^3,v_2^4,v_4^4)$, $(v_1^3,v_3^3;v_2^1,v_2^2,v_2^4,v_4^1)$, $(v_1^4,v_3^4;v_2^1,v_2^2,v_2^3,v_4^2)\}$.

□

Lemma 2.5. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_4 \times K_5$ for some admissible pairs (α,β) such that $\alpha+\beta=30$.*

Proof. We can write, $K_4 \times K_5 = K_4 \times K_4 \oplus 12K_{1,4}$. By Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α,β) such that $\alpha+\beta=18$. Then the graph $K_4 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α,β) such that $\alpha+\beta=30$. In this case we have, $0 \le \alpha \le 18$ and $12 \le \beta \le 30$ such that $\alpha+\beta=30$. □

Lemma 2.6. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_4 \times K_6$ for some admissible pairs (α,β) such that $\alpha+\beta=45$.*

Proof. We can write, $K_4 \times K_6 = 2(K_3 \times K_4) \oplus 27K_{1,4}$. By Lemma 2.2, the graph $K_3 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α,β) such that $\alpha+\beta=9$. Then the graph $K_4 \times K_6$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α,β) such that $\alpha+\beta=45$. In this case we have, $0 \le \alpha \le 18$ and $27 \le \beta \le 45$ such that $\alpha+\beta=45$. □

Lemma 2.7. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_4 \times K_7$ for some admissible pairs (α,β) such that $\alpha+\beta=63$.*

Proof. We can write, $K_4 \times K_7 = K_3 \times K_4 \oplus K_4 \times K_4 \oplus 36K_{1,4}$. By Lemma 2.2, the graph $K_3 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs such that $\alpha+\beta = 9$. By Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs such that $\alpha+\beta = 18$. Hence in this case we have, $0 \leq \alpha \leq 27$ and $36 \leq \beta \leq 63$ such that $\alpha+\beta = 63$. Thus the graph $K_4 \times K_7$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α, β) such that $\alpha+\beta = 63$. □

Lemma 2.8. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_5 \times K_5$ for some admissible pairs (α, β) such that $\alpha+\beta = 50$.*

Proof. Let $K_5 \times K_5 = K_4 \times K_4 \oplus 32K_{1,4}$, then by Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α, β) such that $\alpha+\beta = 18$. Hence in this case we have, $0 \leq \alpha \leq 18$ and $32 \leq \beta \leq 50$ such that $\alpha+\beta = 50$. Thus the graph $K_5 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α, β) such that $\alpha+\beta = 50$. □

Lemma 2.9. *There exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_5 \times K_6$ for some admissible pairs (α, β) such that $\alpha+\beta = 75$.*

Proof. Let $K_5 \times K_6 = K_4 \times K_4 \oplus 57K_{1,4}$. By Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α, β) such that $\alpha+\beta = 18$. Hence in this case we have, $0 \leq \alpha \leq 18$ and $57 \leq \beta \leq 75$ such that $\alpha+\beta = 75$. Thus the graph $K_5 \times K_6$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α, β) such that $\alpha+\beta = 75$. □

Remark 2.1. *The necessary condition given in Theorem 2.1 can be classified into 3 cases according to the values of m and n as stated below:*
Case 1. *Both m and n are even.*
(i) $m, n \equiv 0(mod\ 4)$
(ii) $m \equiv 0(mod\ 4)$, $n \equiv 2(mod\ 4)$
Case 2. *m is even and n is odd.*
(i) m is even, $n \equiv 1(mod\ 4)$
(ii) $m \equiv 0(mod\ 4)$, $n \equiv 3(mod\ 4)$
Case 3. *Both m and n are odd.*
(i) $m \equiv 1(mod\ 4)$, $n \equiv 1(mod\ 2)$
For other values of m and n, the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \times K_n$ will not be satisfied.

Lemma 2.10. *If $m, n \equiv 0(mod\ 4)$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m \times K_n$.*

Proof. Let $m = 4p$ and $n = 4q$, where $p, q \geq 1$. Then we can write, $K_{4p} \times K_{4q} = pq(K_4 \times K_4) \oplus 2pq(p+q-2)(K_{4,12}) \oplus 2pq(p-1)(q-1)(K_{4,16})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α, β) such that $\alpha+\beta = 18$. Also from Theorem 1.4, the graphs $K_{4,12}$ and $K_{4,16}$ have an S_4-decomposition. Thus, the graph $K_m \times K_n$ has the desired decomposition for some admissible pairs (α, β). □

Lemma 2.11. *If $m \equiv 0(mod\ 4)$ and $n \equiv 2(mod\ 4)$, then there exists a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition of $K_m \times K_n$.*

Proof. Let $m = 4p$ and $n = 4q+2$, where $p, q \geq 1$. Then we can write, $K_{4p} \times K_{4q+2} = p(q-1)(K_4 \times K_4) \oplus p(K_4 \times K_6) \oplus 4p(q-1)(K_{4,18}) \oplus 2q(p-1)(q-1)(K_{4,12}) \oplus 3p(p-1)(K_{4,20}) \oplus 4p(p-1)(q-1)(K_{4,24})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. By Lemma 2.6, the graph $K_4 \times K_6$ admits a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition such that $\alpha + \beta = 45$. Also from Theorem 1.4, the graphs $K_{4,8}$, $K_{4,12}$, K_{20} and $K_{4,24}$ admits an S_4-decomposition. Thus, the graph $K_m \times K_n$ has the desired decomposition for some admissible pairs (α, β). □

Lemma 2.12. *If m is even and $n \equiv 1(mod\ 4)$, then there exists a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition of $K_m \times K_n$.*

Proof. The proof of this theorem follows from two cases.
Case 1: $m = 0(mod\ 4)$ and $n \equiv 1(mod\ 4)$.
Let $m = 4p$ and $n = 4q+1$, where $p, q \geq 1$. Then we can write, $K_{4p} \times K_{4q+1} = p(q-1)(K_4 \times K_4) \oplus p(K_4 \times K_5) \oplus 4p(q-1)(K_{4,15}) \oplus 2p(p-1)(q-1)(K_{4,18}) \oplus \frac{p(p-1)}{2}(5K_{4,16}) \oplus 4p(p-1)(q-1)(K_{4,20})$. From Lemma 2.4, the graph $K_4 \times K_4$ has a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. From Lemma 2.5, the graph $K_4 \times K_5$ has a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 30$. Also, from Theorem 1.4, the graphs $K_{4,15}$, $K_{4,18}$, $K_{4,16}$ and $K_{4,20}$ admits an S_4-decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β).
Case 2: $m \equiv 2(mod\ 4)$ and $n \equiv 1(mod\ 4)$.
Let $m = 4p+2$ and $n = 4q+1$, where $p, q \geq 1$. Then we can write, $K_{4p+2} \times K_{4q+1} = (p-1)(q-1)(K_4 \times K_4) \oplus (p-1)(K_4 \times K_5) \oplus 4(p-1)(q-1)(K_{4,15}) \oplus (q-1)(K_6 \times K_4) \oplus (K_6 \times K_5) \oplus 6(q-1)(K_{4,25}) \oplus 4(p-1)(q-1)(K_{4,18}) \oplus 5(p-1)(K_{4,24}) \oplus (p-1)(q-1)(5K_{4,24} \oplus 4K_{4,30})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. From Lemma 2.5, the graph $K_4 \times K_5$ admits a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 30$. From Lemma 2.6, the graph $K_6 \times K_4$ admits a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 45$. From Lemma 2.9, the graph $K_6 \times K_5$ admits a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 75$. Then from Theorem 1.4, the graphs $K_{4,15}, K_{4,25}, K_{4,18}, K_{4,24}$ and $K_{4,30}$ have an S_4-decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β). □

Lemma 2.13. *If $m \equiv 0(mod\ 4)$ and $n \equiv 3(mod\ 4)$, then there exists a $\{\mathcal{K}^{\alpha}, S_4^{\beta}\}$-decomposition of $K_m \times K_n$.*

Proof. Let $m = 4p$ and $n = 4q+3$, where $p \geq 1$ and $q \geq 0$. When $q = 0$, we can write, $K_{4p} \times K_3 = p(4p-1)(P_3 \times K_3)$. By Observation 1.2, the graph $P_3 \times K_3$ admits an S_4-decomposition. When $q \neq 0$, we can write, $K_{4p} \times K_{4q+3} = p(q-1)(K_4 \times$

$K_4) \oplus p(K_4 \times K_7) \oplus 4p(q-1)(K_{4,21}) \oplus 2p(p-1)(q-1)(K_{4,12}) \oplus 7\frac{p(p-1)}{2}(K_{4,24}) \oplus 4p(p-1)(q-1)(K_{4,28})$. From Lemma 2.4, the graph $K_4 \times K_4$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 18$. From Lemma 2.7, the graph $K_4 \times K_7$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 63$. Then from Theorem 1.4, the graphs $K_{4,21}, K_{4,12}, K_{4,24}$ and $K_{4,28}$ have an S_4-decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β). □

Lemma 2.14. *If $m \equiv 1(mod\ 4)$ and $n \equiv 1(mod\ 2)$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m \times K_n$.*

Proof. We discuss the proof in two cases.
Case 1: $m = 5$ and $n \equiv 1(mod\ 2)$.
Let $m = 5$ and $n = 2p + 1$, where $p \geq 1$. For $p = 1$, the graph $K_5 \times K_3$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α, β) by Lemma 2.3. For $p > 1$, we can write, $K_5 \times K_{2p+1} = (p-2)(K_{5,5} - I) \oplus K_5 \times K_5 \oplus 5(p-2)(K_{2,20})$. From Observation 1.1, the graph $K_{5,5} - I$ admits an S_4-decomposition. From Lemma 2.8, the graph $K_5 \times K_5$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α, β) such that $\alpha + \beta = 50$. Also, from Theorem 1.4, the graph $K_{2,20}$ has an S_4-decomposition and hence the desired decomposition exists in this case, for some of the admissible pairs (α, β).
Case 2: $m > 5$ and $n \equiv 1(mod\ 2)$.
Let $m = 4p + 1$ and $n = 2q + 1$, where $p \geq 2$ and $q \geq 1$. Then $K_{4p+1} \times K_{4q+1} = (2q+1)q(K_{4p+1,4p+1} - I)$, where I is a 1-factor of distance zero in $K_{4p+1,4p+1}$. From Observation 1.1, the graph $K_{4p+1,4q+1} - I$ admits an S_4-decomposition and hence the proof. □

Now we prove our main result, which is the sufficient condition for the existence of $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \times K_n$.

Theorem 2.15. *Let $m \geq 3$ and $n \geq 4$. The graph $K_m \times K_n$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α, β), if $mn(m-1)(n-1) \equiv 0(mod\ 8)$.*

Proof. The proof follows from Lemmas 2.10 to 2.14. □

3. $\{\mathcal{K}^\alpha, S_4^\beta\}$-DECOMPOSITION OF $K_m \otimes \overline{K_n}$

In this section, the necessary and sufficient conditions for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \otimes \overline{K_n}$ have been discussed.

Theorem 3.1. *Let $m \geq 3$ and $n \geq 4$ be the given integers. Then the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m \otimes \overline{K_n}$ is $m(m-1)n^2 \equiv 0(mod\ 8)$.* □

The following lemmas are useful in proving our main result which is the sufficient condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \otimes \overline{K_n}$ for $m \geq 3$ and $n \geq 4$.

Lemma 3.2. *The graph $K_3 \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α, β) such that $\alpha + \beta = 12$.*

Proof. The decomposition $K \cup S$ of $K_3 \otimes \overline{K_4}$ into α kites and β stars on 4 edges for all admissible pairs (α, β) such that $\alpha + \beta = 12$ is discussed in the following cases:

Case 1: $(\alpha, \beta) = (12, 0)$.
K=$\{(v_3^1,v_2^2,v_1^1;v_3^3), (v_2^1,v_3^2,v_1^1;v_3^4), (v_2^1,v_1^2,v_3^1;v_1^4), (v_1^2,v_3^2,v_2^3;v_1^1), (v_1^2,v_2^2,v_3^3;v_2^1), (v_2^2,v_3^2,v_1^3;v_3^1), (v_1^3,v_3^4,v_2^3;v_3^1), (v_2^3,v_3^3,v_1^4;v_2^1), (v_2^4,v_3^3,v_1^3;v_2^1), (v_1^4,v_3^2,v_2^4;v_3^1), (v_1^2,v_3^4,v_2^4;v_1^1), (v_1^4,v_2^2,v_3^4;v_2^1)\}$.

Case 2: $(\alpha, \beta) = (11, 1)$.
K=$\{(v_2^2,v_3^1,v_1^1;v_3^4), (v_2^1,v_3^2,v_1^1;v_2^3), (v_1^2,v_3^1,v_2^1;v_1^3), (v_1^2,v_2^2,v_3^3;v_2^1), (v_1^3,v_3^2,v_2^2;v_3^4), (v_1^2,v_3^2,v_2^3;v_3^1), (v_1^3,v_3^3,v_2^4;v_3^1), (v_2^3,v_3^4,v_1^3;v_3^1), (v_1^4,v_2^3,v_3^3;v_1^1), (v_1^4,v_3^2,v_2^4;v_1^1), (v_2^4,v_1^2,v_3^4;v_2^1)\}$ and S=$\{(v_1^4;v_2^1,v_2^2,v_3^1,v_3^4)\}$.

Case 3: $(\alpha, \beta) = (10, 2)$.
K=$\{(v_1^1,v_3^2,v_2^1;v_1^3), (v_1^2,v_3^1,v_2^1;v_1^4), (v_1^1,v_3^1,v_2^2;v_1^4), (v_1^2,v_2^2,v_3^3;v_2^1), (v_2^2,v_3^2,v_1^3;v_3^1), (v_1^2,v_3^2,v_2^3;v_3^1), (v_1^3,v_2^3,v_3^4;v_1^4), (v_2^3,v_3^3,v_1^4;v_3^1), (v_1^3,v_3^3,v_2^4;v_1^2), (v_1^4,v_3^2,v_2^4;v_3^1)\}$ and S=$\{(v_1^1;v_2^3,v_3^3,v_2^4,v_3^4), (v_3^4;v_2^1,v_2^2,v_1^2,v_2^4)\}$.

Case 4: $(\alpha, \beta) = (9, 3)$.
K=$\{(v_2^1,v_3^2,v_1^1;v_2^3), (v_2^1,v_3^1,v_1^2;v_3^4), (v_2^2,v_3^1,v_1^1;v_3^4), (v_1^2,v_2^2,v_3^2;v_2^3), (v_2^3,v_3^1,v_1^3;v_2^1), (v_1^2,v_2^3,v_3^3;v_1^1), (v_1^3,v_2^2,v_3^3;v_2^1), (v_2^4,v_3^3,v_1^4;v_3^1), (v_2^4,v_3^4,v_1^3;v_3^2)\}$ and S=$\{(v_1^4;v_2^1,v_2^2,v_2^3,v_3^2), (v_2^4;v_1^1,v_1^2,v_3^1,v_3^2), (v_3^4;v_2^1,v_2^2,v_2^3,v_1^4)\}$.

Case 5: $(\alpha, \beta) = (8, 4)$.
K=$\{(v_1^1,v_3^2,v_2^1;v_1^3), (v_1^2,v_3^1,v_2^1;v_3^3), (v_1^1,v_2^2,v_3^1;v_2^4), (v_2^2,v_3^2,v_1^2;v_2^4), (v_1^3,v_3^1,v_2^3;v_1^1), (v_1^2,v_2^3,v_3^3;v_1^1), (v_2^2,v_3^3,v_1^3;v_2^4), (v_2^4,v_3^3,v_1^4;v_3^4)\}$ and S=$\{(v_3^4;v_2^1,v_2^2,v_2^3,v_2^4), (v_3^4;v_1^1,v_1^2,v_1^3,v_1^4), (v_1^4;v_2^1,v_2^2,v_2^3,v_3^1), (v_3^2;v_1^3,v_2^3,v_1^4,v_2^4)\}$.

Case 6: $(\alpha, \beta) = (7, 5)$.
K=$\{(v_1^1,v_3^2,v_2^1;v_3^3), (v_2^1,v_3^1,v_1^2;v_2^4), (v_1^1,v_3^1,v_2^2;v_1^4), (v_1^2,v_2^2,v_3^2;v_2^3), (v_1^3,v_2^3,v_3^1;v_1^4), (v_2^3,v_1^2,v_3^3;v_1^4), (v_1^3,v_3^3,v_2^2;v_3^4)\}$ and S=$\{(v_1^4;v_2^1,v_2^3,v_2^4,v_3^2), (v_2^4;v_3^1,v_3^2,v_3^3,v_3^4), (v_3^4;v_2^1,v_2^3,v_1^2,v_1^4), (v_1^3;v_2^1,v_3^2,v_3^4,v_2^4), (v_1^1;v_2^4,v_2^3,v_3^3,v_3^4)\}$.

Case 7: $(\alpha, \beta) = (6, 6)$.
K=$\{(v_1^1,v_3^2,v_2^1;v_3^4), (v_1^2,v_3^1,v_2^1;v_3^3), (v_1^1,v_2^2,v_3^1;v_2^4), (v_1^2,v_2^2,v_3^2;v_1^4), (v_1^3,v_2^3,v_3^1;v_1^4), (v_2^3,v_3^3,v_1^2;v_3^4)\}$ and S=$\{(v_1^4;v_2^1,v_2^2,v_2^3,v_2^4), (v_2^4;v_3^1,v_3^2,v_3^3,v_3^4), (v_1^3;v_2^1,v_2^2,v_2^4,v_3^2), (v_2^3;v_1^1,v_3^1,v_3^2,v_3^4), (v_3^3,v_3^4;v_1^1,v_1^3,v_1^4,v_2^2)\}$.

Case 8: $(\alpha, \beta) = (5, 7)$.
K=$\{(v_1^1,v_2^2,v_3^1;v_2^4), (v_1^1,v_3^2,v_2^1;v_3^4), (v_1^2,v_3^1,v_2^1;v_3^3), (v_1^2,v_2^2,v_3^2;v_1^4), (v_1^3,v_2^3,v_3^1;v_1^4)\}$ and S=$\{(v_1^4; v_2^1,v_2^2,v_2^3,v_2^4), (v_1^1;v_2^3,v_2^4,v_3^3,v_3^4), (v_1^3;v_2^1,v_2^2,v_2^4,v_3^2), (v_2^3,v_2^4;v_1^2,v_3^2,v_3^3,v_3^4), (v_3^3,v_3^4;v_1^2,v_1^3,v_1^4,v_2^2)\}$.

Case 9: $(\alpha, \beta) = (4, 8)$.
K=$\{(v_1^1,v_3^1,v_2^2;v_3^3), (v_1^1,v_3^2,v_2^1;v_3^3), (v_1^2,v_3^1,v_2^1;v_3^4), (v_1^2,v_3^2,v_2^2;v_3^4)\}$ and S=$\{(v_1^3,v_1^4;v_2^1,v_2^2,v_2^3,v_2^4), (v_1^1,v_1^2;v_2^3,v_2^4,v_3^3,v_3^4), (v_1^3,v_1^4,v_2^3,v_2^4;v_3^1,v_3^2,v_3^3,v_3^4)\}$.

Case 10: $(\alpha, \beta) = (3, 9)$.
K=$\{(v_1^1,v_3^1,v_2^2;v_1^3), (v_1^1,v_2^1,v_3^2;v_2^2), (v_2^1,v_1^2,v_3^1;v_1^3)\}$ and S=$\{(v_1^4,v_2^3,v_2^4;v_3^1,v_3^2,v_3^3,v_3^4), (v_1^3,$

$v_1^4;v_2^1,v_2^2,v_2^3,v_2^4)$, $(v_3^3,v_3^4;v_2^1,v_2^2,v_1^2,v_1^3)$, $(v_1^1;v_2^3,v_2^4,v_3^3,v_3^4)$, $(v_1^2;v_2^2,v_2^3,v_2^4,v_3^2)\}$.
Case 11: $(\alpha,\beta)=(2,10)$.
K=$\{(v_1^1,v_3^2,v_2^1;v_3^3)$, $(v_1^1,v_3^1,v_2^2;v_3^4)\}$ and S=$\{(v_1^2,v_1^3,v_1^4;v_2^1,v_2^2,v_2^3,v_2^4)$, $(v_3^1,v_3^4;v_1^2,v_1^3,v_1^4,v_2^1)$, $(v_3^2,v_3^3;v_1^2,v_1^3,v_1^4,v_2^2)$, $(v_2^3,v_2^4;v_3^1,v_3^2,v_3^3,v_3^4)$, $(v_1^1;v_2^3,v_2^4,v_3^3,v_3^4)\}$.
Case 12: $(\alpha,\beta)=(1,11)$.
K=$\{(v_2^1,v_3^2,v_1^1;v_3^3)\}$ and S=$\{(v_1^1;v_3^1,v_2^2,v_2^3,v_3^4)$, $(v_1^2,v_1^3,v_1^4;v_2^1,v_2^2,v_2^3,v_2^4)$, $(v_2^2,v_2^3,v_2^4;v_3^1,v_3^2,v_3^3,v_3^4)$, $(v_3^1,v_3^3,v_3^4;v_1^2,v_1^3,v_1^4,v_2^1)$, $(v_3^2;v_1^2,v_1^3,v_1^4,v_2^4)\}$.
Case 13: $(\alpha,\beta)=(0,12)$.
S=$\{(v_1^1,v_1^2,v_1^3,v_1^4;v_2^1,v_2^2,v_2^3,v_2^4)$, $(v_1^1,v_1^2,v_1^3,v_1^4;v_3^1,v_3^2,v_3^3,v_3^4)$, $(v_2^1,v_2^2,v_2^3,v_2^4;v_3^1,v_3^2,v_3^3,v_3^4)\}$. □

Remark 3.1. *The necessary condition given in Theorem 3.1 can be classified into 2 cases according to the values of m and n as stated below:*
Case 1. *$m \geq 3$ and n is even.*
Case 2. *n is odd and $m \equiv 0$ or $1 (mod\ 8)$.*
For other values of m and n, the necessary condition for the existence of a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \otimes \overline{K_n}$ will not be satisfied.

Lemma 3.3. *For $m \geq 4$, $K_m \otimes \overline{K_4}$ has a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α,β) such that $\alpha+\beta = 2m(m-1)$.*

Proof. **Case 1.** For $m = 4$, we can write, $K_4 \otimes \overline{K_4} = K_3 \otimes \overline{K_4} \oplus K_{4,12}$. By Lemma 3.2, the graph $K_3 \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for all admissible pairs (α,β) such that $\alpha+\beta = 12$. By Theorem 1.4, the graph $K_{4,12}$ admits an S_4-decomposition. Then the graph $K_4 \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α,β) such that $\alpha+\beta = 24$. Hence in this case we have, $0 \leq \alpha \leq 12$ and $12 \leq \beta \leq 24$ such that $\alpha+\beta = 24$.
Case 2. For $m > 4$, we can write $K_m \otimes \overline{K_4} = K_4 \otimes \overline{K_4} \oplus \frac{(m-4)(m-5)}{2} K_{4,4} \oplus (m-4)K_{4,16}$. By Case 1, the graph $K_4 \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α,β) such that $\alpha+\beta = 24$. By Theorem 1.4, the graphs $K_{4,4}$ and $K_{4,16}$ admits an S_4-decomposition and hence $K_m \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α,β). Therefore, in this case we have, $0 \leq \alpha \leq 12$ and $2[6+(m-4)(m+3)] \leq \beta \leq 2m(m-1)$ such that $\alpha+\beta = 2m(m-1)$. □

Lemma 3.4. *For all $m \geq 3$ and $n > 4$ is even, there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m \otimes \overline{K_n}$.*

Proof. When n is even, we have the proof in two cases as given below:
Case 1. When $n \equiv 0 (mod\ 4)$.
We can write, $K_m \otimes \overline{K_n} = q(K_m \otimes \overline{K_4}) \oplus \frac{q(q-1)}{2} m(m-1)K_{4,4}$. By Lemmas 3.2 and 3.3, the graph $K_m \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α,β). By Theorem 1.4, the graph $K_{4,4}$ admits an S_4-decomposition and hence the graph $K_m \otimes \overline{K_n}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α,β).
Case 2. When $n \equiv 2 (mod\ 4)$.
We can write, $K_m \otimes \overline{K_n} = q(K_m \otimes \overline{K_4}) \oplus \frac{q(q-1)}{2} m(m-1)K_{4,4} \oplus 2K_m \oplus K_{m,m} - I \oplus$

$2qm(m-1)K_{1,4}$. By Lemmas 3.2 and 3.3, the graph $K_m \otimes \overline{K_4}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α, β). By Theorem 1.3, the graph K_m admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α, β). By Theorem 1.4, the graphs $K_{4,4}$ and $K_{1,4}$ admits an S_4-decomposition. By Observation 1.1, the graph $K_{m,m} - I$ admits an S_4-decomposition. Hence the graph $K_m \otimes \overline{K_n}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some admissible pairs (α, β). □

Lemma 3.5. *If $m \equiv 0$ or $1(mod\ 8)$ and $n \equiv 1(mod\ 2)$, then there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition of $K_m \otimes \overline{K_n}$ for some admissible pairs (α, β).*

Proof. We can write, $K_m \otimes \overline{K_n} = nK_n \oplus (K_m \times K_n)$. By Theorem 1.3 and 2.15, the graphs K_m and $K_m \times K_n$ have the desired decomposition for some admissible pairs (α, β) and hence the proof follows. □

Theorem 3.6. *Let $m \geq 3$ and $n \geq 4$ be the given integers. Then the graph $K_m \otimes \overline{K_n}$ admits a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition for some of the admissible pairs (α, β), if $m(m-1)n^2 \equiv 0(mod\ 8)$.*

Proof. The proof follows from Lemmas 3.4 and 3.5. □

4. CONCLUDING REMARKS

In this paper, it is proved that there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \times K_n$ for some of the admissible pairs (α, β), whenever $mn(m-1)(n-1) \equiv 0(mod\ 8)$, for $m \geq 3$ and $n \geq 4$. Also, it is proved that there exists a $\{\mathcal{K}^\alpha, S_4^\beta\}$-decomposition in $K_m \otimes \overline{K_n}$ for some admissible pairs (α, β), whenever $m(m-1)n^2 \equiv 0(mod\ 8)$, for $m \geq 3$ and $n \geq 4$.

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