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SOLVABILITY OF COUPLED SYSTEMS OF FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We present existence theorems for coupled systems of quadratic integral equations of fractional order. As applications we deduce existence theorems for two coupled systems of Cauchy problems. Also, an example illustrating the existence theorem is given.

Key words: Quadratic integral equation of fractional-order, coupled system, Cauchy problems, Schauder fixed point.

Abstrak. Makalah ini membahas teorema tentang eksistensi solusi dari sistem persamaan integral kuadratik orde fraksional. Sebagai aplikasi, teorema eksistensi untuk sistem masalah Cauchy diturunkan. Selain itu, sebuah contoh diberikan untuk menggambarkan teorema eksistensi tersebut.

Kata kunci: Persamaan integral kuadratik orde fraksional, sistem kopel, masalah Cauchy, titik tetap Schauder.

1. Introduction

Systems occur in various problems of applied nature, for instance, see (Bashir Ahmad, Juan Nieto [9]- Y. Chen, H. An [11], El-Sayed and Hashem [22], Gafiychuk, Datsko, Meleshko [24], Gejji [25] and Lazarevich [27]). Recently, X. Su [32] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [33] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations. The solvability of the coupled systems of integral equations in reflexive Banach space proved in El-Sayed and Hashem [18]- El-Sayed and Hashem [20]. Also, a comparison between the classical method

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of successive approximations (Picard) method and Adomian decomposition method of coupled system of quadratic integral equations proved in El-Sayed, Hashem and Ziada [21].
Let $\mathbb{R}$ be the set of real numbers whereas $I = [0,1]$, $L_1 = L_1[I]$ be the space of Lebesgue integrable functions on I.

Firstly, we prove the existence of at least one continuous solution for the coupled system of quadratic functional integral equation of fractional order

$$\begin{aligned} x(t) &= a_1(t) + g_1(t, y(\psi_1(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(\phi_1(s)))\, ds,\ t \in I,\ \alpha > 0 \\ y(t) &= a_2(t) + g_2(t, x(\psi_2(t))) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(\phi_2(s)))\, ds,\ t \in I,\ \beta > 0. \end{aligned} \tag{1}$$

Quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. Argyros [1]- Banaś, Rzepka [8] and El-Sayed, Hashem[13]- El-Sayed, Rzepka[23]). However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness we use Tychonoff fixed point theorem. The existence of continuous solutions for some quadratic integral equations was proved by using Schauder-Tychonoff fixed point theorem Salem [31].
Also, the existence of solutions of the two Cauchy problems

$$\begin{aligned} {}_RD^{\alpha}x(t) &= f_1(t, y(\phi_1(t))),\ t \in (0,1) \text{ and } x(0) = 0,\ \alpha \in (0,1) \\ {}_RD^{\beta}y(t) &= f_2(t, x(\phi_2(t))),\ t \in (0,1) \text{ and } y(0) = 0,\ \beta \in (0,1) \end{aligned} \tag{2}$$

(where ${}_RD^{\alpha}$ is the Riemann-Liouville fractional order derivative)
and

$$\begin{aligned} \frac{dx(t)}{dt} &= f_1(t, y(\phi_1(t))),\ t \in (0,1),\ x(0) = x_0, \\ \frac{dy(t)}{dt} &= f_2(t, x(\phi_2(t))),\ t \in (0,1),\ y(0) = y_0, \end{aligned} \tag{3}$$

will be proved.

The proof of the main result will be based on the following fixed-point theorem.

Theorem 1.1. *(Schauder Fixed Point Theorem)Curtain and Pritchard* [12].
Let Q be a nonempty, convex, compact subset of a Banach space X, and $T: Q \to Q$ be a continuous map. Then T has at least one fixed point in Q.

Definition 1.2. *The fractional-order integral of order β (positive real number) of the function f is defined on $[a,b]$ by (see Kilbas, Srivastava and Trujillo* [26]*, Podlubny* [28]*, Miller and Ross* [29] *and Samko, Kilbas, Marichev* [30]*)*

$$I_a^{\beta} f(t) \ = \ \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\ ds, \tag{4}$$

and when $a=0$, we have $I^{\beta} f(t) \ = \ I_0^{\beta} f(t)$.

Definition 1.3. *The Riemann-Liouville fractional-order derivative of order $\alpha \in (0,1)$ of the function f is given by (see Kilbas, Srivastava and Trujillo* [26]*, Podlubny* [28]*, Miller and Ross* [29] *and Samko, Kilbas, Marichev* [30]*)*

$$_RD^{\alpha} f(t) \ = \ \frac{d}{dt} I^{1-\alpha} f(t).$$

For the properties of fractional calculus (see Kilbas, Srivastava and Trujillo [26], Podlubny [28], Miller and Ross [29] and Samko, Kilbas, Marichev [30] for example).

2. Existence of Continuous Solutions

Now, the coupled system (1) will be investigated under the assumptions:

(i) $a_i : \ I \to \mathbb{R}, \ i=1,\ 2$ are continuous and bounded with $M_i = \sup_{t\in I} \ |a_i(t)|$.

(ii) $g_i : \ I \ \times \ \mathbb{R} \ \to \mathbb{R}, \ i=1,\ 2$ are continuous and bounded with $N_i = \sup_{(t,x)\in I\times\mathbb{R}} \ |g_i(t,x)|, \ i=1,\ 2$.

(iii) There exist constants $h_i, \ l_i, \ i=1,\ 2$ respectively satisfying

$$|g_i(t,x) - g_i(s,y)| \le l_i \ |t-s| + h_i \ |x-y|$$

for all $t, \ s \in I$ and $x, \ y \in \mathbb{R}$.

(iv) $f_i : \ I \times \mathbb{R} \ \to \ \mathbb{R}, \ i=1,\ 2$ satisfy Carathèodory condition (i.e. measurable in t for all $x : I \to \mathbb{R}$ and continuous in x for all $t \ \in \ I$).

(v) There exist two functions $m_i \ \in \ L_1$ and positive constants b_i such that $|f_i(t,x)| \le m_i(t) + b_i \ |x| \ (\forall \ (t,x) \in I\times\mathbb{R}$) and $k_i = \sup_{t\in I} I^{\gamma_i} \ m_i(t), \ i=1,\ 2$ for any $\gamma_1 \le \alpha, \ \gamma_2 \le \beta$.

(vi) $\psi_i, \ \phi_i : \ I \to \ I$ are continuous .

Let $C(I)$ be the class of all real functions defined and continuous on I with the norm

$$||\ x\ || \ = \ \sup\{ \ |\ x(t)\ | : \ t \ \in \ I \ \}.$$

Now, we define the Banach space $X \ = \ \{x(t)|x(t) \in C(I) \ \}$ endowed with the norm $||x||_X \ = \ \sup_{t\in I} |x(t)|, \ Y \ = \ \{y(t)|y(t) \ \in \ C(I)\}$ endowed with the norm $||y||_Y = \sup_{t\in I} |y(t)|$. For $(x,y) \in X \times Y$, let $||(x,y)||_{X\times Y} \ = \ \sup\{||x||_X, ||y||_Y\}$. Clearly, ($X \times Y, ||(x,y)||_{X\times Y}$) is a Banach space.

Define the operator T by

$$T(x,y)(t) \; = \; (T_1 y(t), \; T_2 x(t)),$$

where

$$\begin{aligned} T_1 y(t) \; &= \; a_1(t) \; + \; g_1(t, y(\psi_1(t))) \; I^{\alpha} f_1(t, y(\phi_1(t))), \; \alpha > 0, \; t \in I, \\ T_2 x(t) \; &= \; a_2(t) \; + \; g_2(t, x(\psi_2(t))) \; I^{\beta} f_2(t, x(\phi_2(t))), \; \beta > 0, \; t \in I, \end{aligned}$$

Theorem 2.1. *Let the assumptions (i)-(vi) be satisfied, then the coupled system of quadratic integral equations of fractional order (1) has at least one solution in* $X \times Y$.

Proof. Define

$$U = \{u = (x(t), y(t)) | (x(t), y(t)) \in X \times Y : \; ||(x,y)||_{X\times Y} \leq r\}.$$

For $(x,y) \in U$, we have

$$|\; T_1 y(t) \;| \; \leq \; |a_1(t)| \; + \; |g_1(t, y(\psi_1(t)))| \int_0^t \frac{(t \; - \; s)^{\alpha-1}}{\Gamma(\alpha)} \; |f_1(s, y(\phi_1(s)))| \; ds$$

$$|T_1 y(t)| \; \leq \; M_1 \; + \; N_1 \; I^{\alpha-\gamma_1} \; I^{\gamma_1} \; m_1(t) \; + \; N_1 \; b_1 \int_0^t \frac{(t \; - \; s)^{\alpha-1}}{\Gamma(\alpha)} \; |y(\phi(s))| \; ds.$$

Also, from assumption (v) we obtain

$$|T_1 y(t)| \; \leq \; M_1 \; + \; N_1 \; k_1 \int_0^t \frac{(t \; - \; s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} \; ds$$

$$+ \; N_1 \; b_1 \; r_1 \int_0^t \frac{(t \; - \; s)^{\alpha-1}}{\Gamma(\alpha)} \; ds, \;\; ||y||_Y = \sup_{t\in I} |y(t)| \leq r_1.$$

Then

$$||T_1 y(t)|| \; \leq \; M_1 \; + \; \frac{N_1 \; k_1}{\Gamma(\alpha \; - \; \gamma_1 \; + \; 1)} \; + \; \frac{N_1 \; b_1 \; r_1}{\Gamma(1+\alpha)}.$$

From the last estimate we deduce that $r_1 = (M_1 \; + \; \frac{N_1 \; k_1}{\Gamma(\alpha \; - \; \gamma_1 \; + \; 1)})(1 - \frac{N_1 \; b_1}{\Gamma(1+\alpha)})^{-1}$.

By a similar way as done above we have

$$||T_2 x(t)|| \; \leq \; M_2 \; + \; \frac{N_2 \; k_2}{\Gamma(\beta \; - \; \gamma_2 \; + \; 1)} \; + \; \frac{N_2 \; b_2 \; r_2}{\Gamma(1+\beta)}$$

and $r_2 = (M_2 \; + \; \frac{N_2 \; k_2}{\Gamma(\beta \; - \; \gamma_2 \; + \; 1)})(1 - \frac{N_2 \; b_2}{\Gamma(1+\beta)})^{-1}$.
Therefore,

$$\begin{aligned} ||Tu(t)|| \; = \; ||T(x,y)(t)|| \; &= \; ||(T_1 y(t), T_2 x(t))|| \\ &= \; \max_{t\in I}\{ \; ||T_1 y(t)|| \; , \; ||T_2 x(t)|| \; \} = \; r. \end{aligned}$$

From the last estimate we can choose

$$r = \max_{t\in I}\{r_1, \; r_2\},$$

then, for every $u = (x, y) \in U$ we have $Tu \in U$ and hence $TU \subset U$.
It is clear that the set U is closed and convex.
Assumptions (ii) and (iv) imply that $T : U \to C(I) \times C(I)$ is a continuous operator. Now, for $u = (x, y) \in U$, and for each $t_1, t_2 \in I$ (without loss of generality assume that $t_1 < t_2$), we get

$$|(T_2x)(t_2) - (T_2x)(t_1)| = |a_2(t_2) - a_2(t_1)$$

$$+ g_2(t_2, x(\psi_2(t_2)))\, I^\beta\, f_2(t_2, x(\phi_2(t_2))) - g_2(t_1, x(\psi_2(t_1)))\, I^\beta\, f_2(t_1, x(\phi_2(t_1)))$$

$$+ g_2(t_1, x(\psi_2(t_1)))\, I^\beta\, f_2(t_2, x(\phi_2(t_2))) - g_2(t_1, x(\psi_2(t_1)))\, I^\beta\, f_2(t_2, x(\phi_2(t_2)))|$$

$$\leq |a_2(t_2) - a_2(t_1)| + |g_2(t_2, x(\psi_2(t_2))) - g_2(t_1, x(\psi_2(t_1)))|\, I^\beta\, |f_2(t_2, x(\phi_2(t_2)))|$$

$$+ |g(t_1, x(\psi_2(t_1)))|\; |\, I^\beta\, f_2(t_2, x(\phi_2(t_2))) - I^\beta\, f_2(t_1, x(\phi_2(t_1)))\, |,$$

but

$$|I^\beta\, f_2(t_2, x(\phi_2(t_2))) - I^\beta\, f_2(t_1, x(\phi_2(t_1)))| = |\int_0^{t_1} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)}\, f_2(s, x(\phi_2(s)))\, ds$$

$$+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)}\, f_2(s, x(\phi_2(s)))\, ds - \int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)}\, f_2(s, x(\phi_2(s)))\, ds|$$

$$\leq |\int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)}\, f_2(s, x(\phi_2(s)))\, ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)}\, f_2(s, x(\phi_2(s)))\, ds$$

$$- \int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)}\, f_2(s, x(\phi_2(s)))\, ds| \leq \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)}\, |f_2(s, x(\phi_2(s)))|\, ds.$$

Then

$$\begin{aligned}
|I^\beta\, f_2(t_2, x(\phi_2(t_2))) &- I^\beta\, f_2(t_1, x(\phi_2(t_1)))| \\
&\leq I^\beta_{t_1} |f_2(t_2, x(\phi_2(t_2)))| \\
&\leq I^\beta_{t_1} m_2(t_2) + b_2\, I^\beta_{t_1} |x(\phi_2(t_2))| \\
&\leq I^{\beta-\gamma_2}_{t_1} I^{\gamma_2}_{t_1} m_2(t_2) + b_2\, I^\beta_{t_1} |x(\phi_2(t_2))| \\
&\leq k_2\, \frac{(t_2 - t_1)^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + b_2\, r_2\, \frac{(t_2 - t_1)^\beta}{\Gamma(\beta + 1)}.
\end{aligned}$$

Then we get

$$\begin{aligned}
&|(T_2x)(t_2) - (T_2x)(t_1)| \\
&\quad \leq |a_2(t_2) - a_2(t_1)| \\
&\quad\quad + [\, l_2|t_2 - t_1| + h_2|x(\psi_2(t_2)) - x(\psi_2(t_1))\,|\,]\, I^\beta\, |\, f_2(t_2, x(\phi_2(t_2)))\,| \\
&\quad\quad + |\, g_2(t_1, x(\psi_2(t_1)))\,|\, (k_2\, \frac{(t_2 - t_1)^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + b_2\, r_2\, \frac{(t_2 - t_1)^\beta}{\Gamma(\beta + 1)})
\end{aligned}$$

i.e.,

$$\begin{aligned}
&|\ (T_2x)(t_2)\ -\ (T_2x)(t_1)\ |\\
&\leq\ |\ a_2(t_2)\ -\ a_2(t_1)\ |\\
&\quad+[l_2|t_2-t_1|\ +\ h_2|x(t_2)-x(t_1)|]\ I^{\beta}\ (m_2(t_2)\ +\ b_2\ |x(\phi_2(t_2))|)\\
&\quad+k_2\ N_2\ \frac{(t_2-t_1)^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}\ +\ N_2\ b_2\ r_2\ \frac{(t_2-t_1)^{\beta}}{\Gamma(\beta+1)}\\
&\leq\ |\ a_2(t_2)\ -\ a_2(t_1)\ |\ +\ \frac{k_2}{\Gamma(\beta-\gamma_2+1)}\ [l_2|t_2-t_1|\ +\ h_2|x(t_2)-x(t_1)|]\\
&\quad+\frac{b_2\ r_2}{\Gamma(\beta+1)}\ [l_2|t_2-t_1|\ +\ h_2|x(t_2)-x(t_1)|]\\
&\quad+\frac{k_2\ N_2}{\Gamma(\beta-\gamma_2+1)}\ (t_2-\ t_1)^{\beta-\gamma_2}\ +\ \frac{N_2\ b_2\ r_2}{\Gamma(\beta+1)}\ (t_2-\ t_1)^{\beta}.
\end{aligned}$$

As done above we can obtain

$$\begin{aligned}
&|\ (T_1y)(t_2)\ -\ (T_1y)(t_1)\ |\\
&\leq\ |\ a_1(t_2)\ -\ a_1(t_1)\ |\\
&\quad+\frac{k_1}{\Gamma(\alpha-\gamma_1+1)}\ [l_1|t_2-t_1|\ +\ h_1|y(t_2)-y(t_1)|]\\
&\quad+\frac{b_1\ r_1}{\Gamma(\alpha+1)}\ [l_1|t_2-t_1|\ +\ h_1|y(t_2)-y(t_1)|]\\
&\quad+\frac{k_1\ N_1}{\Gamma(\alpha-\gamma_1+1)}\ (t_2-\ t_1)^{\alpha-\gamma_1}\ +\ \frac{N_1\ b_1\ r_1}{\Gamma(\alpha+1)}\ (t_2-\ t_1)^{\alpha}.
\end{aligned}$$

Now, from the definition of the operator T, we get

$$\begin{aligned}
Tu(t_2)\ -\ Tu(t_1) &= T(x,y)(t_2)\ -\ T(x,y)(t_1)\\
&= (T_1y(t_2),\ T_2x(t_2))\ -\ (T_1y(t_1),\ T_2x(t_1))\\
&= (T_1y(t_2)\ -\ T_1y(t_1),\ T_2x(t_2)\ -\ T_2x(t_1)),
\end{aligned}$$

and

$$\begin{aligned}
&||Tu(t_2)\ -\ Tu(t_1)|| = \max_{t_1,t_2\in I}\{||T_1y(t_2)\ -\ T_1y(t_1)|| + ||T_2x(t_2)\ -\ T_2x(t_1)||\}\\
&\leq\ ||\ a_1(t_2)-a_1(t_1)\ ||\ +\ \frac{k_1}{\Gamma(\alpha-\gamma_1+1)}\ [l_1|t_2-t_1|\ +\ h_1|y(t_2)-y(t_1)|]\\
&+\frac{b_1\ r_1}{\Gamma(\alpha+1)}\ [l_1|t_2-t_1|\ +\ h_1|y(t_2)-y(t_1)|]\ +\ \frac{k_1\ N_1}{\Gamma(\alpha-\gamma_1+1)}\ (t_2-\ t_1)^{\alpha-\gamma_1}\\
&+||\ a_2(t_2)\ -\ a_2(t_1)\ ||\ +\ \frac{k_2}{\Gamma(\beta-\gamma_2+1)}\ [l_2|t_2-t_1|\ +\ h_2|x(t_2)-x(t_1)|]\\
&+\frac{b_2\ r_2}{\Gamma(\beta+1)}\ [l_2|t_2-t_1|\ +\ h_2|x(t_2)-x(t_1)|]\ +\ \frac{k_2\ N_2}{\Gamma(\beta-\gamma_2+1)}\ (t_2-\ t_1)^{\beta-\gamma_2}\\
&+\frac{N_2\ b_2\ r_2}{\Gamma(\beta+1)}\ (t_2-\ t_1)^{\beta}\ +\ \frac{N_1\ b_1\ r_1}{\Gamma(\alpha+1)}\ (t_2-\ t_1)^{\alpha}.
\end{aligned}$$

Hence

$$| \ t_2 - t_1 \ | < \delta \Longrightarrow \ || \ Tu(t_2) \ - \ Tu(t_1) \ || < \varepsilon(\delta).$$

This means that the functions of TU are equi-continuous on I. Then by the Arzela-Ascoli Theorem Curtain and Pritchard [12] the closure of TU is compact. Since all conditions of the Schauder Fixed-point Theorem hold, then T has a fixed point in U which completes the proof. ■

Example 2.2. *Consider the following coupled system of functional equations for* $t \ \in \ I$

$$\begin{aligned} x(t) &= 1 + \ [\sqrt{t^2+5} + t(|log(|y(t)|+3)|+1)]I^{2/3} \ [2t \ + \ \frac{1}{3-t} \ y(\sin(t^2+3t))], \\ y(t) &= 1 + \ [\frac{1+2t}{10} + \frac{x^2(t)}{30}e^{-t}]I^{2/3}[\ -\ln(1-t) \ + \ \frac{1}{3-t} \ x(\sin(t^2+4t \ )) \ ]. \end{aligned} \tag{5}$$

Set

$$f_1(t,x) \ = \ 2 \ t \ + \ \frac{1}{3-t} \ x, \ g_1(t,y) = \sqrt{t^2+5} + \ t(|log(|y(t)|+3)|+1), \ t \in I$$

$$f_2(t,x) \ = \ -\ln(1-t) \ + \ \frac{1}{3-t} \ x, \ g_2(t,x) = \frac{1+2t}{10} + \frac{x^2}{30}e^{-t}, \ t \in I.$$

Then easily we can deduce that:

- $M_1 = M_2 = 1.$
- $|f_2(t,x)| \ \leq \ \ln(1-t) \ + \ 1/2 \ x$ *and* $|f_1(t,x)| \ \leq \ 2 \ t \ + \ 1/2 \ x.$
- $\phi_1(t) = \sin(t^2+3 \ t), \ \phi_2(t) = \ \sin(t^2+4 \ t), \ \psi_1(t) = \psi_2(t) = t,$

$$\begin{aligned} |g_1(t,z) - g_1(s,y)| &= |\sqrt{t^2+5} + t(|log(|z(t)|+3)|+1) \\ &\quad -\sqrt{s^2+5} - s(|log(|y(s)|+3)|+1)| \\ &\leq |\sqrt{t^2+5} - \sqrt{s^2+5}| + t|log(|z(t)|+3) \\ &\quad -log(|y(s)|+3)| + |t-s| \\ &+ |t \ log(|y(s)|+3) - s \ log(|y(s)|+3)| \\ &\leq \frac{6}{5} \ |t-s| + |z-y| + |t-s| + r.|t-s| \\ &\leq (2+r) \ |t-s| + |z-y|, \ r > 0 \end{aligned}$$

and

$$
\begin{aligned}
|g_2(t,z)-g_2(s,x)| &= |\frac{1+2t}{10}+\frac{z^2}{30}e^{-t}-\frac{1+2s}{10}-\frac{x^2}{30}e^{-s}| \\
&\leq \frac{2}{10}|t-s|+\frac{1}{30}|e^{-t}z^2-e^{-t}x^2|+\frac{1}{30}|e^{-t}x^2-e^{-s}x^2| \\
&\leq \frac{1}{5}|t-s|+\frac{|x+z|}{30}|x-z|+\frac{r^2}{30}|e^{-t}-e^{-s}| \\
&\leq \frac{1}{5}|t-s|+\frac{2r}{30}|x-z|+\frac{r^2}{30}|t-s| \\
&\leq \frac{6+r^2}{30}\ |t-s|+\frac{r}{15}\ |x-z|,\ r>0.
\end{aligned}
$$

Then all the assumptions of Theorem 2.1 are satisfied so the coupled system of the functional equations (5) possesses at least one solution in $X\times Y$.

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Corollary 3.1. *Let the assumptions of Theorem 2.1 be satisfied (with* $\psi_i(t)=\phi_i(t)=t,\ i=1,2$*), then the coupled system of the fractional-order quadratic integral equations*

$$
\begin{aligned}
x(t) &= a_1(t)+g_1(t,y(t))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s,y(s))\ ds,\ t\in I,\ \alpha>0 \\
y(t) &= a_2(t)+g_2(t,x(t))\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,x(s))\ ds,\ t\in I,\ \beta>0
\end{aligned}
\tag{6}
$$

has at least one solution in $X\times Y$.

Corollary 3.2. *Let the assumptions of Theorem 2.1 be satisfied (with* $g_i(t,x)=1,\ i=1,2$ *), then the coupled system of the fractional-order integral equations*

$$
\begin{aligned}
x(t) &= a_1(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s,y(\phi_1(s)))\ ds,\ t\in I,\ \alpha>0 \\
y(t) &= a_2(t)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,x(\phi_2(s)))\ ds,\ t\in I,\ \beta>0
\end{aligned}
\tag{7}
$$

has at least one solution in $X\times Y$.

Now, letting $\alpha,\ \beta\to 1$, we obtain

Corollary 3.3. *Let the assumptions of Theorem 2.1 be satisfied (with* $g_i(t,x) = 1,\ a_1(t) = x_0,\ a_2(t) = y_0$ *and letting* $\alpha,\ \beta \to 1$ *), then the coupled system of the integral equations*

$$\begin{aligned} x(t) &= x_0 + \int_0^t f_1(s, y(\phi_1(s)))\ ds,\ t \in I,\ \alpha > 0 \\ y(t) &= y_0 + \int_0^t f_2(s, x(\phi_2(s)))\ ds,\ t \in I,\ \beta > 0 \end{aligned}$$

has at least one solution in $X \times Y$ *which is equivalent to the coupled system of the initial value problems (3).*

4. The coupled system of the fractional order functional differential equations

For the coupled system of the initial value problems of the nonlinear fractional-order differential equations (2) we have the following theorem.

Theorem 4.1. *Let the assumptions of Theorem 2.1 be satisfied (with* $a_i(t) = 0$ *and* $g_i(t, x(t)) = 1,\ i = 1, 2$*), then the coupled system of the Cauchy problems (2) has at least one solution in* $X \times Y$*.*

Proof. Integrating (2) we obtain the coupled system of the integral equations

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s, y(\phi_1(s)))ds,\ t \in I,\ \alpha > 0 \\ y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s, x(\phi_2(s)))\ ds,\ t \in I,\ \beta > 0, \end{aligned} \tag{8}$$

which by Theorem 2.1 has the desired solution.
Operating with ${}_RD^{\alpha}$ on the first equation of the coupled system (8) and with ${}_RD^{\beta}$ on the second equation of the coupled system (8) we obtain the coupled system of the initial value problems (2). So the equivalence between the coupled system of the initial value problems (2) and the coupled system of the integral equations (8) is proved and then the results follow from Theorem 2.1.■

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