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CERTAIN TYPES OF DERIVATIONS IN RINGS: A SURVEY

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Abstract. In this overview article, we provide a historical account on derivations, Jordan derivations, (α, β)-derivations, left derivations, pre-derivations, homoderivations, nilpotent derivations, and other variants, drawing from the contributions of multiple researchers. Additionally, we delve into recent findings and suggest potential avenues for future investigation in this area. Furthermore, we offer pertinent examples to illustrate that the assumptions underlying various results are indeed necessary and not redundant.

Key words and Phrases: (α, β)- derivation, bi-derivation, Jordan derivation, invertible derivation, n-Jordan homoderivation.

1. INTRODUCTION

Numerous scholarly works delve into the specific conditions under which a ring exhibits commutativity. Jacobson's groundbreaking contribution was the initial foundation in this area. His seminal research demonstrated that for every element x in a ring R, there exists a positive integer $n(x)$ such that $x^{n(x)} = x$, then R is commutative. Herstein was among the first mathematicians who pursued Jacobson's findings. Beginning in 1951 [80], he initially demonstrated that if there is a positive integer n in a ring R such that $x^n - x$ lies in the center, then R is commutative. In 1953 [81], he enhanced this outcome by relaxing the requirement, allowing the exponent n to vary with x rather than being universal. Later in [82],

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he significantly refined the theorem by establishing that for every x in R, there exists a polynomial $p_x(x)$ such that $x^2p_x(x) - x$ is in the center, leading to the commutativity of R.

Various conditions equivalent to commutativity for a ring can be extended and generalized in numerous directions. Since the conditions above relating x are polynomial constraints, it is natural to consider rings which satisfy general polynomial identities where the polynomial is not defined. An early result of this type was done by Bell [29] who proved that if R is a ring in which for every ordered pair (x, y) of elements of R there exists a polynomial $p(X) \in XZ[X]$ such that $xy = yxp(x)$, then R is commutative. The primary approach to studying the commutativity of a certain class of rings was through the utilization of polynomial identities or by constructing general expressions using polynomials, whose evaluation lies within the center for all elements. In the recent years, tools and techniques for investigating the commutativity of rings and algebras have shifted towards the use of specific maps, such as endomorphisms or automorphisms.

The most recent tool and technique to describe the commutativity of rings and algebras is the use of certain kinds of derivations. That's why we have choosen the title of the manuscript, i.e., certain types of derivations in rings, to provide an alternative avenue to investigate the commutativity of rings. Moreover, these tools are also applicable to find better codes in the field of algebraic coding theory (see [42, 43] for details).

At the very beginning of a calculus class one learns the idea of derivatives, its rules and finds its applications in most of the parts of mathematics and not only there, its wide applications in other sciences like physics, engineering, economics is also evident. Properties of a derivative function, especially the famously known Leibniz rule caught attention of many mathematicians, which we know is defined as

$$\frac{d}{dx}(fg) = \left(\frac{d}{dx}f\right)g + f\left(\frac{d}{dx}g\right) \tag{1}$$

for any two differentiable functions f and g.

Over the years the idea of Leibniz rule has been used tremendously but with different notations, the idea being the same. Like the notation used by Issac Newton was

$$\dot{(fg)} = \dot{f}g + f\dot{g} \tag{2}$$

and that by Lagrange,

$$(fg)^{'} = f^{'}g + fg^{'}. \tag{3}$$

Starting with a concept familiar from our early encounters with mathematics in school we embark on a journey into the realm of ring theory, specifically exploring its counterpart: derivations within rings. A lot of attention has been given to the idea of creating something which behaves like a derivative in case of rings and a lot of work has already been done in this direction. This survey paper seeks to navigate through the intricacies of certain types of derivations in rings, drawing parallels to the foundational principles from calculus while unraveling the complexities of algebraic structures and their properties. So we can happily define an additive map d on any ring R, R need not be commutative nor associative, there is no restriction on the ring, and name it a derivation on R if

$$d(ab) = d(a)b + ad(b), \tag{4}$$

for all $a, b \in R$.

Now suppose the ring R is an associative one, then defining a map $ad(a)$, for any $a \in R$ as $ad(a) : b \to [a, b] = ab - ba$, for all $b \in R$ forms a derivation on the ring R, and such a type of derivation is called an *inner derivation* and is usually denoted by I_a. It can be easily verified that the sum of derivations will fetch us a derivation and the product zd for any $z \in Z(R)$ (where $Z(R)$ denotes centre of the ring), d a derivation on R would do the same. The set of all derivations on R, denoted by $Der(R)$ can be thought of as a vector space over the ring R and in addition to that it also forms a Lie algebra with respect to the operation

$$[d_1, d_2] = d_1d_2 - d_2d_1$$

for $d_1, d_2 \in Der(R)$.

Example 1.1. *Consider R to be a commutative ring, define d: $R[x] \longrightarrow R[x]$ by*

$$d(f(x)) = f'(x)$$

where '$'$' denotes the usual differentiation on $R[x]$. Clearly, d so defined forms a derivation on $R[x]$.

As we venture into this exploration, we aim to elucidate the significance and applications of these derivations within ring theory. From Lie derivations to Jordan derivations and beyond, each type offers unique insights into the behavior of rings. By examining their definitions, properties, and applications, we provide readers with a comprehensive understanding of these essential algebraic constructs.

Moreover, this survey endeavors to bridge the gap between the familiar concepts of calculus and the abstract world of ring theory, offering a pathway for readers to traverse between these seemingly disparate fields of mathematics. Through examples, illustrations, and intuitive explanations, we strive to make the study of derivations in rings accessible and engaging to mathematicians at all levels.

In summary, this survey paper serves as a valuable resource for researchers, educators, and students alike, offering a panoramic view of certain types of derivations in rings. While the establishment of functional identities remains a fundamental aspect of ring theory, it is the deeper understanding of commutativity facilitated by derivations that often unveils the true nature of a ring's structure. By consolidating existing knowledge, identifying key trends, and pointing out future research directions, we hope to contribute to the advancement of ring theory and its associated fields.

2. Derivations

The idea of derivations is not new, derivations have been extensively studied from a long period of time. In this span, derivations have proved to have numerous applications as it beautifully relates with the structure of rings and unfolds many a properties of various other structures be it algebras, Lie algebras, ideals to name a few.

Definition 2.1. *In a ring R if we define a map d from R to R satisfying*

(i) $d(a+b) = d(a) + d(b)$ and

(ii) $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$,

then such a map d is termed as a derivation on R.

The set of all derivations on R, denoted by $Der(R)$ can be thought of as a Z-module (where Z denotes the center of R), as sum of derivations will fetch us a derivation and zd for any $z \in Z$, $d \in Der(R)$ would do the same. Let's modify the properties of derivation and see if something interesting pops up. For instance, if we consider the case $a = b$, property (i) becomes,

$$d(a^2) = d(a)a + ad(a) \text{ for all } a \in R, \tag{5}$$

which opens up a new class of derivations, known as *Jordan derivations*. Like derivations help us to know the structure of the ring, Jordan derivations also help to reveal some properties of rings.

Example 2.2. *Consider any 2-torsion free ring R in which $a^2 = 0$, $\forall\ a \in R$, but for some $a, b \in R$ $ab \neq 0$, if we characterize $d : R \to R$ as the identity function then easily d will be a Jordan derivation but it is not a derivation. Instead of the identity map we can take any additive map as well.*

Another such example would be an additive map defined on a ring A, as

$$(ab)^* = b^*a + ba^* \text{ for all } a, b \in A.$$

$*$ is not a derivation but we can see easily that it is a Jordan derivation, such a derivation is famously known as a reverse derivation. One might think Jordan derivations were defined like this, but that's not true, instead Herstein constructed a Jordan ring $(R, +, \circ)$ then defined an additive map $d{:}R \longrightarrow R$ satisfying

$$d(a \circ b) = d(a) \circ b + a \circ d(b) \text{ for all } a, b \in R$$

and named such a map as Jordan derivation. As seen above that a Jordan derivation may not necessarily be a derivation, a natural question ought to be asked when does it imply, that is when a Jordan derivation forms a derivation. In this direction the first result goes to Herstein [84], who established that each Jordan derivation on any prime ring having characteristic not equal to 2 is a derivation. Later in the year 1975 Cusack [61] extended it to a larger class of semiprime rings in which $2x = 0$ implies $x = 0$, i.e., to a 2-torsion free semiprime ring. Now if we go on generalizing the idea of derivations and endomorphisms, we come up with many interesting and important type of derivations. Let's start with an automorphism σ defined on R and let Q be its Martindale ring of quotient, defining an additive map $d : R \to R$ which satisfies

$$d(ab) = d(a)b + \sigma(a)d(b) \text{ for all } a, b \in R$$

is what we know as a *Skew derivation* (or a *σ-derivation*). We can think of any derivation as a skew derivation considering σ as the identity map. Yet another simple example will be the map $\sigma - 1_R$. If a map $D : R \to R$ is defined by

$$D(x) = bx - \sigma(x)b \text{ for all } x \in R$$

for a fixed $b \in Q$, then clearly D forms a σ derivation and such a type of derivation is termed as an inner skew derivation, and if no such b exists then D is outer. The existence of skew derivations helps us to relate rings with the theory of GPIs, like Chuang and Lee [60] studied polynomial identities with skew derivations. They proved that if $\phi(x_i, D(x_i))$ is a generalized polynomial identity for a prime ring R and D is any outer skew derivation of R, then such a ring R also satisfies the generalized polynomial identity $\phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Also, they proved in case $\phi(x_i, D(x_i), \sigma(x_i))$ is a generalized polynomial identity for a prime ring R, D an outer skew derivation of R, and σ an outer automorphism of R, then R also satisfies the generalized polynomial identity $\phi(x_i, y_i, z_i)$, where x_i, y_i and z_i are distinct indeterminates. To this Vincenzo and Ajda [153] proved an interesting result to further link the two.

Theorem 2.3. *Suppose m, n, r are non-zero fixed positive integers, R is a 2-torsion free prime ring, L is a non-central Lie ideal of R, $D : R \to R$ a skew derivation of R, and*

$$E(x) = D(x^{m+n+r}) - D(x^m)x^{n+r} - x^m D(x^n)x^r - x^{m+n}D(x^r), \ x \in R.$$

If $E(x) = 0$, $\forall\, x \in L$, then D is a usual derivation of R or R satisfies $s_4(x_1, ..., x_4)$, the standard identity of degree 4.

Till now we used a single endomorphism, what if we use two endomorphisms θ and ϕ defined on R and then generalize the definition of derivation, an additive map $D{:}R \longrightarrow R$ which satisfies

$$D(xy) = D(x)\theta(y) + \phi(x)D(y) \text{ for all } x, y \in R$$

is known as a (θ, ϕ) *derivation*. Similarly, we can define a *Jordan* (θ, ϕ) *derivation* as an additive map $D : R \to R$, satisfying

$$D(x^2) = D(x)\theta(x) + \phi(x)D(x) \text{ for all } x \in R.$$

Clearly every derivation is a (θ, ϕ) derivation by taking θ and ϕ both as identity maps. Kaya [101] and Guven [79] gave a couple of results which linked (θ, ϕ) derivations with the structure of rings, and being a very general form of derivations these type of derivations turn out to be equally important. Yet another significant class of derivations are left derivations and left Jordan derivations. An additive mapping $D : R \to X$, where R is a ring and X is a left R-module is called a *left derivation* if

$$D(ab) = aD(b) + bD(a),\ a, b \in R.$$

Similarly, an additive mapping $D : R \to X$ will be called a *Jordan left derivation* if

$$D(a^2) = 2aD(a),\ a \in R.$$

The concept of *Jordan left derivations* turns out to be closely related to commuting mappings, which leads us to a number of intriguing conclusions., see [69] and [17]. We know that the Lie product of any two elements of a ring R is defined as $[a, b] = ab - ba$. Müller in [141] introduced the idea of *pre-morphisms* and in particular of *pre-derivations*, that clubbed the concepts of derivations with Lie products. He showed that, if G is a Lie group endowed with a bi-invariant semi-Riemannian metric and g its Lie algebra, then the Lie algebra of the group of isometries of G fixing the identity element is a subalgebra of $pDer(g)$ (where $pDer(g)$ is the set of all pre-derivations on g). Thus, the study of the algebra of *pre-derivations* is not only interesting from the algebraic point of view but also from its applications. Let g be a Lie algebra we define a linear map $p : g \to g$ satisfying

$$p[x, [y, z]] = [px, [y, z]] + [x, [py, z]] + [x, [y, pz]],$$

for every $x, y, z \in R$ to be a *pre-derivation*. Clearly, $Der(g)$ is contained in $pDer(g)$. From derivations to prederivations, what if we generalize this concept. In order to do this we define *Leibniz Algebras* first and then move on to what is called *Leibniz derivations*. Let V be a vector space and $k \in \mathbb{N}$. A $k-$ *Leibniz algebra structure* on V is a $(k + 1)$-linear map

$$V \times \cdots \times V \to V : (x_1, \ldots, x_{k+1}) \mapsto [x_1, \ldots, x_{k+1}]$$

satisfying the identity,

$$[[x_1, \ldots, x_{k+1}], y_1, \ldots, y_k] = \sum_{i=1}^{i=n} [x_1, \ldots, x_{i-1}, [x_i, y_1, \ldots, y_k], x_{i+1}, \ldots, x_{k+1}].$$

We define the multilinear bracket $[x_1, \ldots, x_{k+1}]_{k+1}$, or when the length of the bracket is clear from the context, $[x_1, \ldots, x_{k+1}]$, as the nested expression

$$[x_1, [x_2, [x_3, \ldots, [x_{k-1}, [x_k, x_{k+1}]] \cdots]]].$$

Lie algebras and Leibniz algebras, are examples of *1-Leibniz algebras*. The above defined $(k + 1)$ linear map defines a *k-Leibniz algebra structure* on g, denoted by $\mathcal{L}_k(g)$. It's subalgebras and ideals are defined in the natural way. A subspace $\mathcal{T}$ of a *Leibniz k-algebra* $\mathcal{L}$ is a subalgebra of $\mathcal{L}$ if it satisfies $[\mathcal{T}, \ldots, \mathcal{T}]_{k+1} \subseteq \mathcal{T}$. Similarly, a subspace $\mathcal{I}$ is an ideal, if it satisfies $[\mathcal{I}, \mathcal{L}, \ldots, \mathcal{L}]_{k+1} + \cdots + [\mathcal{L}, \ldots, \mathcal{L}, \mathcal{I}]_{k+1} \subseteq \mathcal{I}$.

Now let's define *Leibniz-k derivation* for any $k \in \mathbb{N}$. A Leibniz-derivation of order k for a Lie algebra g is an endomorphism $P : g \to g$ satisfying the identity

$$P[x_1, \ldots, x_{k+1}] = [P(x_1), x_2, \ldots, x_{k+1}] + [x_1, P(x_2), \ldots, x_{k+1}] + \cdots + [x_1, x_2, \ldots, P(x_{k+1})],$$

for every $x_1, x_2, \ldots, x_{k+1} \in g$. Let $\mathcal{L}Der_k(g)$ be the set of all Leibniz-derivations for g of order k and let $\mathcal{L}Der(g)$ be the set of all Leibniz-derivations, so we have

$\mathcal{L}Der(g) = \cup_{k\in\mathbb{N}}\mathcal{L}Der_k(g)$. Derivations defined on g, pre-derivations defined on g will be simple examples of Leibniz derivation of order 1 and 2 respectively. Jacobson [96] proved that if a Lie algebra admits an invertible derivation, it must be nilpotent. He also suspected, that the converse might be true, that is every nilpotent Lie algebra has an invertible derivation but the latter is not true. In the year 2013, Wolfgang Alexander Moens[164] proved that a Lie algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. This was the motivation behind defining these generalizations of derivations.

We are much familiar with the idea of nilpotency be it nilpotent elements, nilpotent ideals or for that matter nilpotent rings, what if we talk about nilpotent derivations. On the similar lines as the other definitions a derivation d defined on a ring R is said to form a nilpotent derivation if

$$d^n R = 0 \text{ for some natural number } n.$$

For $n = 2$, R be a prime ring having characteristic not equal to 2, Posner [144] has shown d to be zero. For $n > 2$, the situation is not that straight forward. Herstein in [85] shows that for simple rings with some characteristic restrictions, a nilpotent derivation is identical to an inner derivation $x \to [a, x]$, induced by a nilpotent element a. Thus showing a strong relationship between these types of derivations. As a consequence, for a simple ring the index of a nilpotent derivation must be an odd number and precisely equal to $2m - 1$, where m is the nilpotent index of the element a above. Consider the example of the ring of 2×2 upper triangular matrices over real field, the derivation induced by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ forms a nilpotent derivation of index 2. Further A. Kovacs [111] asks whether the result of odd nilpotency index holds for all prime rings. But the result is not true in general, which is clear from the above example. This holds for prime rings as well. In the year 1983, L.O. Chung and Jiang Luh [64] were able to show that nilpotency of a derivation on a 2-torsion free semiprime ring is always an odd number, thus proving it for a much bigger class.

Equipping the idea of derivation with nilpotency gave us an interesting class of derivations, experimenting further we have various other derivations like star derivations, generalized derivations and many more. Lets see how star derivations look like. A map $* : R \to R$ on a ring R is said to be an involution if (i) $*$ is additive, (ii) $(a^*)^* = a$ and (iii) $(ab)^* = b^*a^* \ \forall\ a, b \in R$. Any ring equipped with an involution is said to be a *-ring or a ring with involution. Rings with involution have been of interest to several authors especially while generalizing various existing results on *-rings. Various subsets of *-rings have proved useful time and again, like the set of hermitian and skew hermitian elements. $H(R) = \{x \in R : \ x^* = x\}$ and $S(R) = \{x \in R : \ x^* = -x\}$ defined so are called the set of hermitian and skew hermitian elements respectively. Involutions are further classified into involution of first and second kind. This division is based on how an involution behaves, an involution is said to be of first kind if $H(R) \subseteq Z(R)$, otherwise it is said to be of second kind. Having said that a lot of work on *-rings has already been accomplished like proving *-versions of Vukman's[156] results, Posner's theorems and many more (see [4], [142]). In this direction, Ali et al.[5] proved some interesting results related to prime ideals of *-rings. Few of these results are mentioned below:

Theorem 2.4. *Let R be a ring with involution * of the second kind and P a prime ideal of R such that $S(R) \cap Z(R) \nsubseteq P$. If R admits a derivation d such that $d(xx^*) - d(x^*)d(x) \in P \ \forall \ x \in R$, then one of the following holds:*

(1) char(R/P)=2
(2) $d(R) \subseteq P$.

Theorem 2.5. *"Let R be a ring with involution * of the second kind and P a prime ideal of R such that $S(R) \cap Z(R) \nsubseteq P$. If R admits a derivation d such that $[[d(x), x^*], x^*] \in P \ \forall \ x \in R$, then one of the following holds:*

(1) char(R/P)=2
(2) $d(R) \subseteq P$
(3) R/P is a commutative integral domain.

In the same paper they posed certain problems which could be of interest. Some of these open problems are included below,

(1) Let R be a ring of suitable characteristic with involution * of the second kind and P a prime ideal of R such that $S(R) \cap Z(R) \nsubseteq P$. Next, let $f : R \to R$ be a mapping satisfying $[f(x), (x^*)^m]^n \in Z(R)$ or $P \ \forall \ x \in R$. Then, what we can say about the structure of R and f?
(2) Let R be a ring of suitable characteristic with involution * of the second kind and P a prime ideal of R such that $S(R) \cap Z(R) \nsubseteq P$. Next, let $d : R \to R$ be a derivation satisfying $([d(x^{k_1}), (x^*)^{k_2}]_n)^m \in Z(R)$ or $P \ \forall \ x \in R$. Then, what we can say about the structure of R and d?"

Let's now focus our attention to some specific structures and how derivations behave on them.

2.1. Derivations on prime rings.

Recall that a ring R is said to be prime if $aRb = 0$ implies either $a = 0$ or $b = 0$, there are many equivalent ways of defining prime rings, for example, right(left) annihilator of a nonzero right(left) ideal should be zero, whichever fits to the context and feels applicable can be used. The story of derivations with prime ring dates back to 1957 when Posner[144] proved two remarkable results, first one focusing on the product of derivations and the other one related to the existence of a centralizing derivation with the commutativity of the ring. The result states;

Theorem 2.6. *"Let R be a prime ring of characteristic not 2 and d_1, d_2 derivations of R such that the iterate $d_1 d_2$ is also a derivation, then at least one of d_1, d_2 is zero."*

Proof. Since d_1, d_2 are derivations, we have

$$\begin{aligned} d_1 d_2(ab) &= d_1\{d_2(a)b + a d_2(b)\} \\ &= d_1 d_2(a)b + d_2(a)d_1(b) + d_1(a)d_2(b) + a d_1 d_2(b), \end{aligned} \tag{6}$$

$\forall a, b \in R$. Also,

$$d_1 d_2(ab) = d_1 d_2(a)b + a d_1 d_2(b), \text{ for all } a, b \in R. \tag{7}$$

On compairing Equations (6) and (7), we obtain

$$d_2(a)d_1(b) + d_1(a)d_2(b) = 0, \text{ for all } a, b \in R. \tag{8}$$

Replace a by $ad_1(c)$ in Equation (8) and using properties of derivations, we have

$$d_2(a)d_1(c)d_1(b) + ad_2d_1(c)d_1(b) + d_1(a)d_1(c)d_2(b) + ad_1^2(c)d_2(b) = 0, \tag{9}$$

$\forall a, b, c \in R$.
Now, replace a by $d_1(c)$ in Equation (8), we get

$$d_2(d_1(c))d_1(b) + d_1(d_1(c))d_2(b) = 0,$$

implies

$$ad_2(d_1(c))d_1(b) + ad_1(d_1(c))d_2(b) = 0, \ \forall \ a, b, c \in R. \tag{10}$$

On compairing Equations (9) and (10)

$$d_2(a)d_1(c)d_1(b) + d_1(a)d_1(c)d_2(b) = 0, \text{ for all } a, b, c \in R. \tag{11}$$

From Equation (8), we can write

$$d_2(a)d_1(b) = -d_1(a)d_2(b), \text{ for all } a, b \in R.$$

Therefore,

$$d_2(c)d_1(b) = -d_1(c)d_2(b).$$

Using the above equation in equation (11), we have

$$(d_2(a)d_1(c) - d_1(a)d_2(c))d_1(b) = 0.$$

We know that for a derivation d on R, if $ad(x) = 0$, $\forall \ x \in R$ then either $a = 0$ or $d = 0$. Thinking of $d_2(a)d_1(c) - d_1(a)d_2(c)$ as a, we can deduce

$$\text{either } d_2(a)d_1(c) - d_1(a)d_2(c) = 0 \text{ or } d_1(b) = 0, \text{ for all } a, b, c \in R.$$

If $d_1(b) \neq 0$, implies

$$d_2(a)d_1(c) - d_1(a)d_2(c) = 0.$$

Also, from Equation (8), we have

$$d_2(a)d_1(c) + d_1(a)d_2(c) = 0.$$

Adding the above Equations, we have

$$2d_2(a)d_1(c) = 0, \text{ for all } a, c \in R.$$

Since R is a prime ring with characteristic not 2, it implies R is a 2-torsion free ring. Therefore,

$$d_2(a)d_1(c) = 0, \text{ for all } a, c \in R.$$

Think of $d_2(a)$ as a and using the fact that $ad(x) = 0,$ for all $x \in R$ implies either $a = 0$ or $d = 0$, so we conclude

$$\text{either } d_2(a) = 0 \text{ or } d_1(c) = 0, \text{ for all } a, c \in R,$$

hence the desired result. □

The condition of characteristic not 2 in the above result is indispensable, to see this consider a ring S of 2×2 matrices with entries from the Galois field $\{0, 1, w, w^2\}$, with inner derivations λ and δ defined by

$\lambda(X) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X \right]$ and $\delta(X) = \left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, X \right]$. The characteristic of S is 2, and we have $\lambda \neq 0$, $\delta \neq 0$ but $\lambda\delta = 0$, making the iterate a derivation but none of the derivations are zero. Therefore, we cannot dispose off characteristic not 2 criteria.

With Posner's first theorem, the debate for the composition of derivations to be a derivation just began. Over the period of time there have been a lot many generalizations to it, and restrictions on characteristic of the ring to make the iterate a derivation. Also, various authors focused on whether product of inner derivations is inner or for that matter the product of few inner and few outer derivations. Some important results pertaining to the above discussion are as following:

Theorem 2.7. [112] *"Let Z be a commutative domain, $charZ > n > 1$. If $d_1, d_2, \ldots, d_n$ are such derivations of Z that the composition $d_1 d_2 \cdots d_n$ is a derivation, then $d_i = 0$ for some $1 \leq i \leq n$."*

Theorem 2.8. [112] *"Let R be a prime ring with non-trivial center Z, $charR > n > 1$, and let $d_1, \ldots, d_n$ be such derivations of R that $d_1 \cdots d_n$ is a derivation. Then $d_1 \cdots d_n$, is $Z-linear$. Moreover, if $d_v \neq 0$ for any proper subset V of $U = \{1, \ldots, n\}$, then d_i is $Z-linear$ $\forall\ i \in U$."*

The proof of Posner's first theorem is easily extendable to ideals of R, Lanski attempted to generalize Posner's first theorem to Lie ideal L of R, assuming that the product of derivations d and h is a Lie derivation on L while many authors have attempted to prove that the product dh either takes the Lie ideal L to 0 or takes it in the center of the ring R. In this pursuit Lanski in the year 1988[113], attempted to obtain Posner's results using the concept of differential identities which was developed by V.Kharchenko in [106].

Theorem 2.9. [113] *"If L is a noncommutative Lie ideal of R and d and h are nonzero derivations of R so that dh is a Lie derivation of L into R, then $charR = 2$ and either R satisfies S_4 or $h = dc$ for c in the extended centroid of R."*

In addition to this Lanski was able to prove the corresponding results to Posner's theorems for (skew) symmetric elements in rings with involution.

Theorem 2.10. [113] *"If R, a prime ring has an involution $*$, $J = J^*$ is a nonzero ideal of R, and d and h are nonzero derivations of R so that dh is a Lie derivation from the skew-symmetric elements of J to R, then R satisfies S_4, or d and h are inner and R must satisfy a nonzero generalized polynomial identity, unless $charR = 2$ and $h = dc$ for c in the extended centroid of R."*

The following results proved helpful in understanding whether product of few inner and few outer derivations is a derivation;

Theorem 2.11. [58] *"Let R be a prime ring of characteristic $p > 0$. Suppose that $\delta_1, \delta_2, \dots, \delta_p$ are derivations of R such that $\phi(x) = x^{\delta_1 \delta_2 \cdots \delta_p}$ defines a derivation on R. Then (1) if one of $\delta_1, \delta_2, \dots, \delta_p$ is inner, then $\phi(x)$ is also inner, (2) if all $\delta_1, \delta_2, \dots, \delta_p$ are outer, then there exists $0 \neq \alpha \in C$ such that $\phi(x) = \alpha(x)^{\delta_1^p}$.*

Theorem 2.12. [114] *Let R be prime ring of characteristic different from 2 and $d_1, d_2, D \in Der(R)/\{0\}$ so that $d_1 d_2 D = E \in Der(R)$. Then either $d_1, d_2, D \in Inn(R)$, or else $char(R) = 3$, d_1 is outer, $d_2 = d_1 z_1, D = d_1 z_2$ and $(z_1)^{d_1} = 0$, where $z_i \in C$, so $E = {d_1}^3 z_1 z_2$.*

Theorem 2.13. [97] *Let λ and δ be derivations of a prime ring R, and let $\lambda \delta^m = 0$ where m is a positive integer. Then either $\lambda = 0$ or $\delta^r = 0$ where $r < 4m - 1$.*

Theorem 2.14. [59] *Let R be a prime GPI-ring with extended centroid C. Suppose that $dim_C RC \geq 3$ and that C is a finite field with q elements. Then there exist $2q+3$ nonzero derivations d_i of R, $i = 1, 2, ..., 2q+3$, such that $d_1(x) d_2(x) \cdots d_{2q+3}(x) = 0$, $\forall\ x \in R$.*

Corollary 2.15. [59] *Let R be a prime ring with infinite extended centroid C and L a non-commutative Lie ideal of R. Suppose that $d_1, ..., d_n$ are derivations of R satisfying $d_1(x) d_2(x) ... d_n(x) = 0\ \forall\ x \in L$. Then $d_i = 0$ for some i."*

No doubt Posner's first theorem helped to understand derivations and their compositions better but the arrival of Posner's second theorem revolutionized the study of derivations. It started the discussion of a much larger an important idea of centralizing and commuting mappings [2] which in turn laid the foundation of functional identities and as of now functional identities have proved to be of great importance for solving some hundred year old problems. Lets first define certain notions to be used to understand Posner's result. A derivation $d : R \to R$ is *centralizing* if $xd(x) - d(x)x$ is in the centre, Z, of R $\forall\ x \in R$, and *commuting* if $d(x)x = xd(x)\ \forall\ x \in R$. For every $x \in S$, a derivation is said to be centralizing on a subset S if $[x, d(x)] \in Z$.

Theorem 2.16. *"Let R be a prime ring and d is a derivation of R such that, $\forall\ a \in R$, $ad(a) - d(a)a$ is in the center of R. Then, if d is not the zero derivation, R is commutative.*

Matheiu in [134] proved Posner's theorem in a more general setting under the assumption that the derivation is centralizing on a non zero ideal only instead of the whole ring. The proof goes like;

Proof. Let I be a nonzero ideal of the prime ring R with $char \neq 2$, and let $d{:}R \longrightarrow R$ be a derivation such that $[a, d(a)] \in Z(R)\ \forall\ a \in I$. Replacing a by $a + b$, $b \in I$, we obtain

$$[a, d(b)] + [b, d(a)] \in Z(R), \tag{12}$$

in particular, taking $b = a^2$,

$$[a, d(a^2)] + [a^2, d(a)] \in Z(R), \tag{13}$$

as d is a derivation, we have

$$[a, d(a^2)] - [a^2, d(a)] = 0. \tag{14}$$

Combining Equations (13) and (14) gives,

$$[a^2, d(a)] \in Z(R).$$

Since $[a, d(a)]$ is also central, we have

$$2[a, d(a)]^2 = [2[a, d(a)]a, d(a)] = [[a^2, d(a)], d(a)] = 0.\text{"}$$

As R is prime, we conclude $[a, d(a)] = 0$. This shows that in Equation(12) we in fact have,

$$[a, d(b)] + [b, d(a)] = 0 \text{ for all } a, b \in I. \tag{15}$$

Now, let d_c denote the inner derivation $d_c(x) = [x, c]$, $x \in R$. Then Equation(15) implies, $d_b \circ d = d_{db}$ on I $\forall\ b \in I$. So the product of these two derivations d_b and d is a derivation, using Posner's first theorem, we conclude that either $d = 0$ or $d_b = 0\ \forall\ b \in I$, i.e., $xb - bx = 0\ \forall\ x \in R$, implying $I \subset Z(R)$. Since R is prime, the latter case entails that R is commutative. □

This opened up a whole new discussion of whether some other derivations or some other identities involving derivations will help us reveal the properties of rings. Results analogous to that of Posner's second theorem were established, the result motivated mathematicians to look for such centralizing automorphisms as well and one such result is due to Luh.

Theorem 2.17. [122] *"Let R be a prime ring. If R possesses a nontrivial commuting automorphism, then R is a commutative integral domain."*

In 1976, Mayne strengthened this by proving that a prime ring with a nontrivial centralizing automorphism is an integral domain. The reslut stated;

Theorem 2.18. [136] *"If R is a prime ring with a nontrivial centralizing automorphism, then R is a commutative integral domain."*

The study of derivations on prime rings did not stop here but many authors time and again related this concept with other existing and interesting ideas. In the year 1981 Herstein, Bergen and Kerr[40] tried to relate prime rings with derivations to Lie ideals and came up with certain cool results. From taking a nonzero Lie ideal U with $d^2(U) = 0$ in the center of its parent ring R, for any derivation d on R to proving a close relationship between a non central Lie ideal with the center of the ring. In totality they bridged the concept of derivations and that of Lie ideals for prime rings.

Theorem 2.19. [40] *"Let R be a prime ring, char $R \neq 2$, and let $U \not\subset Z$ be a Lie ideal of R. Suppose that δ and d are derivations of R such that $\delta d(U) = 0$. Then either $d = 0$ or $\delta = 0$."*

In the years 1982-84, Mayne (see [137], [138]) also proved that an automorphism or a derivation need only be centralizing and invariant on a nonzero ideal in the prime ring in order to ensure that the ring is commutative. Also, if R is of characteristic not two, then the mapping need only be centralizing and invariant on a nonzero Jordan ideal. Some of his interesting work is stated below;

Theorem 2.20. [137] *"Let R be a prime ring and U be a nonzero ideal of R. If R has a nontrivial automorphism or derivation T such that $uu^T - u^T u$ is in the center of R and $u^T \in U$ for every $u \in U$, then R is commutative."*

Theorem 2.21. [138] *"Let R be a prime ring and I be a nonzero ideal in R. If L is a nontrivial automorphism or derivation of R such that $xL(x) - L(x)x$ is in the center of R for every $x \in I$, then the ring R is commutative."*

It is easy to extend the above theorem to the case where the centralized ideal is quadratic Jordan, which generalizes Awtar's theorem for centralizing derivations.

Theorem 2.22. [138] *"Let R be a prime ring and U be a nonzero quadratic Jordan ideal of R. If L is a nontrivial automorphism or derivation of R which is centralizing on U, then R is commutative."*

Till now we were talking about two sided ideals, the trend did follow, in 1987 Bell and Martindale showed that instead of a two sided ideal the result holds for a one sided ideal as well;

Theorem 2.23. [34] *"Let R be a prime ring and U a nonzero left ideal. If R admits a nonzero derivation which is centralizing on U, then R is commutative."*

Further Herstein in the year 1978, related the commutativity of a derivation with that of the commutativity of the ring giving us a very special result;

Theorem 2.24. [86] *"Let R be a prime ring, $d \neq 0$ a derivation of R such that $d(x)d(y) = d(y)d(x)\ \forall\ x, y \in R$. Then, if char$R \neq 2$, R is a commutative integral domain, and if char$R = 2$, R is commutative or is an order in a simple algebra which is 4-dimensional over its center."*

So people thought of identities similar to this, in this direction Bell and Daif proved some interesting results for derivations satisfying $d(xy) = d(yx)$ but for some specific subsets of the ring. They showed that;

Theorem 2.25. [32] *"Let R be a prime ring and U a nonzero two-sided ideal of R. If R admits a nonzero derivation d such that $d(xy) = d(yx)\ \forall\ x, y \in U$, then R is commutative."*

In this case being a two-sided ideal is important, as can be seen in the example below. Consider a 2×2 ring R with entries from a field F, and take $U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}$. Let's define a derivation on R as $d(A) = A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}A$, for every $A \in R$. Clearly d satisfies $d(xy) = d(yx)$, $\forall\ x, y \in U$. But the ring is non commutative, this is because U is a one sided ideal. These ideas connecting special forms of derivations with commutativity was taken forward by many authors who showed some parallel results for some additive mappings. Some other fascinating results are:

Theorem 2.26. [156] *"Let R be a noncommutative prime ring of characteristic different from two and three. Suppose R contains the identity element 1. Let $D : R \rightarrow R$ be an additive mapping, such that $D(x^3) = 3xD(x)x$ holds $\forall\ x \in R$. In this case $D = 0$.*

Theorem 2.27. [45]*Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $F : R \rightarrow R$ satisfies the relation $[[f(x), x], x] = 0\ \forall\ x \in R$. In this case F is commuting on R.*

Theorem 2.28. [46]*Let R be a prime ring and U be a nonzero left ideal of R. Suppose derivations d and g of R satisfy $d(u)u - ug(u) \in Z(R)$, $\forall\ u \in U$. If $d \neq 0$ then R is commutative.*

Theorem 2.29. [117]*Let R be a prime ring with center Z and I a nonzero ideal of R. Suppose that d is a derivation on R such that $d^n(I) \subset Z$ for some natural number n. Then either $d^n = 0$ or R is commutative."*

It is evident from the description above that some additive mappings, such as endomorphisms, automorphisms, and derivations, cannot centralize on specific subsets of a noncommutative prime ring. Brešar was inspired by this to reveal the structure of every additive mapping that centralizes on a prime ring, which he did;

Theorem 2.30. [46] *"Let R be a prime ring. Suppose an additive mapping F of R into itself is centralizing on R. If either R has a characteristic different from 2 or F is commuting on R, then F is of the form $F(x) = \lambda x + \zeta(x)$, $x \in R$, where λ is an element from the extended centroid C of R and ζ is an additive mapping of R into C."*

In [116] Lanski proved certain important results regarding rings with involution and also his emphasis was mainly on lie ideals. He was able to establish the following results;

Theorem 2.31. *Let R be prime with $char(R) \neq 2$. Let d be a derivation of R and L a Lie ideal of R with $L^* = L$ and $d(L) \subset K$ or $d(L) \subset S$. Then either $L \subset Z$, $d(R) = 0$, or R satisfies S_4.*

Theorem 2.32. *Let R be prime with $char(R) = 2$. Let d be a derivation of R and L a Lie ideal of R such that $L^* = L$ and $d(L) \subset S$. Then either $L \subset Z$, $d(R) = 0$, or R satisfies S_4.*

Motivated by Posner's second theorem Khan and Khan[105] proved that a derivation d on a non commutative prime ring R with characteristic not 2 or 3 such that the map $x \longrightarrow [[[[d(x), x], x], x], x]$ becomes centralizing makes the derivation d as the zero map. In the same paper they provided the following conjectures:

Conjecture 2.33. *Let R be a non commutative semiprime ring with suitable characteristic of R and $d : R \rightarrow R$ be a derivation. Assume that $g_n(x) = 0$ for some integer $n, x \in R$, where $g_1(x) = [d(x), x]$ and $g_{n-1}(x) = [g_n(x), x]$, then $d = 0$.*

Conjecture 2.34. *Let B be Banach algebra. If $f, g : B \rightarrow B$ are continuous linear derivations and $f(x^m)x^n + x^n g(x^m) \in Q(B)$, $x \in B$ and some integers m, n, then*

f and g maps B into $rad(B)$, where $Q(B)$ stands for the set of all quasinilpotent elements in B and $rad(B)$ denotes for radical of a Banach algebra B.

Finally, we conclude this section with some open problems related to Herstein's result of theorem 2.24 proposed by Khan, Ali and Ayedh[104]:

Problem 2.1. *"Suppose m and n are fixed positive integers. Next, let us consider R is a ring with suitable involution and P is a semi(prime) ideal of R. If R admits derivations d_1 and d_2 like that $[d_1(x)^m, d_2((x)^*)^n] \in P$, $\forall\ x \in R$, then what we can say about the structures of R, d_1 and d_2?"*

Problem 2.2. *"Let m and n be fixed positive integers. Next, let R be a ring with suitable involution and P a semi(prime) ideal of R. If R admits derivations d_1 and d_2 such that $[d_1(x)^m, d_2((x)^*)^n] - [(x)^m, ((x)^*)^n] \in P$, $\forall\ x \in R$, then what we can say about the structures of R, d_1 and d_2?"*

2.2. **Derivations on semiprime rings.**

In an analogous way like prime rings, various mathematicians were interested in knowing whether the existence of a special type of map guarantees for a ring to have a central ideal or to be commutative. Many tried to generalize the already known results as semiprime rings are just a bigger class of rings containing prime rings. Some achieved such generalizations but few of the results had to be modified to fit in the semiprime setting. As observed in the previous section maps like derivations, automorphisms, endomorphisms do play a vital role in knowing the structure of the ring, so a natural question does arise whether this trend continues for semiprime rings as well. Following results will help us to answer this:

Corollary 2.35. [34] *"Let R be semiprime and U a nonzero left ideal; and suppose that R admits an endomorphism T which is one-to-one on U, centralizing on U, and not the identity on U. If $U^T \subset U$, then R contains a nonzero central ideal."*

Theorem 2.36. [34] *"Let R be a semiprime ring and U a nonzero left ideal. If R admits a derivation D which is nonzero on U and centralizing on U, then R contains a nonzero central ideal."*

In the year 1992, Daif and Bell tried to prove that the existence of a specific identity like that of Hersteins' unfolds the structure of a semiprime ring, the result states;

Theorem 2.37. [30] *"If R is a semiprime ring admitting a derivation d such that there exists a nonzero ideal K of R such that either $xy + d(xy) = yx + d(yx)$, $\forall\ x, y \in K$, or $xy - d(xy) = yx - d(yx)$ $\forall\ x, y \in K$, then K is a central ideal.*

Theorem 2.38. [30] *Let R be a semiprime ring admitting a derivation d for which either $xy + d(xy) = yx + d(yx)$, $\forall\ x, y \in R$ or $xy - d(xy) = yx - d(yx)$, $\forall\ x, y \in R$. Then R is commutative."*

Consider $R = R_1 \oplus R_2$, where R_1 is an integral domain, R_2 is a prime ring which is not commutative, and d is the direct sum of derivations on the summands

R_1 and R_2. That means we cannot conclude from the above result that the ring is commutative.

Now let's talk about the rings admitting a special kind of commutativity-preserving maps. If R is a ring and $S \subset R$, a map f: $R \longrightarrow R$ is called *strong commutativity preserving (scp)* on S if $[x, y] = [f(x), f(y)]\ \forall\, x, y \in S$. Bell and Daif were able to establish a near-commutativity property for semiprime rings admitting a derivation which is scp on a nonzero right ideal.

Theorem 2.39. [31] *"Let R be a semiprime ring and U a nonzero right ideal. If R admits a derivation d which is scp on U, then $U \subset Z$."*

Corollary 2.40. [31] *"If R is a semiprime ring admitting a derivation d which is strong commutativity preserving on R, then R is commutative."*

Example 2.41. *"Suppose R is a 3-dimensional algebra over a field having characteristic 2, with basis $\{w_0, w_1, w_2\}$ and multiplication defined by*

$$w_i w_j = \begin{cases} w_0 & \text{if } (i,j) = (1,2); \\ 0 & \text{if } (i,j) \neq (1,2). \end{cases}$$

Suppose T is a linear transformation on R defined by $T(w_0) = 0, T(w_1) = w_1, T(w_2) = w_2$. We can easily see that T is a derivation which is scp on R. This shows that the hypothesis of semi-primeness cannot be omitted from the above corollary. The derivation T in this case is not an inner derivation. Indeed, it is easy to write that any ring R admitting an inner derivation that is a scp on R must be commutative.

Example 2.42. *Let $R = R_1 \oplus R_2$, where R_1 is a non-commutative prime ring with derivation d_1 and R_2 is a commutative domain. Define d:$R \longrightarrow R$ by $d((r_1, r_2)) = (d_1(r_1), 0)$. Then R is a semiprime ring, and d is a derivation which is scp on the ideal U consisting of elements of form $(0, r_2)$. Thus, under the hypothesis of the above theorem we cannot prove that R must be commutative."*

Inspired by the work on scp maps and that of Bell and Daif [31], Deng and Ashraf initiated the study of a more general concept than scp mappings by considering the situation when mappings F and G of a ring R satisfy $[F(a), G(b)] = [a, b]$, $\forall\, a, b$ in some subset of R. They were able to show;

Theorem 2.43. [70] *"Let R be a semiprime ring, and U a nonzero ideal of R. If R admits a mapping F and a derivation d such that $[F(x), d(y)] = [x, y]$, $\forall\, x, y \in U$, then R contains a nonzero central ideal."*

Similarly in 1997, Motoshi Hongan generalized the results of Bell and Daif by taking the identities in center instead of mapping it to the zero element, so what was proved is;

Theorem 2.44. [88] *"Let R be a 2-torsion free semiprime ring, and let I be a nonzero ideal of R. Then the following conditions are equivalent:*

(1) R admits a derivation d such that $d[x, y] - [x, y] \in Z\ \forall\, z, y \in I$.
(2) R admits a derivation d such that $d[x, y] + [x, y] \in Z\ \forall\, x, y \in I$.

(3) R admits a derivation d such that $d[x,y]+[x,y] \in Z$ *or* $d[x,y]-[x,y] \in Z$ $\forall\ x,y \in I$

(4) $I \subset Z$*."*

From the above result we cannot exclude the condition of being 2-torsion free, which is elaborated by the following example;

Example 2.45. *"Suppose* $R = \left(\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix} \right)$, $x = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$, *and d the inner derivation induced by x that is,* $d(z) = [x,z]\ \forall\ z \in R$. *Then R is a non-commutative prime ring with* $char R = 2$, *and* $d[z,y] \pm [z,y] \in Z\ \forall\ z \in R$.

Motivated by his work on prime rings, Vukman showed the following:

Theorem 2.46. [159] *For integers* m,n *with* $m \geq 0, n \geq 0$, *and* $m+n \neq 0$, *let R be an* $(m+n+2)!$*-torsion free semiprime ring with identity element. Suppose there exists an additive mapping* $d : R \to R$, *such that* $d(x^{m+n+1}) = (m+n+1)x^m d(x) x^n$ *is fulfilled* $\forall\ x \in R$. *In this case, d is a derivation, which maps R into its center. In case R is a noncommutative prime ring, we have* $d = 0$.

In the same paper Vukman proposed the following conjecture,

Conjecture 2.47. *Let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping* $d : R \to R$ *satisfying the relation*

$$(m+n)d(x^2) = 2nd(x)x + 2mxd(x),$$

$\forall\ x \in R$ *and some integers* $m \geq 0, n \geq 0$, $m+n \neq 0$. *In case* $m \neq n$, *the mapping d is a derivation which maps R into* $Z(R)$*."*

Extensions of Herstein's theorems:

Theorem 2.48. [67] *"Let R be a semiprime ring and d a derivation of R with* $d^3 \neq 0$. *If* $[d(x), d(y)] = 0\ \forall\ x,y \in R$, *then R contains a nonzero central ideal."*

Daif tried to extend the theorem of Herstein in the situation when the ring was semiprime and the condition $[d(x), d(y)] = 0$ was satisfied on just an ideal of the ring.

Theorem 2.49. [67] *"Let R be a two-torsion-free semiprime ring and U a nonzero two-sided ideal of R. If R admits a derivation d which is nonzero on U and* $[d(x), d(y)] = 0\ \forall\ x,y \in U$, *then R contains a nonzero central ideal."*

What if the two sided ideals are replaced by one sided ideals? Is the above result still valid?

To understand this let us suppose a ring R to be the set all 2×2 matrices over a field F; let $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$. Suppose d is the inner derivation given by $d(a) = a\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} a\ \forall\ a \in R$. For any two elements a and $b \in U$, we have that $[d(a), d(b)] = 0$, but the conclusion of the theorem is not true.

Recently Fosner, Baydar and Strasek proved:

Theorem 2.50. [76] *"Let n be a fixed positive integer, let R be a $(2n)!$-torsion free semiprime ring, let α be an automorphism or an antiautomorphism of R, and let $d_1, d_2 : R \to R$ be derivations. Suppose that*

$$(d_1^2(x) + d_2(x)) \circ \alpha(x)^n = 0$$

holds $\forall\ x \in R$. Then $d_1 = d_2 = 0$.

Theorem 2.51. [76] *"Let R be a 2-torsion-free semiprime ring, let α be an automorphism or an antiautomorphism of R, and let $d_1, d_2 : R \to R$ be derivations. Suppose that $F : R \to R$ is a mapping defined by*

$$F(x) = (d_1^2(x) + d_2(x)) \circ \alpha(x),\ \ x \in R.$$

If $F(x) \circ \beta(x) = 0$ holds $\forall\ x \in R$ and some automorphism or antiautomorphism β of R, then $d_1 = d_2 = 0$."

In the year 2020, Jabel Atteya[21] gave various identities by generalizing some of the earlier results on a nonzero ideal of the ring and guaranteed the existence of a central ideal. The result states:

Theorem 2.52. [21] *"Let R be a 2-torsion free semiprime ring with the centre $Z(R)$, U be a non-zero ideal and $d : R \to R$ be a derivation mapping. Suppose that R admits*
(1) a derivation d satisfying one of the following conditions:

(1) $[d(x), d(y)] - [x, y] \in Z(R),\ \forall\ x, y \in U,$
(2) $[d^2(x), d^2(y)] - [x, y] \in Z(R),\ \forall\ x, y \in U,$
(3) $[d(x)^2, d(y)^2] - [x, y] \in Z(R),\ \forall\ x, y \in U,$
(4) $[d(x^2), d(y^2)] - [x, y] \in Z(R),\ \forall\ x, y \in U,$
(5) $[d(x), d(y)] - [x^2, y^2] \in Z(R),\ \forall\ x, y \in U.$

(2) a non-zero derivation d satisfying one of the following conditions:

(1) $d([d(x), d(y)]) - [x, y] \in Z(R),\ \forall\ x, y \in U,$
(2) $d([d(x), d(y)]) + [x, y] \in Z(R),\ \forall\ x, y \in U.$

Then R contains a non-zero central ideal."

3. Left derivations

The concepts of a left derivation, (an additive mapping $d_1 : R \to X$ satisfying $d_1(ab) = ad_1(b) + bd_1(a),\ \forall\ a \in R$) and that of a Jordan left derivation (additive mapping $d_2 : R \to X$ such that $d_2(a^2) = 2ad_2(a),\ a \in R$) were introduced by Brešar and Vukman in [54]. In the past three decades, there has been a significant amount of work around Jordan left derivations and related mappings, as these are in a close connection with the so-called commuting mappings. The main motivation comes from Posner's second theorem. Way back in 1955, a fundamental result was obtained by Singer and Wermer[147] while studying the range of derivations on Banach algebras, which is famously called the Singer-Wermer theorem. It states that any continuous derivation on a commutative Banach algebra maps the algebra into the Jacobson radical and in the same paper they conjectured that the assumption of continuity was unnecessary. This conjecture was later on proved in 1988 by Thomas[151]. So what was obtained was that every derivation on a commutative semisimple Banach algebra is zero. Various authors time and again have been able to prove many non-commutative forms of the Singer-Wermer theorem (see, e.g.,[52], [108] [118]). So the question arised that under what conditions all derivations on a Banach algebra are zero attracted a lot of attention. In this direction the range of left derivations was reviewed. In the year 1998, Jung [99] proved that every spectrally bounded left derivation maps algebra into its Jacobson radical. Vukman [161] showed that every Jordan left derivation on a semisimple Banach algebra is identically zero. In the same paper he conjectured that every Jordan left derivation on a Banach algebra maps the algebra into its radical.

Thinking of Posner's results for left derivations, Brešar and Vukman came up with some interesting results in 1990 where they guaranteed the commutativensss of the ring once a Jordan left derivation was found. The result states;

Theorem 3.1. [54] *"Let R be a ring and X be a 2-torsion free and 3-torsion free left $R-module$. Suppose that $aRx = 0$ with $a \in R$, $x \in X$ implies that either $a = 0$ or $x = 0$. If there exists a nonzero Jordan left derivation $d : R \to X$ then R is commutative.*

Corollary 3.2. [54] *Let R be a prime ring of characteristic different from 2 and 3. If R admits a nonzero Jordan left derivation $d : R \longrightarrow R$ then R is commutative."*

In the same paper they hinted that the condition for X to be 3-torsion free can be omitted. This was generalized by Ashraf, Rehman and Ali in [19] for a 2-torsion free prime ring;

Theorem 3.3. *"Let R be a 2-torsion free prime ring. Suppose J is a nonzero Jordan ideal and a subring of R. If $d : R \to R$ is an additive mapping such that $d(u^2) = 2ud(u)\ \forall\ u \in J$, then either $J \subseteq Z(R)$ or $d(J) = (0)$.*

In the year 1992, Deng [69] proved the commutativity of a prime ring using a left module, precisely he proved;

Theorem 3.4. *Let R be a prime ring with characteristic $\neq 2$, and X a nonzero left R-module. Suppose that X is faithful and prime. If there exists a nonzero Jordan left derivation $d : R \longrightarrow X$, then R is commutative.*

In 1992, Vukman [157] proved the result for semiprime rings. He was able to accomplish the following;

Theorem 3.5. *Let R be a 2-torsion free semiprime ring and $d : R \longrightarrow R$ be a Jordan left derivation. If there exists a positive integer n such that $d(x)^n = 0\ \forall\ x \in R$, then $D = 0$."*

In 1998, Jung [99] investigated the above study for semiprime Banach algebra and was able to establish;

Theorem 3.6. *Every left derivation on a semiprime Banach algebra A is a derivation which maps A into the intersection of the center of A and the Jacobson radical of A.*

Corollary 3.7. *If d is a left derivation on a semi-simple Banach algebra, then $d = 0$.*

This is evident that every left derivation forms a Jordan left derivation but the converse need not be true in general. In this direction Ashraf and Nadeem [16] in 2000 obtained a general result which established that under appropriate restrictions on a Lie ideal U of a 2-torsion free prime ring, every Jordan left derivation on U turns out to be a left derivation on U (see [3, 10] for more recent results).

Corollary 3.8. [16] *"Let R be a 2-torsion free prime ring and $d : R \to R$ be a Jordan left derivation. Then d is a left derivation.*

Further, Vukman in 2008 proved the following results;

Theorem 3.9. [161] *Let R be a noncommutative 2-torsion free prime ring and let $d : R \longrightarrow R$ be a left Jordan derivation. In this case $d = 0$.*

Theorem 3.10. [161] *Let R be a 2-torsion free semiprime ring and let $d : R \to R$ be a left Jordan derivation. In this case d is a derivation which maps R into $Z(R)$."*

Recently in 2017, Hosseini [89] gave certain conditions under which the left derivations become zero. Few of these interesting results are:

Theorem 3.11. *If B be a semi-simple Banach algebra and let $d : B \to B$ is a Jordan left derivation, then d is identically zero.*

Theorem 3.12. *If B is a Banach algebra, $d : B \to B$ is a Jordan left derivation, and P is a primitive ideal of B. If $d(P) \subset P$, then $d(B) \subseteq P$.*

Theorem 3.13. *Let R be a 2-torsion free semiprime ring, and let $d : R \to R$ be a Jordan left derivation. Then, $d(x)[y, z] = 0 \ \forall \ x, y, z \in R$.*

Corollary 3.14. *"If R is a 2-torsion free semiprime ring such that $ann\{[b, c] \mid b, c \in R\} = \{0\}$, then every Jordan left derivation $d : R \to R$ is identically zero.*

Theorem 3.15. *If R is a non-commutative 2-torsion free simple ring, and $d : R \to R$ is a Jordan left derivation, then $d = 0$."*

Brešar and Vukman [54] proved that every Jordan left derivation on a non-commutative 2-torsion free and 3-torsion free prime ring is identically zero. But the condition of 3-torsion free can be omitted, this was achieved by Hosseini;

Theorem 3.16. [89] *"Let R be a non-commutative 2-torsion free prime ring, and let $d : R \to R$ be a Jordan left derivation. In this case, d is zero.*

Corollary 3.17. [89] *Let R be a (commutative or non-commutative) 2-torsion free prime ring, I be a non-zero Jordan ideal of R, and let $d : R \to R$ be a Jordan left derivation such that $ad(a)a = 0 \ \forall \ a \in I$. Then, d is identically zero.*

Theorem 3.18. [90] *Let I be a semiprime ideal of an algebra $\mathcal{A}$, and let $d : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation such that $d(I) \subset I$. If $dim\{d(a) + I \mid a \in A\} \leq 1$, then $d(\mathcal{A}) \subset I$.*

Let's end this section with a conjecture given by Vukman;

Conjecture 3.19. [161] *Let A be a Banach algebra and let $d : A \to A$ be a linear left Jordan derivation. In this case d maps A into its radical."*

4. (α, β)-derivations

The union of the idea of derivations with that of endomorphisms on R lets us dive in a new and interesting type of derivations. "An additive map $d : R \to R$ defined by the identity

$$d(ab) = d(a)b + \sigma(a)d(b) \ \forall \ a, b \in R,$$

where σ is an automorphism on R is what we know as a *Skew derivation* (or a *σ-derivation*). Clearly, every derivation is a skew derivation. Introducing another endomorphism on R and defining an additive map $d : R \longrightarrow R$ by

$$d(xy) = d(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in R$$

will give us what we call a (α, β)-derivation. Similarly, we define a *Jordan (α, β)-derivation* as an additive map $d : R \to R$, satisfying

$$d(x^2) = d(x)\alpha(x) + \beta(x)d(x) \text{ for all } x \in R.$$

In [86], Herstein showed that an element a of R satisfying $ad(x) = d(x)a \ \forall \ x \in R$, for any derivation d should be central. Later on in [56], Chang extended this result by assuming that $[a, \delta(x)] = 0 \ \forall \ x \in R$, where δ is an (α, β)-derivation of R such that $\delta\alpha = \alpha\delta$, $\delta\beta = \beta\delta$." Argac, Kaya and Kisir tried to generalize these results by proving;

Theorem 4.1. [8] *"Let U be a nonzero ideal of R and $d : R \to R$ a nonzero (α, β)-derivation such that $d\alpha = \alpha d$, $d\beta = \beta d$. If $a \in R$ and $[a, d(U)]_{\lambda,\mu} = 0$ then $a \in C_{\lambda,\mu}$, where $C_{\alpha,\beta} = \{c \in R | c\alpha(r) = \beta(r)c \ for \ all \ r \in R\}$.*

Theorem 4.2. [8] *Let U be a nonzero ideal of R and $d_1 : R \to R$, a nonzero (σ, τ) derivation and $d_2 : R \to R$, a (α, β) derivation such that $d_2\alpha = \alpha d_2$, $d_2\beta = \beta d_2$. If $d_1 d_2(U) = 0$ then $d_1 = 0$ or $d_2 = 0$.*

Corollary 4.3. [8] *Let U be a nonzero ideal of R and d: $R \longrightarrow R$, a nonzero (σ, τ)-derivation such that $d\sigma = \sigma d$, $d\tau = \tau d$. If $[d(R), U]_{\alpha,\beta} = 0$ then R is commutative.*

Generalizations to centralizing theorems:

Theorem 4.4. [23] *Let d be a nonzero (α, β)-derivation, U an ideal of R and $a \in R$. If $[d(U), a]_{\alpha,\beta} = 0$ then $a \in Z(R)$.*

Theorem 4.5. [23] *Let d be a nonzero (α, β)-derivation and U an ideal of R. $[d(U), d(U)]_{\alpha,\beta} = 0$ then R is commutative.*

Theorem 4.6. [23] *Let d be a nonzero (α,β)-derivation. If $[d(R),d(R)]_{\alpha,\beta} \subseteq C_{\alpha,\beta}$ then R is commutative."*

In 1997, Chang gave a description of automorphisms δ, g, h and (α,β)-derivations of a prime ring R satisfying $\delta(x) = ag(x) + h(x)b \; \forall \; x \in U$, where a and b are some fixed noncentral elements of R and U a nonzero ideal of R. He accomplished a rather fascinating result relating (α,β)-derivations with that of elements from symmetric Martindale quotient ring.

Theorem 4.7. [57] *"Let R be a prime ring, U a nonzero ideal of R, Q the symmetric Martindale quotient ring of R and C the extended centroid of R. Further, let δ, g and h be (α,β)-derivations of U into R and $a, b \in Q \setminus C$. Suppose that either $g \neq 0$ or $h \neq 0$. Then the following conditions are equivalent:*

(1) $\delta(x) = ag(x) + h(x)b \; \forall \; x \in U$.
(2) There exists an invertible element $s \in Q$ such that

$$\begin{aligned}
\beta(x) &= s^{-1}\alpha(x)s, \\
\delta(x) &= [asbs^{-1}, \alpha(x)]s, \\
g(x) &= s[b, \beta(x)] = [sbs^{-1}, \alpha(x)]s, \\
h(x) &= [a, \alpha(x)]s
\end{aligned}$$

$\forall \; x \in U$."

Also they showed under what conditions two (α,β)-derivations are same,

Corollary 4.8. [57] *"Let R be a commutative domain and let $0 \neq \delta_1$ be an (α_1,β_1)-derivation, $0 \neq \delta_2$ be an (α_2,β_2)-derivation of R. If $\delta_1 = \delta_2$, then we have either $\alpha_1 = \alpha_2\beta_1$ or $\alpha_1 = \beta_2$ or $\alpha_2 = \beta_1$.*

Theorem 4.9. [57] *Let R be a prime ring, U a nonzero ideal of R, and δ a nonzero (α,β)-derivation of R. If $[\delta(U),\delta(U)]_{\alpha,\beta} = 0$, then*

(1) if $charR \neq 2$, then R is commutative. In this case, if $\alpha \neq \beta$, then $\alpha^2 = \beta^2$, $\alpha\beta = \beta\alpha$ and $\delta(x) = \lambda(\alpha(x) - \beta(x))$, for some $\lambda \in C$ such that $\alpha(\lambda) + \beta(\lambda) = 0$. Also $\delta^2 = 0$.
(2) if $charR = 2$, then R is an S_4-ring.

Generalizing Posner's theorem to (α,β)-derivations:

Theorem 4.10. [15] *Let R be a 2-torsion free prime ring. Suppose there exists a (α,β)-derivation $d : R \to R$ such that $[d(x),x]_{\alpha,\beta} = 0, \; \forall \; x \in R$. Then either $d = 0$ or R is commutative.*

Results from Herstein [86], Bell and Daif [32] paved way for proving similar looking identities for (α,β)-derivations. Motivated by these results Ashraf and Nadeem [15] proved;

Theorem 4.11. [15] *Let R be a 2-torsion free prime ring, and I a nonzero ideal of R. If R admits a (α,β)-derivation d such that $[d(x),d(y)] = 0, \; \forall \; x,y \in I$ and d commutes with both α,β, then either $d = 0$ or R is commutative.*

Theorem 4.12. [15] *Let R be a 2-torsion free prime ring, and I a nonzero ideal of R. If R admits a nonzero (α,β)-derivation d such that $d(xy)=d(yx)$, $\forall\ x,y\in I$ and d commutes with β, then R is commutative.*

Theorem 4.13. [15] *Let R be a 2-torsion free prime ring and α,β be automorphisms of R. Suppose that d_1 and d_2 are two (α,β)-derivations of R such that $d_1\alpha=\alpha d_1$, $d_1\beta=\beta d_1$, $d_2\alpha=\alpha d_2$ and $d_2\beta=\beta d_2$. If $d_1d_2(R)=0$, then either $d_1=0$ or $d_2=0$."*

Motivated by the above results Kaya, Guven and Soyturk were able to generalize these results as;

Theorem 4.14. [102] *"Let d_1 be a (σ,τ) derivation and d_2 an (α,β) -derivation on R such that $d_2\alpha=\alpha d_2$, $d_2\beta=\beta d_2$. If $d_1d_2(R)=0$ then $d_1=0$ or $d_2=0$.*

Theorem 4.15. [102] *Let $0\neq d_1\colon\ R\longrightarrow R$ be a (σ,τ)-derivation and $0\neq d_2\colon\ R\longrightarrow R$ an (α,β)-derivation such that $d_2\alpha=\alpha d_2$ and $d_2\beta=\beta d_2$. If $(d_1(R),d_2(R))=0$ then R is commutative."*

An additive mapping $\delta:R\to R$ is called a left (θ,ϕ)-derivation (resp., Jordan left (θ,ϕ)-derivation) on S if $\delta(xy)=\theta(x)\delta(y)+\phi(y)\delta(x)$ (resp., $\delta(x^2)=\theta(x)\delta(x)+\phi(x)\delta(x)$) holds $\forall\ x,y\in S$. In 2004, Zaidi, Ashraf and Ali [165] worked with left (θ,ϕ)-derivations and Jordan left (θ,ϕ)-derivations and were able to prove that for a Jordan ideal J which is also a subring of a 2-torsion-free prime ring R if $\delta(x^2)=2\theta(x)\delta(x)\ \forall\ x\in J$, for any automorphism θ of R then either $J\subseteq Z(R)$ or $\delta(J)=(0)$. Further, they also talked about (θ,θ)-derivations and deduced that left (θ,θ)-derivations of a prime ring R acts either as a homomorphism or as an antihomomorphism on the ring R. Some interesting results given in [165] are stated;

Theorem 4.16. *"Let R be a 2-torsion-free prime ring and let J be a Jordan ideal and a subring of R. If θ is an automorphism of R and $\delta:R\to R$ is an additive mapping satisfying $\delta(u^2)=2\theta(u)\delta(u)$, $\forall\ u\in J$, then either $J\subseteq Z(R)$ or $\delta(J)=(0)$."*

In the above result if we consider J to be a subring only then the result may not hold. To visualize this consider a ring S such that the square of each element of S is zero, but the product of some elements in S is nonzero. Also, suppose that $R=\left\{\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}\mid x,y\in S\right\}$, consider $J=\left\{\begin{pmatrix} 0 & y\\ 0 & 0\end{pmatrix}\mid y\in S\right\}$. Define maps δ: $R\longrightarrow R$ and θ: $R\longrightarrow R$ by
$\delta\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & y\\ 0 & 0\end{pmatrix}$ and $\theta\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix} x & -y\\ 0 & 0\end{pmatrix}$. It can be verified that δ forms a Jordan left (θ,θ)-derivation, but neither $J\subseteq Z(R)$ nor $\delta(J)=(0)$. So the subring J has to be an ideal as well.

In 1989, Bell and Kappe[33] were able to show that if d, a derivation of a prime ring R acts as a homomorphism or as an antihomomorphism on a nonzero right ideal I of R, then the derivation d must be zero. Later on in [18] this result was extended for (α,β)-derivations as;

Theorem 4.17. [18] *"Let R be a prime ring, I a nonzero right ideal of R, and let α, β be automorphisms of R. Suppose that $\delta : R \to R$ is a (α, β)-derivation of R.*

(1) If δ acts as a homomorphism on I, then $\delta = 0$ on R.
(2) If δ acts as an antihomomorphism on I, then $\delta = 0$ on R."

Zaidi, Ashraf and Ali [165] were able to extend it to left (θ, θ)-derivation of a prime ring R, with appropriate restrictions, the result states;

Theorem 4.18. [165] *"Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal and a subring of R. Suppose that θ is an automorphism of R and $\delta : R \to R$ is a left (θ, θ)-derivation of R. (i) If δ acts as a homomorphism on J, then $\delta = 0$ on R. (ii) If δ acts as an antihomomorphism on J, then $\delta = 0$ on R."*

Taking Lanski's[115] result forward,

Theorem 4.19. [160] *"Let R be a noncommutative prime ring of characteristic different from two and let D and $G \neq 0$ be (α, β)-derivations of R such that G commutes with α and β. If $[D(x), G(x)] = 0$ holds $\forall\ x \in R$ then $D = \lambda G$ where λ is an element from $C(R)$."*

In case we have α- derivations D and G on a prime ring R of characteristic different from two satisfying the relation $D(x)G(x) = 0\ \forall\ x \in R$ then either $D = 0$ or $G = 0$ (see Theorem 3 in [158]). Our next result is in the spirit of the result we have just mentioned.

Theorem 4.20. [160] *"Let R be a prime ring of characteristic different from two and let $D, G : R \to R$ be (α, β)-derivations such that $D(x)G(x) = 0$ holds $\forall\ x \in R$. Suppose that either D or G commutes with α and β. In both cases either $D = 0$ or $G = 0$."*

Various conjectures related to (α, β) derivations can be tried to absorb the concept properly. Some of these conjectures are:

Conjecture 4.21. [160] *"Let R be a prime ring of characteristic different from two and let d and g be (α, β)-derivations of R into itself with suitable additional assumptions. Suppose that the relation $d(x)g(x) + g(x)d(x) = 0$ holds $\forall\ x \in R$. In this case either $d = 0$ or $g = 0$."*

Conjecture 4.22. [163] *"Let R be a semiprime ring with a suitable torsion restriction and α, β be automorphisms of R. Suppose $d : R \to R$ is an additive mapping such that*

$$2d(x^n) = d(x^{n-1})\alpha(x) + \beta(x^{n-1})d(x) + d(x)\alpha(x^{n-1}) + \beta(x)d(x^{n-1}),$$

holds $\forall\ x \in R$ and some fixed integer $n \geq 2$. Then d is a (α, β)-derivation."

Conjecture 4.23. [163] *"Let R be a semiprime ring with a suitable torsion restriction and α, β be automorphisms of R. Suppose $d : R \to R$ is an additive mapping such that*

$$d(x^n) = \sum_{j=1}^{n} \alpha(x^{n-j})d(x)\beta(x^{j-1}),$$

holds $\forall\ x \in R$ and some fixed integer $n \geq 2$. Then d is a (α, β)-derivation."

5. Bi-derivations

After the concept of derivation was defined in [144] by Posner, a lot of researchers studied derivations in ring theory in different manners and having various forms. In 1980, Gy. Maksa introduced the concept of symmetric bi-derivation in [124] (see also [125]) where he also gave the representation theorem of bi-derivations in Hilbert spaces. A lot of work has been done in this direction. Let's first equip ourselves with the terminology associated with bi-derivations.

Definition 5.1. *Any function $D : R \times R \to R$ is known as bi-additive if it is additive in each argument, i.e., $D(u+w, v) = D(u, v) + D(w, v)$ and $D(u, v+w) = D(u, v) + D(u, w)$ $\forall\ u, v, w \in R$.*

Definition 5.2. *Any function $D : R \times R \to R$ is known as symmetric if $D(u, v) = D(v, u)$ $\forall\ u, v \in R$.*

For example, let R be a ring. The mapping $D : R \times R \to R$ defined by $D(u, v) = u + v$ $\forall\ u, v \in R$ is a symmetric mapping.

Definition 5.3. *A function $D : R \times R \to R$ is called symmetric bi-additive if it is both bi-additive and symmetric.*

Something that helped the transition of various results of ordinary derivations to that of symmetric bi-derivations is the idea of trace of a symmetric bi-derivation. The way it is defined is,

Definition 5.4. *A function "$d : R \to R$ such that $d(x) = D(x, x)$ is known as a trace of D where $D : R \times R \to R$ is a symmetric function.*

Clubbing all these concepts we say a symmetric bi-additve mapping $D : R \times R \to R$ is called a symmetric bi-derivation of R if it is a derivation in each argument; that is; for every $u \in R$, the maps $v \to D(u, v)$ and $v \to D(v, u)$ are derivations of R into R or if it satisfies the relation $D(uv, z) = D(u, z)y + uD(v, z)$ $\forall$ $u, v, z \in R$. It is evident that in this case the relation $D(u, vz) = D(u, v)z + vD(u, z)$ $\forall\ u, v, z \in R$ also holds. If $D : R \times R \to R$ is a symmetric mapping, which is also bi-additive then d, the trace of D satisfies the relation

$$d(u + v) = d(u) + d(v) + 2D(u, v).$$

Since we have

$$D(0, v) = D(0 + 0, v) = D(0, v) + D(0, v)$$

$\forall\ v \in R$, we obtain $D(0, v) = 0$ $\forall\ v \in R$. Hence, we get $D(-u, v) = -D(u, v)$, similarly $D(u, -v) = -D(u, v)$ $\forall\ u, v \in R$. This tells us that d is an even function.

Example 5.5. *For a commutative ring R, let $M := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R \right\}$. We can easily see that M is a non-commutative ring under the operation of matrix $+$ and matrix $\cdot$. We define a map $D : M \times M \to M$ by $\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1a_2 \\ 0 & 0 \end{pmatrix}$. It is easy to see that D forms a symmetric bi-derivation.*

Example 5.6. *Let R be a commutative ring. Then a map $D : R \times R \to R$ defined by*

$$(x_1, x_2) \mapsto d(x_1)d(x_2)$$

where d is a derivation on R. Then this D is a symmetric bi-derivation on R.

Example 5.7. *A typical example of bi-derivation are mappings of the form $(x, y) \mapsto c[x, y]$ where c is an element of the center of R.*

In [125], functional equations for the trace of symmetric bi-derivations were introduced. Also the non-negative solution for functional equations were characterized. When studying additive commuting maps, the concept of bi-derivation naturally develops. On linearizing $[f(x), x] = 0 \,\forall\, x \in R$, we get $[f(x), y] = [x, f(y)]$, $x, y \in R$ and hence the map $(x, y) \mapsto [f(x), y]$ is a bi-derivation where f is a commuting map. In [50] it was shown that every bi-derivation D on a non-commutative prime ring R is of the form $D(x, y) = \lambda[x, y]$ for some $\lambda \in C$. In 1995, the same author has given a more general version of this result (see, [49]). Indeed, he showed:

Theorem 5.8. *Let $D : I \times I \to Q_r$ be a bi-derivation. If R is non-commutative, then there exists $\lambda \in C$ such that $D(x, y) = \lambda[x, y] \,\forall\, x, y \in I$.*

As a consequence of the above result, a slight generalization of the main result of [46] has been obtained.

Corollary 5.9. *Let $f : I \to Q_r$ be additive modulo C. If f is commuting then there exists $\lambda \in C$ and a map $\zeta : I \to C$ such that $f(x) = \lambda x + \zeta(x) \,\forall\, x \in I$."*

Some results concerning symmetric bi-derivations in prime and semiprime rings can be found in [146], [154], [155]. In [154], the author has generalized the very famous and classical Posner's result for bi-derivations. The following result is in the spirit of Posner's second result [144] which states that the existence of a non-zero centralizing derivation on a prime ring forces the ring to be commutative. He showed that:

Theorem 5.10. *"Suppose R is a non-commutative prime ring having characteristic not $\neq 2$ and 3 and also $D : R \times R \to R$ and d are symmetric bi-derivation and trace of D, respectively. Suppose that d is centralizing on R. Then, $D = 0$."*

Posner in [144] showed that in case of a prime ring R having characteristic $\neq 2$ and D_1, D_2 non-zero derivations on R, the function $x \mapsto D_1(D_2(x))$ cannot form derivation. The result below was motivated by Posner's result mentioned above.

Theorem 5.11. *"Suppose R is a prime ring of characteristic not two and three and $D_1 : R \times R \to R$ and $D_2 : R \times R \to R$ are symmetric bi-derivations. Furthermore next, suppose that there exists a symmetric bi-additive mapping $B : R \times R \to R$, such that $d_1(d_2(x)) = f(x)$ holds $\forall\, x \in R$, where d_1 and d_2 are the traces of D_1 and D_2, respectively, and f is the trace of B. Then either $D_1 = 0$ or $D_2 = O$."*

The above result was also proved for semiprime rings for $D_1 = D_2$ which is as follows;

Corollary 5.12. *Suppose R is a semiprime ring which is 2-torsion and 3-torsion free and $D : R \times R \to R$ and $B : R \times R \to R$ are symmetric bi-derivation and a symmetric bi-additive mapping, respectively. Suppose that $d(d(x)) = f(x)$ holds for all $x \in R$, where d is the trace of D and f is the trace of B. Then, $D = 0$.*

One could have worked on the necessity of characteristic of R not equal to three in theorem 5.10. Infact, in 1996, Hongan [87] has proved the result without assuming that R is of characteristic different from three. He generalized the above mentioned result to non-zero two sided ideals of prime ring of characteristic not two.

Theorem 5.13. *"Suppose R is a non-commutative prime ring of char $R \neq 2$ and I a nonzero two-sided ideal of R. Next, let us consider that $D : R \times R \to R$ be a symmetric bi-derivation and d the trace of D. If d is centralizing on I, then $D = 0$.*

The following result is an addition to the theory of centralizing maps. In [155], Vukman has generalized theorem 5.10 as;

Theorem 5.14. *Suppose R is a non-commutative prime ring of characteristic different from two and three. Next, let us suppose there exists a symmetric bi-derivation $D : R \times R \to R$, such that the mapping $x \mapsto [f(x), x]$, for f stands for the trace of D, is centralizing on R. Then, $D = 0$."*

Based on the above theorem, Vukman proposed the conjecture for arbitrary n, where $n \in \mathbb{N}$, which he felt, would be very hard to prove. The conjecture states:

Conjecture 5.15. *"Let R be a non-commutative prime ring of characteristic different from two and three, and let $D : R \times R \to R$ and $f : R \to R$ be a symmetric bi-derivation and its trace, respectively. Suppose that for some $n \geq 1$, we have $f_n(x) \in Z(R)\ \forall\ x \in R$, where $f_{k+1}(x) = [f_k(x), x]$ for $k = 1, 2, 3 \ldots$ and $f_1(x)$ stands for $f(x)$. Then $D = 0$."*

After 7 years, in 1997, Deng [71] proved the above mentioned Vukman's conjecture for arbitrary n. The result states,

Theorem 5.16. *Suppose n is a fixed positive integer and R is a prime ring having char $R = 0$ or char $R > n+2$. Let $f_{k+1} = [f_k(x), x]$ for $k > 1$, and $f_1(x) = f(x)$, the trace of a symmetric bi-derivation D of R. If $f_n(x) \in Z(R)\ \forall\ x \in R$, then either $D = 0$ or R is a commutative ring.*

Proof. Linearizing $f_n(x) \in Z(R)$, we obtain

$$[[\ldots[f(x) + f(y) + 2D(x, y), x + y], \ldots, x + y], x + y] \in Z(R);$$

and using [71, Proposition], we get

$$[\ldots[[f(x), y], x], \ldots, x] + [\ldots[[f(x), x], y], \ldots, x] + \cdots + [\ldots[f(x), x], \ldots, y] \\ + 2[\ldots[[D(x, y), x], x], \ldots, x] \in Z(R),$$

equivalently,

$$\begin{aligned}(-1)^{n-2}I_x^{n-2}([f_1(x),y])+(-1)^{n-3}I_x^{n-3}([f_2(x),y])+\ldots+[f_{n-1}(x),y]+\\ 2(-1)^{n-1}I_x^{n-1}(D(x,y))\in Z(R)\end{aligned} \tag{16}$$

Noting that

$$\begin{aligned}(-1)^{n-2}I_x^{n-2}([f_1(x),x^2])&=(-1)^{n-3}I_x^{n-3}([f_2(x),x^2])=\ldots=\\ &=[f_{n-1}(x),x^2]=(-1)^{n-1}I_x^{n-1}(D(x,x^2))=2f_n(x)x,\end{aligned}$$

and replacing y by x^2 in (16), we get $2(n+1)f_n(x)x\in Z(R)$. Since $f_n(x)\in Z(R)$, it follows that $f_n(x)=0$. The linearization of $f_n(x)=0$ gives

$$\begin{aligned}(-1)^{n-2}I_x^{n-2}([f_1(x),y])+(-1)^{n-3}I_x^{n-3}([f_2(x),y])+\ldots+[f_{n-1}(x),y]+\\ 2(-1)^{n-1}I_x^{n-1}(D(x,y))=0.\end{aligned} \tag{17}$$

Since $I_x^{n-k}([f_{k-1}(x),xy])=xI_x^{n-k}([f_{k-1}(x),y])+I_x^{n-k}(f_k(x)y)$ for $k=2,3,\ldots,n$ and $I_x^{n-1}(D(x,xy))=xI_x^{n-1}(D(x,y))+I_x^{n-1}(f_1(x)y)$. Substituting xy for y in (17), we have

$$\begin{aligned}(-1)^{n-2}I_x^{n-2}(f_2(x)y)+(-1)^{n-3}I_x^{n-3}(f_3(x)y)+\ldots\\ +(-1)I_x(f_{n-1}(x)y)+2(-1)^{n-1}I_x^{n-1}(f_1(x)y)=0.\end{aligned}$$

Taking $y=f_{n-2}(x)$, applying $I_x^k(ab)=\sum_{j=0}^{k}\binom{k}{j}I_x^{k-j}(a)I_k^j(b)$ and noting $I_x^i(f_j(x))=0$ for $i+j>n$, we then conclude that

$$\begin{aligned}2(-1)^{n-1}\binom{n-1}{1}I_x^{n-2}(f_1(x))I_x(f_{n-2}(x))+(-1)^{n-2}\binom{n-2}{1}I_x^{n-3}(f_2(x))\\ I_x(f_{n-2}(x))+\ldots+(-1)(f_{n-1}(x))I_x(f_{n-2}(x))=0.\end{aligned}$$

But $(-1)^kI_x^{k-1}(f_{n-k}(x))I_x(f_{n-2}(x))=(f_{n-1}(x))^2$, so $(n+2)(n-1)(f_{n-1}(x))^2=0$, and by the hypotheses on the characteristic, we get $(f_{n-1}(x))^2=0$. Suppose that $D\neq 0$. By [[71], Theorem 1], $f_{n-1}(x)=0$, and by induction, $f_2(x)=[f(x),x]=0$. Using Vukman's result [[154], Theorem 1], R is commutative, thus the proof of theorem is complete. □

Let σ be an automorphism of R. Following Brešar [46], a bi-additive map $\Delta:R\times R\to R$ will be called a σ-bi-derivation of R if for every $x\in R$, the maps $y\to\Delta(x,y)$ and $y\to\Delta(y,x)$ are σ-derivations of R into R. In this paper, structure of an arbitrary σ-bi-derivation of a prime ring R has been characterized. Basically he proved:

Theorem 5.17. *Let R be a non-commutative prime ring, σ be an automorphism of R and $\Delta:R\times R\to R$ be a nonzero σ-bi-derivation of R. Then σ is X-inner and there exists an invertible element $b\in Q_s$, such that $\sigma(x)=bxb^{-1}$ and $\Delta(x,y)=b[x,y]$ $\forall\ x,y\in R$.*

Ashraf in 1999 [9], studied bi-derivations and characterized the commutativity of a ring satisfying certain identities involving symmetric bi-derivations. Also, he showed that,

Theorem 5.18. *"Let R be a 2-torsion free and 3-torsion free semiprime ring. Suppose that there exists a symmetric bi-derivation $D : R \times R \to R$ such that the mapping $x \mapsto [f(x), x]$ is centralizing on R, where f denotes the trace of D. Then f is commuting on R."*

In [109], commutativity of a prime ring involving symmetric bi-derivation has been discussed. In [1], the authors have considered a relationship between Lie ideal L of a ring and bi-additive symmetric map D fulfilling certain identities and proved that L must be central. Furthermore, it is shown that if R permits a symmetric bi-derivation D such that the trace d of D is n-centralizing on L, then d is n-commuting on L. This was driven by Posnerâs well known theorem and a work of Deng and Bell. Very recently, in 2021, Leerawat and Khun-in [119], looked into some results on the concept of symmetric bi-derivations on rings, which expand some of Vukman's ideas. They also look into various circumstances involving symmetric bi-derivations, derivations and endomorphisms in a prime ring that change it into a commutative ring.

6. n-Derivations

The notion of n-derivation was first introduced by Kyoo-Hong Park [143] in the year 2009. The idea resonates with that of bi-derivations.

Definition 6.1. *Let $n \geq 2$ be a fixed positive integer and $R^n = R \times R \times \cdots \times R$. A map $\Delta : R^n \to R$ is said to be symmetric (or permuting) if the equation $\Delta(x_1, x_2, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ holds $\forall\ x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$.*

Definition 6.2. *Let $n \geq 2$ be a fixed positive integer. An n-additive map $\Delta : R^n \to R$ (i.e., additive in each component) is called an $n-derivation$ if the relations*

$$\Delta(x_1x_1^{'}, x_2, \ldots, x_n) = \Delta(x_1, x_2, \ldots, x_n)x_1^{'} + x_1\Delta(x_1^{'}, x_2, \ldots, x_n)$$
$$\Delta(x_1, x_2x_2^{'}, \ldots, x_n) = \Delta(x_1, x_2, \ldots, x_n)x_2^{'} + x_2\Delta(x_1, x_2^{'}, \ldots, x_n)$$
$$\vdots$$
$$\Delta(x_1, x_2, \ldots, x_nx_n{'}) = \Delta(x_1, x_2, \ldots, x_n)x_n^{'} + x_n\Delta(x_1, x_2, \ldots, x_n^{'})$$

are valid, $\forall\ x_i, x_i^{'} \in R$.

n-derivation behaves like an ordinary derivation in each component. One can easily see that a 1-derivation is a derivation and a 2-derivation will give us a bi-derivation. To the above definition of n-derivation if we add the condition of Δ to be symmetric, it will fetch us derivations what we call as symmetric n-derivations, due to the symmetric nature of Δ the relations will of course be equivalent to one another and the validity of one will give us the rest of the identities.

Example 6.3. *Let* $R := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$. *Define a map* $\Delta : R^n \to R$ *by*

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1 a_2 \cdots a_n \\ 0 & 0 \end{pmatrix}.$$

Δ *so defined gives us a symmetric* n*-derivation.*

Definition 6.4. *A map* $\delta : R \to R$ *is called trace of a symmetric* n*-additive map* Δ *and is defined by* $\delta(x) = \Delta(x, x, \ldots, x)\ \forall\ x \in R$.

The trace of a symmetric n-derivation plays an important role as it helps to bridge the gap between an n-derivation and that of an ordinary derivation. It becomes useful while generalizing the results already proved for derivations or bi-derivations to that of n-derivations. In this direction Park in [143] proved various interesting results for symmetric n-derivations like generalization to the famous Posner's result and many more. Some of his initial work on n-derivations which paved way for future developments in this area are stated under:

Lemma 6.5. *"Let* n *be a fixed positive integer and let* R *be a* $n!$*-torsion free ring. Suppose that* $y_1, y_2, \ldots, y_n \in R$ *satisfy* $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ *for* $\lambda = 1, 2, \ldots, n$. *Then* $y_i = 0\ \forall\ i$.

Lemma 6.6. *Let* n *be a fixed positive integer and let* R *be a* $n!$*-torsion free ring. Suppose that* $y_1, y_2, \ldots, y_n \in R$ *satisfy* $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n \in Z$ *for* $\lambda = 1, 2, \ldots, n$. *Then* $y_i \in Z\ \forall\ i$.

Theorem 6.7. *Let* $n \geq 2$ *be a fixed positive integer and let* R *be a* $n!$*- torsion free prime ring. Suppose that there exists a nonzero symmetric* n*-derivation* $\Delta : R^n \to R$ *such that the trace* δ *of* Δ *is centralizing on* R. *Then* R *is commutative.*

In [12], Ashraf and Jamal gave certain interesting identities which would tell us about the structure of the ring, like Daif and Bell did for semiprime rings in [68]. Ashraf and Jamal were able to relate that to the trace δ of an n-derivation and proved that a ring R is commutative if there exists a permuting n-additive map $\Delta : R^n \to R$ such that $xy + \delta(xy) = yx + \delta(yx)$, $xy - \delta(xy) = yx - \delta(yx)$, $xy - yx = \delta(x) \pm \delta(y)$ and $xy + yx = \delta(x) \pm \delta(y)$ holds $\forall\ x, y \in R$. In addition to this they proved that if R is a prime ring with suitable torsion restriction then R is commutative whenever there exist non-zero permuting n-derivations Δ_1 and Δ_2 from $R^n \to R$ such that $\Delta_1(\delta_2(x), x, \ldots, x) = 0\ \forall\ x \in R$, where δ_2 is the trace of Δ_2. Finally, they showed that in a prime ring R of suitable torsion restriction, if $\Delta_1, \Delta_2 : R^n \to R$ are non-zero permuting n-derivations with traces δ_1, δ_2, respectively, and $B : R^n \to R$ is a permuting n-additive map with trace f such that $\delta_1 \delta_2(x) = f(x)$ holds $\forall\ x \in R$, then R is commutative. Park proved Posner's result for n-derivation in case of prime rings, this was taken forward by Ashraf, Parveen and Jamal in [14] with a specific condition.

Theorem 6.8. *For any fixed integer* $n \geq 2$, *let* R *be a* $n!$*-torsion free semiprime ring. Suppose that* R *admits a nonzero permuting* n*-derivation* $\Delta : R^n \to R$ *with trace* $\delta : R \to R$. *If* δ *satisfies* $[\delta(x), \delta(y)] = 0\ \forall\ x, y \in R$. *Then* δ *is commuting on* R.

The condition of semiprimeness is important, as is supported by the below example;

Example 6.9. *Let* $R := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$*, R so defined is a noncommutative ring under matrix addition and matrix multiplication. We define* $\Delta : R^n \to R$ *as in example* (6.3)*, then* Δ *is a symmetric n-derivation with trace*

$$\delta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^n \\ 0 & 0 \end{pmatrix}.$$

Though δ *satisfies* $[\delta(x), \delta(y)] = 0$, $\forall\ x, y \in R$ *but* δ *is not commuting, making semiprimeness condition indispensable in this case.*

In the same paper they were able to prove similar sort of results for any ring but satisfying different identities. One of such result is given below.

Theorem 6.10. *For any fixed integer* $n \geq 2$*, let R be a n!- torsion free ring. If R admits a nonzero permuting n-derivation* $\Delta : R^n \to R$ *with trace* $\delta : R \to R$ *satisfying any one of the following conditions:*

(1) $[\delta(x), \delta(y)] = [x, y]\ \forall\ x, y \in R$
(2) $[\delta(x), \delta(y)] = [y, x]\ \forall\ x, y \in R$.

Then R is commutative.

Ashraf, Jamal and Mozumder in [13] used properties of an ideal and the trace of a permuting n-derivation to achieve results for the existence of central ideals which we know eventually helps to make the ring commutative. Few of their results are:

Theorem 6.11. *For a fixed integer* $n \geq 2$*, let R be an n!-torsion free semiprime ring and I be a nonzero ideal of R. Let* $\Delta : R^n \to R$ *be a permuting n-derivation such that* $\Delta(I, \ldots, I) \neq \{0\}$ *and* $[\delta(x), x] = 0\ \forall\ x \in R$*, where* δ *is the trace of* Δ*. Then, R contains a nonzero central ideal.*

Theorem 6.12. *For a fixed integer* $n \geq 2$*, let R be a (2n-1)!-torsion free semiprime ring and I be a nonzero left ideal of R. Suppose that there exists a permuting n-derivation* $\Delta : R^n \to R$ *with trace* δ*. Then, R is commutative if* $\delta(x^2) = \pm x^2\ \forall\ x, y \in R$.

What if we introduce the idea of involution in case of n-derivations? Ofcourse, various authors have had interest in this amalgam, like Ashraf and Siddeeque in [20] tried to define what we call *-n- derivations.

Definition 6.13. *Let n be any fixed positive integer. An n-additive mapping* $D : R \times R \times \cdots \times R \to R$ *is called a *-n-derivation of R if the relations*

$$D(x_1x_1', x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)(x_1')^* + x_1D(x_1', x_2, \ldots, x_n)$$
$$D(x_1, x_2x_2', \ldots, x_n) = D(x_1, x_2, \ldots, x_n)(x_2')^* + x_2D(x_1, x_2', \ldots, x_n)$$
$$\vdots$$

$$D(x_1, x_2, \ldots, x_n x_n{}') = D(x_1, x_2, \ldots, x_n)(x_n^{'})^* + x_n D(x_1, x_2, \ldots, x_n^{'})$$

hold $\forall\ x_1, x_1^{'}, x_2, x_2^{'}, \ldots, x_n, x_n^{'} \in R$.

They were able to establish a couple of interesting generalizations.

Theorem 6.14. *Let R be a prime *-ring. If it admits a nonzero *-n-derivation (resp. reverse *-n-derivation) D, then R is commutative.*

The above theorem can't be applied to *-semiprime rings, an example to support this argument is stated as under;

Example 6.15. *Let $R = Q \times \mathbb{C}$ be the ring of cartesian product of Q, the ring of real quaternions and $\mathbb{C}$, the ring of complex numbers with respect to componentwise addition and multiplication. Let $*_1, *_2$ and $*$ denote the involutions of the rings Q, $\mathbb{C}$ and R, respectively and are defined by*

$$q^{*_1} = a - bi - cj - dk \text{ where } q = a + bi + cj + dk \in Q,$$

$$z^{*_2} = x - yi \text{ where } z = x + yi \in \mathbb{C},$$

$$(q, z)^* = (q^{*_1}, z^{*_2}) \text{ for all } q \in Q,\ z \in \mathbb{C}.$$

*Let d be a $*_2$-derivation of $\mathbb{C}$ defined by $d(z) = \alpha(z - z^{*_2})$ where α is any fixed complex number. Define $D : R\times R\times\cdots\times R \to R$ such that $D((q_1, z_1), (q_2, z_2), \ldots, (q_n, z_n))$ $= (0, d(z_1)d(z_2)\cdots d(z_n))$. It can be easily verified that R is a semiprime ring but not a prime ring and D is a nonzero *-n-derivation of R. However, R is not commutative.*

Conjecture 6.16. [11] *Let R be a semi-prime ring with suitable torsion restrictions and D be a non-zero permuting n-derivation. Suppose that for some integer $m \geq 1$, $d_m(x) \in Z(R)$, $\forall\ x \in R$ where $d_{k+1}(x) = [d_k(x), x]$ for $k > 1$ and $d_1(x) = d(x)$ stands for the trace of D. Then $[d(x), x] = 0$, $\forall\ x \in R$."*

7. Nil and nilpotent derivations

A derivation d on a ring R is called nilpotent if $d^n R = 0$ for some positive integer n. For $n = 2$ and R a prime ring of characteristic $\neq 2$, a basic theorem of Posner [144] implies that d is actually zero. For $n > 2$, the situation is less clear cut. Herstein in [85] showed that for simple rings with some characteristic restrictions, a nilpotent derivation is identical to an inner derivation $x \longrightarrow [a, x]$ induced by a nilpotent element a. As a consequence, the index of a nilpotent derivation (for simple rings) must be an odd number; it is equal to $2m - 1$ where m is the nilpotent index of the element a above. A. Kovacs thus asks the possibility of generalizing these to prime rings. Chung and Luh in 1983 [62] showed that the index of nilpotent derivation is indeed an odd number even for semiprime rings without 2-torsion. They proved the following:

Theorem 7.1. *Let R be a 2-torsion free semiprime ring and d be a derivation of R. If $d^{2n} R = 0$ then $d^{2n-1} R = 0$.*

The restriction of semiprimeness and that of 2-torsion free are both necessary, as can be seen from the following examples:

Example 7.2. *The ring of all two by two upper triangular matrices over the real field is not semiprime but the inner derivations determined by* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *has nilpotency two. Thus the assumption of semiprimeness is necessary.*

Example 7.3. *In the simple ring of two by two matrices over* $\mathbb{Z}_2$ *the inner derivation determined by* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *has nilpotency two, thus the necessity of the restriction on characteristics.*

Taking this idea forward Chung and Luh in the year 1984 [63] generalized this result to any semiprime ring without any torsion restrictions and were able to prove;

Theorem 7.4. *Let R be a semiprime ring and δ a nilpotent derivation of R. Then the nilpotency of δ is either a power of 2 or an odd number.*

The hypothesis that R is semiprime is essential. It can be seen from the following example; let R be the ring of 6×6 upper triangular matrices over $GF(2)$ and δ the inner derivation determined by the matrix

$$A = E_{12} + E_{23} + E_{34} + E_{45} + E_{56},$$

where E_{ij} denotes the unit matrix having 1 at the (i, j)-position and zero elsewhere. We can easily see that $A^6 = 0$, $A^5 \neq 0$ and the nilpotency of δ is 6 which is neither a power of 2 nor an odd number.

In 1992, Lanski [114] proved certain interesting results by relating nilpotency of a derivation on a subset to that of its nilpotency on the whole ring. He was able to prove that a derivation nilpotent on any noncommutative Lie ideal of a ring R will be a nilpotent derivation with the same nilpotency. Further he studied the results for one sided ideals and was able to establish the following;

Theorem 7.5. *Let T be a nonzero right ideal of R and D a derivation of R so that $D^n(T) = 0$. If either $char(R) = 0$ or $char(R) > n$, then $D = ad(A)$ for $A \in Q$, and for some $c \in C$, $(A - c)^n = 0$, so in particular, $D^t = 0$ for $t = min\ positive(2n - 1, char(R))$.*

Posner [144], gave the idea of product of derivations to be a derivation in case of prime rings, taking this idea forward, David Jensen [97] proved that a specific product of powers of derivations gives us either a zero derivation or a nilpotent one. He proved the following results;

Theorem 7.6. *Let λ and δ be derivations of a prime ring R, and let $\lambda\delta^m = 0$ where m is a positive integer. Then either $\lambda = 0$ or $\delta^r = 0$ where $r \leq 4m - 1$.*

Theorem 7.7. *Let λ and δ be derivations of a prime ring R, and let $\lambda^n\delta = 0$ where n is a positive integer. Then either $\delta^2 = 0$ or $\lambda^t = 0$ where $t \leq 12n - 9$.*

Theorem 7.8. *Assume λ and δ are derivations of a prime ring R, and assume $\lambda^n\delta^m = 0$ where n and m are positive integers. If λ and δ commute, then at least one of them is nilpotent.*

Till now we have been dealing with nilpotent derivations, let's shift our focus to a broader concept which contains the idea of nilpotent derivations within, called the nil derivations.

7.1. **Nil derivations.** A mapping $f : R \to R$ is said to be nil on an ideal I of R if for each $x \in I$ there exists a number n which depends on x, such that $f^n(x) = 0$. There can be a number of values for n such that $f^n(x) = 0$ holds but we are interested in the least such number n and we call it as the index of nilpotency of f with respect to x, which is denoted by $nil(f, x)$. f is said to be 2-sided nil if for each $x \in R$ there exists a number n such that $x(f^n(y)) = 0 = f^n(x)$ for all $y \in R$. One can observe that nilpotent derivations are always nil but the converse need not be true.

Example 7.9. *Let δ be a derivation on $R[x]$, the ring of polynomials over a ring R satisfying $\delta(x^n) = nx^{n-1}$. Clearly δ forms a nil derivation which is not nilpotent.*

Felzenszwalb and Lanski [75] in 1983 tried to study nil derivations with the aim of seeing whether nonzero nil derivations exist or not. What they achieved was;

Theorem 7.10. *"Let R be a prime ring with a nontrivial idempotent so that R contains no nonzero nil right ideal. If D is a nil derivation on I, then D = 0."*

Similar result was proved for primitive rings and that for semi-simple rings. Further, they were interested to know whether dropping the condition of no nil right ideals will affect the above result, indeed it did alter the result;

Theorem 7.11. *Let R be a prime ring with nontrivial idempotent. If D is a nil derivation on I, then D = 0 if either R is a simple ring, or if $D(x)^N = 0$, $\forall\ x \in I$ and N a fixed integer.*

In the same paper they were able to drop every other restriction be it having a nontrivial idempotent or for that matter of primeness and achieved;

Theorem 7.12. *Let R be a ring containing no nonzero nil right ideal. If D is a nil derivation on I, then D(I) = 0.*

Kharchenko [106] was able to prove an interesting result that all algebraic derivations are X-inner. Various authors tried to generalize and sharpen this result (see [66], [133]). A natural question can be asked whether nil derivations are inner. This was answered by Chung [65] in the year 1985. The result states,

Theorem 7.13. *Let δ be a nil derivation on a prime ring of characteristic 0. Then the following are equivalent:*

(1) there is a nonzero ideal I of R such that δ is E-inner for E = E(I) and δ is induced by a 2-sided nil element in E(I);
(2) there is a nonzero ideal I of R such that exp δ is E-inner for E = E(I) and exp δ is induced by an unipotent element f in E(I) such that f has a nonzero fixed point.

(3) *there is an element* $a \neq 0$ *in* R *such that* $\delta a = 0$ *and that* $a\delta(x)a = 0\ \forall\ x \in R$.

A derivation δ is said to be $E - inner$ if there is an $f \in E(I)$ (here $E(I) = End(I_R, I_R)$) such that $\delta(x)(w) = (adf)(x)(w) = [fx - xf](w)\ \forall\ x \in R$ and $w \in I$. Also $exp\ \delta$ refers to the automorphism generated by nilpotent derivation δ, $exp\ \delta : R \to R$ given by the relation $exp\ \delta = \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n$.

8. Derivations with invertible values

The idea of derivations with invertible values was first introduced by Bergen, Herstein and Lanski [39] in the year 1983 in an attempt to know the properties of a ring which possesses an invertible derivation. By an invertible derivation on a ring R (with unit element) we mean a non zero derivation d which is defined as $d(x) = 0$ or $d(x)$ invertible in R. They were successful in proving that such a ring admitting an invertible derivation had a special structure.

Theorem 8.1. *"Let* R *be a ring with 1 and* $d \neq 0$ *a derivation of* R *such that, for each* $x \in R$, $d(x) = 0$ *or* $d(x)$ *is invertible in* R*. Then* R *is either*

(1) *a division ring* D*, or*
(2) D_2*, or*
(3) $D[x]/(x^2)$*, where* $char D = 2$, $d(D) = 0$ *and* $d(x) = 1 + ax$ *for some* a *in the center* Z *of* D*.*

Furthermore, if $2R \neq 0$ *then* $R = D_2$ *is possible if and only if* D *does not contain all quadratic extensions of* Z*, the center of* D*; equivalently if and only if some element in* Z *is not a square in* D*."*

Later on shifting the idea of invertible derivations from the whole ring to its one sided ideals, they were able to establish an analogous result as the previous one. The result stated;

Theorem 8.2. *Let* R *be a ring with 1 and suppose that* $d \neq 0$ *is a derivation of* R *such that* $d(L) \neq 0$ *for some left ideal of* R *and* $d(x) = 0$ *or* $d(x)$ *is invertible for every* $x \in L$*. Then* $R = D$, $R = D_2$*, or,* $R = D[x]/(x^2)$ *where* $2R = 0$ *for some division ring* D*.*

Taking this idea forward, Bergen and Carini in [38] did generalize the above result by taking a non-central Lie ideal and to there suprise a similar story did follow. The outcome of their thought was;

Theorem 8.3. *Let* R *be a ring with 1,* $U \nsubseteq Z$ *a Lie ideal of* R*, and* d *a derivation of* R *such that* $d(U) \neq 0$ *and* $d(u) = 0$ *or* $d(u)$ *is invertible, for every* $u \in U$*. Then* R *is either*

(1) *a division ring* D*, or*
(2) D_2*,*

unless 2R or 3R is zero, d *is not inner, and* R *is not semiprime. In this case,* $R = M + d(M)$*, where* M *is the unique maximal ideal of* R *and* $M^3 = 0$*.*

Further, they focused on a specific structure, what if $R = D_2$ and they came up with;

Theorem 8.4. *Suppose $R = D_2$, then:*

(1) if D is not commutative and $2R \neq 0$, every derivation of R such that d(u) = 0 or d(u) is invertible, $\forall$ u in a non-central Lie ideal, must be inner.
(2) there exists an inner derivation d such that $d(U) \neq 0$ and d(u) = 0 or d(u) is invertible, $\forall$ u contained in a non-central Lie ideal U, if and only if D does not contain all quadratic extensions of Z or D is a field of characteristic 2.

In 1986 Giambruno, Misso and Milies [78] tried to introduce the idea of involution in the already established idea of invertible derivations and came up with certain interesting results,

Theorem 8.5. *Let R be a 2-torsion free semiprime ring with involution. Let d be a derivation of R such that $d(S) \neq 0$ and the non-zero elements of $d(S)$ are invertible in R. Then R is either:*

(1) a division ring D, or
(2) D_2, the ring of 2×2 matrices over D, or
(3) $D \oplus D^{op}$, the direct sum of a division ring and its opposite relative to the exchange involution, or
(4) $D_2 \oplus D_2^{op}$, with the exchange involution, or
(5) F_4, the ring of 4×4 matrices over a field F with symplectic involution.

Once involution was introduced with invertible derivations there was no looking back, Jer-Shyong Lin in [121] proved something similar for arbitrary semprime rings but using the skew symmetric part (K) of the ring.

Theorem 8.6. *Let R be a semiprime ring with involution *. Let d be a derivation of R such that $d(K) \neq 0$ and the nonzero elements of d(K) are invertible in R. Then R is either:*

(1) a division ring D, or
(2) D_2, the ring of 2×2 matrices over a division ring, or
(3) $D \oplus D^{op}$, the direct sum of a division ring and its opposite, relative to the exchange involution $(x, y)^* = (y, x)$, or
(4) $D_2 \oplus D^{op}$ with the exchange involution, or
(5) F_3, the 3×3 matrices over a field, or
(6) F_4, the 4×4 matrices over a field.

Example 8.7. *Let $R = F_4$, where F is a field in which -1 is not a square, and let * be the transpose involution in F_4. Let d be the inner derivation on F_4 induced by $\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then it can be seen that $d(K) \neq 0$ and the nonzero elements of d(K) are invertible.*

In 2014, Kaygorodov and Popov[103] described all the alternative algebras that admitted derivations with invertible values and proposed a similar problem for Jordan algebras. It stated:

Problem 3. How does one classify all the Jordan (and, more generally, structurable) algebras that admit derivations with invertible values?

9. Lie derivations

The study of Lie and Jordan mappings goes back to Ancochea ([6], [7]), Herstein [83], Hua ([92], [93]), Jacobson and Rickart [95], Kaplansky [100], and Smiley [148]. Since then for almost 30 years the study of Lie isomorphisms and Lie derivations was carried on mainly by Martindale and his students [91], [94], [126], [127], [128], [129], [130], [131], [132], [145]. A Lie derivation d on a ring R is an additive map satisfying the condition,

$$d[a,b] = [d(a),b] + [a,d(b)], \tag{18}$$

$\forall\ a,b \in R$. In 1964, Martindale [127] described the various forms of Lie derivations of primitive rings of characteristic not 2 with nontrivial idempotents which lead to further understanding of such derivations. The result is as following;

Theorem 9.1. *Let L be a Lie derivation of a primitive ring R into itself, where R contains a nontrivial idempotent and the characteristic of R is not 2. Then L is of the form $D + T$, where D is an ordinary derivation of R into a primitive ring $\bar{R}$ containing R and T is an additive mapping of R into the center of $\bar{R}$ that maps commutators into zero.*

Later on he considered several cases of prime rings with involution [94], [107], [149], [150], and of von Neumann algebras [140] under the similar assumptions. Similarly, Brešar [47] in 1993 determined the structure of Lie derivations of prime rings which do not satisfy the standard polynomial identity S_4 and gave the result;

Theorem 9.2. *Let R be a prime ring of characteristic not 2. Let d be a Lie derivation of R. If R does not satisfy S_4 then d is of the form $\delta + \tau$, where δ is a derivation of R into its central closure and τ is an additive mapping of R into C sending commutators to zero.*

This result together with other results initiated the theory of functional identities on rings. For further understanding the concept of functional identities on rings one can see the book [48]. What Brešar did to prime rings Mathieu and Banning [25] tried to generalize the same for semiprime rings and were able to establish the following result;

Theorem 9.3. *Let d be a Lie derivation on a 2-torsion free semiprime ring R. Then $d = \delta + \tau$ where δ and τ are additive mappings from R to $Q_s(R)$ and to eC, respectively, such that $e\delta$ is a derivation and τ sends commutators to zero. Moreover, δ and τ are uniquely determined by these properties.*

In the year 1995, Gordan A. Swain [149] tried to extend Lie derivations defined on the skew symmetric elements of a prime ring with involution to derivations on the associative ring generated by the skew elements. He was able to extend Lie derivations to ordinary derivations in both the cases when the ring satisfied a GPI and when it does not satisfy one. The result stated;

Theorem 9.4. *Let R be a prime ring with involution of the first kind, and characteristic $\neq 2,3$. Let K denote the skew elements of R, and let C denote the extended centroid of R. Assume that $(RC : C) \neq 1,4,16$. Then any Lie derivation of K into itself can be extended to an ordinary derivation of $< K >$.*

Swain gave certain other interesting results as well that lead to the extension of Lie derivations to ordinary derivations,

Theorem 9.5. *Let $A \cong M_3(C)$ or $M_5(C)$ where C is a field with characteristic $\neq 2,3$, and let A have an involution of the first kind. Then any Lie derivation of K into itself can be extended to an ordinary derivation of A.*

Gordan [150] with the help of a result of Blau [41] pertaining to triadditive mappings on skew elements was able to generalize the result to non-GPI rings with involutions of second kind. For the other case of GPI rings with involution of second kind they believe the idea is tractable, and using the socle for the results on prime rings without involution. The result states;

Theorem 9.6. *Let R be a non-GPI prime ring with involution and characteristic $\neq 2,3$. Let K denote the skew elements of R, and C denote the extended centroid of R. Let δ be a Lie derivation of K into itself. Then $\delta = \rho + \epsilon$ where ϵ is an additive map into the skew elements of the extended centroid of R which is zero on $[K,K]$, and ρ is a Lie derivation which can be extended to an ordinary derivation of $< K >$ into RC.*

Lie derivations, as well as other Lie maps have been extensively studied for a long time now (see [26], [36], [48], [51], [135], [152]). In this direction Yu Wang recently in [162] gave the description of Lie(Jordan) derivations in case of arbitrary triangular algebras. By a triangular algebra we mean the following;

Definition 9.7. *Let A and B be algebras with unity. Let M be a faithful (A,B)-bimodule. $U = Tri(A,M,B) := \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in A, b \in B, m \in M \right\}$, under the usual matrix operations is said to be a triangular algebra.*

Upper triangular matrix algebras are usual examples of triangular algebras. Triangular algebras are classical examples of algebras which are not semiprime. Yu Wang's result for triangular algebra with Lie derivation goes like;

Theorem 9.8. *Let U = Tri(A, M, B) be a triangular algebra. Let L be a Lie derivation on U. Then there exists a triangular algebra U^0 such that U is a subalgebra of U^0 having the same unity and L can be written as*

$$L = \delta + \tau,$$

where $\delta : U \to U^0$ is a derivation and $\tau : U \to Z(U^0)$ is a linear map such that $\tau([x,y]) = 0$, $\forall\ x, y \in U$.

One of the corollaries of this result gives a description of Jordan derivations on triangular algebras which will be discussed in the next section.

Conjecture 9.9. [152] *"For every Lie derivation d on a complex Banach algebra A and every primitive ideal P of A we have*

$$[d(P), A] \subset P."$$

10. Jordan derivations

Herstein showed interest in the relationship between associative rings and Jordan(Lie) structures, in this pursuit he defined what we call Jordan derivations, from a ring R he constructed another ring equipped with the operation of multiplication as the Jordan product, i.e., $a \circ b = ab + ba$ and then defined Jordan derivation d as an additive mapping on $(R, +, \circ)$ satisfying,

$$d(a \circ b) = d(a) \circ b + a \circ d(b), \text{ for all } a, b \in R. \tag{19}$$

But the commonly used definition of Jordan derivation for rings became, an additive mapping on a ring R satisfying

$$d(a^2) = d(a)a + ad(a), \text{ for all } a \in R. \tag{20}$$

Both the definitions resonate if the Jordan ring taken in the first definition is two torsion free. Clearly every derivation is by default a Jordan derivation but the converse may not be true in every setting. In this direction Herstein way back in 1957 [84] gave a revolutionary result that paved way for many other interesting results as to when derivations are Jordan derivations. The result states:

Theorem 10.1. *If R is a prime ring of characteristic different from 2, then every Jordan derivation of R is a derivation.*

For a prime ring with characteristic 2 he did manage to prove the same but with a specific condition on its structure;

Theorem 10.2. *Let A be a prime ring of characteristic 2. Then if A is not a commutative integral domain, any Jordan derivation of A is an ordinary derivation.*

Further, Cusack [61] in 1975 tried to generalize Herstein's result, and was able to accomplish several results, some of them are:

Theorem 10.3. *Let D be a Jordan derivation on a ring R in which $2x = 0$ implies $x = 0$. Then $\forall$ a and b in R, $D(ab)$ - aDb - $(Da)b$ is in L, Baer lower radical of R.*

We know that the Baer lower radical which is also known as the prime radical, L of R is the intersection of all the prime ideals of R. One of the corollaries of the above result is what Cusack aimed for, the extension of Herstein's result to semiprime rings, the result is as follows:

Corollary 10.4. *If R is semiprime, then D is a derivation.*

Theorem 10.5. *Let R be a ring where $2x = 0$ implies $x = 0$ and which has a commutator which is not a zero divisor. Then every Jordan derivation on R is a derivation.*

Later on in 1984, Awtar [22] generalized the result of Herstein on Lie ideals and proved:

Theorem 10.6. *Let R be a prime ring of characteristic different from two and U be a square closed Lie ideal of R. If $d : R \to R$ is an additive mapping such that $d(u^2) = d(u)u + ud(u)\ \forall\ u \in U$, then $d(uv) = d(u)v + ud(v)$, $\forall\ u, v \in U$.*

In 1988, Brešar and Vukman [53] gave an alternative proof of Herstein's result. In addition to that Brešar [44] generalized the result of Herstein even further to 2-torsion-free semiprime rings;

Theorem 10.7. *Let R be a 2-torsion free semiprime ring and let $d{:}R \longrightarrow R$ be a Jordan derivation. In this case, d is a derivation.*

So one might think that those Jordan derivations which are not derivations, i.e., proper Jordan derivations are very difficult to find. Indeed that turns out to be true. In 1996, B.E. Johnson [98] thought about those (Banach) algebras $\mathbb{A}$ which had no proper Jordan derivations from $\mathbb{A}$ into any arbitrary (Banach) $\mathbb{A}$-bimodule $\mathbb{M}$. While it turned out that this was true for some important classes of algebras (in particular, for the algebra of all $n \times n$ complex matrices), there were on the other hand simple counterexamples on some algebras and their (special) bimodules. The fundamental example in this direction is given on the algebra $\mathbb{T}_2$ of 2×2 upper triangular matrices over C(a commutative ring with unity). We can make C an $\mathbb{T}_2$-bimodule by defining $a\lambda = a_{22}\lambda$ and $\lambda a = \lambda a_{11}\ \forall\ \lambda \in C,\ a \in \mathbb{T}_2$.

A map $\delta : \mathbb{T}_2 \to C$ defined by $\delta(a) = a_{12}$ is a proper Jordan derivation, which, Johnson hinted would be an antiderivation, i.e., satisfies

$$\delta(ab) = \delta(b)a + b\delta(a),$$

$\forall\ a, b \in \mathbb{T}_2$.

In this direction Benkovic [35] in 2005 gave a very interesting result which helps to decompose a Jordan derivation into a derivation and an antiderivative, the result states;

Theorem 10.8. *Let $\mathbb{M}$ be a $\mathbb{T}_n$-bimodule and let $\Delta : \mathbb{T}_n \to \mathbb{M}$ be a Jordan derivation. Then there exist a derivation $d : \mathbb{T}_n \to \mathbb{M}$ and an antiderivation $\delta : \mathbb{T}_n \to \mathbb{M}$ such that $\Delta = d + \delta$ and $\delta(\mathbb{D}_n) = 0$. Moreover, d and δ are uniquely determined, where $\mathbb{D}_n$ is the subalgebra of all diagonal matrices.*

As a corollary to this result Dominik proved that there does not exist any proper Jordan derivation for $n \times n$ upper triangular matrices.

Corollary 10.9. *Let $2 \leq n \leq m$. There are no proper Jordan derivations from $\mathbb{T}_n$ into $\mathbb{M}_m$. In particular, there are no proper Jordan derivations from $\mathbb{T}_n$ into itself.*

In the previous section we talked about how Yu Wang [162] described Lie derivations on triangular algebras, in a similar manner he did the same for Jordan derivations and the result states;

Corollary 10.10. *Let U = Tri(A, M, B) be a triangular algebra. Let D be a Jordan derivation on U. Suppose that U is 2-torsion free. Then D is a derivation. Otherwise, there exists a triangular algebra U^0 such that U is a subalgebra of U^0 having the same unity and D can be written as $D = \delta + \tau$, where $\delta : U \to U^0$ is a derivation and $\tau : U \to Z(U^0)$ is a linear map such that $\tau(x \circ y) = 0$, $\forall\ x, y \in U$.*

Recently in 2019, Fošner and Jing proved that every Jordan derivation on a triangular ring is a derivation and were also able to establish that, under some conditions, every Jordan derivation on a 2-torsion free ring is a derivation (see [77]).

11. Pre-derivations

The idea of pre-derivations was first introduced by Müller [141] in the year 1989 while studying certain bi-invariant pseudo-Riemannian metrics. In the same paper he initially started with the ideas like pre-morphisms, pre-endomorphisms and that of pre-isomorphisms. On the similar lines he defined pre-derivations that resonated with the rest of the notions, a club of pre-morphism and derivation.

Definition 11.1. *Let g be a Lie algebra. A pre-derivation of g is a linear mapping $P : g \to g$ such that*

$$P[x, [y, z]] = [Px, [y, z]] + [x, [Py, z]] + [x, [y, Pz]], \tag{21}$$

for every $x, y, z \in g$.

Let us denote the set of all pre-derivations on g to be $Pder(g)$. It is easy to verify that this set forms a subalgebra of the Lie algebra $gl(g)$ and also contains the algebra $Der(g)$, the set of all derivations on g. Müller proved that, if G is a Lie group endowed with a bi-invariant semi-Riemannian metric and g its Lie algebra, then the Lie algebra of the group of isometries of G fixing the identity element is a subalgebra of $Pder(g)$. Making the study of the algebra of prederivations not only interesting from the algebraic point of view but also from its applications part. In the same paper, Müller tried to establish when pre-derivations can be ordinary derivations, he was successful in finding the fact that:

Corollary 11.2. *Every pre-derivation of a finite-dimensional semi-simple Lie algebra g over K is a derivation, hence even an inner derivation.*

In the year 1997, Bajo [24] tried to relate Lie algebras having a pre-derivation with that of the nilpotency of that algebra. This was motivated by a known result of Jacobson [96] which stated that a Lie algebra having a non-singular derivation will have to be nilpotent. This characterization of the structure of Lie algebras using a pre-derivation was a significant development for further studies in this area. The result states:

Theorem 11.3. *If g is a real or complex Lie algebra admitting a non-singular pre-derivation then g is nilpotent.*

Jacobson [96] in the year 1955 answered a very interesting question that whether a lie algebra admits a non-singular derivation or not and proved that any

Lie algebra over a field of characteristic zero admitting a non-singular derivation must be nilpotent. He also enquired about its converse, i.e., whether any nilpotent Lie algebra admits a non-singular derivation to which the answer was a negative. Later Dixmier and Lister [72] constructed a nilpotent Lie algebra possessing only nilpotent derivations and such an algebra was called the characteristically nilpotent Lie algebra which was then extensively studied in [120]. A similar question could be asked for pre-derivations and yes we have a partial answer from Bajo who proved that the characteristically nilpotent algebra given by Favre in [74] has only singular pre-derivations. The result states:

Proposition 11.4. *Let g be the real Lie algebra spanned by elements $\{x_l, x_2, \dots, x_7\}$ with non-trivial brackets*

$$\begin{aligned}
&[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6, \\
&[x_2, x_3] = x_6, \\
&[x_2, x_4] = [x_2, x_5] = -[x_3, x_4] = x_7.
\end{aligned}$$

Every pre-derivation of g is nilpotent and hence singular.

Analogous to the definition of characteristically nilpotent Lie algebras we call a nilpotent Lie algebra strongly nilpotent if all its pre-derivations are nilpotent. Burde [55] in 2002 worked extensively on nilpotent Lie algebras and was able to relate dimension of $Pder(g)$ to that of $Der(g)$ and proved,

Proposition 11.5. *Let g nilpotent of degree $p > 1$ and of dimension $n \geq 3$. Then $dim\ Pder(g) > dim\ Der(g)$.*

Burde provided an example of a characteristically nilpotent algebra which is not strongly nilpotent.

Example 11.6. *Consider the following 7-dimensional Lie algebra, defined by*

$$\begin{aligned}
&[e_1, e_i] = e_{i+1}, 2 \leq i \leq 6 \\
&[e_2, e_3] = e_6 + e_7 \\
&[e_2, e_4] = e_7.
\end{aligned}$$

It can be easily verified that all the derivations are nilpotent (see [123]). On the other hand, $P = diag(1, 3, 3, 5, 5, 7, 7)$ is a non-singular pre-derivation. One has $dimPder(g) = 16$ and $dimDer(g) = 11$.

He gave yet another example of a Lie algebra possessing only nilpotent pre-derivations.

Proposition 11.7. *Let g be the n-dimensional Lie algebra with basis $(e_1, \ldots, e_n)$, $n \geq 7$ and defining brackets*

$$\begin{aligned}
&[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1,\\
&[e_2, e_3] = e_{n-1},\\
&[e_2, e_4] = e_n,\\
&[e_2, e_5] = -e_n,\\
&[e_3, e_4] = e_n.
\end{aligned}$$

Then g is strongly nilpotent.

12. HOMODERIVATIONS

In 2000, El Sofy [73] gave an interesting concept which clubs both the concepts of homomorphism and that of derivation and he named such a map as a homoderivation. He showed that such maps are neither homomorphisms nor derivations.

Definition 12.1. *An additive map h from a ring R to itself satisfying the condition*

$$h(ab) = h(a)h(b) + h(a)b + ah(b), \ \forall \ a, b \in R,$$

is termed as a homoderivation.

For example, $h(x) = f(x) - x$ for any endomorphism f on R forms a homoderivation.

Example 12.2. *Let $R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$. Define $h : R \to R$ as $h\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$. h so defined forms a homoderivation on R.*

Bell and Daif [32] in 1998 while extending Hersteinâs [86] results proved that for a derivation d, whenever $d([x, y]) = 0$, $\forall \ x, y$ in a two sided ideal of R then the ring R is commutative. Extending this for homoderivations Asmaa Melaibari et al. [139] showed the following:

Theorem 12.3. *Let R be a prime ring and U a nonzero ideal of R. If R admits a nonzero homoderivation h that satisfies $h([x, y]) = 0$, $\forall \ x, y \in U$, then R is commutative.*

Theorem 12.4. *"Let R be a prime ring with characteristic different from 2. If $0 \neq h$ is a zero-power valued homoderivation, on R such that $h([x, y]) \in Z(R)$ $\forall$ $x, y \in R$, then R is commutative."*

In [27], the focus was shifted to nilpotent homoderivations in order to prove Herstein's [86] results for homoderivations. The composition of homoderivations was also studied in the same paper.

Theorem 12.5. *Let R be a prime ring and $s \geq 1$ be a fixed integer. Next, let h_1 and h_2 be two homoderivations on R such that $[h_1^s(x), h_2^s(y)] = 0$, $\forall\ x, y \in R$. Then either R is a commutative integral domain, or $h_1^{3s-1} = 0$, or $h_2^{3s-1} = 0$.*

Theorem 12.6. *Let R be a prime ring. Next, let h_1 and h_2 be homoderivations of R such that $h_1 \circ h_2 = h_2 \circ h_1$ and $h_1^{s_1} \circ h_2^{s_2} = 0$ for positive integers s_1 and s_2. Then at least one of the homoderivations must be nilpotent.*

For more than 2 homoderivations the above result does not hold, consider the ring $R = M_3(D)$, where D is a division ring. Now, consider 3 maps h_1, h_2, h_3 on R defined by,

$$h_1(X) = AXA^{-1} - X,$$
$$h_2(X) = BXB^{-1} - X,$$
$$h_3(X) = CXC^{-1} - X,$$

where $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. One can observe that h_1, h_2, h_3 so defined are homoderivations on R. Also, these homoderivations are commuting with each other such that $h_1 \circ h_2 \circ h_3 = 0$. But, none of them is a nilpotent homoderivation. In 2023, Sögütcü [110] characterized semiprime rings with homoderivations and generalized some of the existing results that included derivations on prime rings to that of semiprime rings. Few of their results are as:

Theorem 12.7. *"Let R be a semiprime ring and I a nonzero ideal of R. Then, R contains a nonzero central ideal if R admits a nonzero homoderivation h on I such that $h(I) \subseteq Z$.*

Theorem 12.8. *Let R be a semiprime ring and I a nonzero ideal of R. Then, R contains a nonzero central ideal, if R admits a nonzero homoderivation h on I such that*

(1) $h([I, I]) = (0)$ or
(2) $[h(I), I] \subseteq Z$ or
(3) $[h(I), h(I)] = (0)$, $Ih^2(I) \neq (0)$, and $h(I) \subseteq I$."

Recently, Belkadi et. al. [28] introduced the concepts of Jordan homoderivations and n-Jordan homoderivations, as follows:

Definition 12.9. *Let R be a ring. An additive mapping $h : R \to R$ is called a Jordan homoderivation if $h(x^2) = h^2(x) + h(x)x + xh(x)$ holds $\forall\ x \in R$.*

Definition 12.10. *Let R be a ring, let $n \geq 1$ be a fixed integer and let $h : R \to R$ be an additive mapping. Then h is said to be an n-Jordan homoderivation if it satisfies the following relation*

$$h(x^n) = (h(x) + x)^n - x^n,$$

$\forall\ x \in R$.

They were able to accomplish that every homoderivation forms an n-Jordan homoderivation. Also, for a 2-torsion free commutative ring any Jordan homoderivation was proved to be a homoderivation. Further they showed that under suitable torsion conditions n-Jordan homoderivation forms a homoderivation. The result states;

Theorem 12.11. *Let $n \geq 2$ be a fixed integer, let R be a unital ring with identity e and let $h : R \to R$ be an n-Jordan homoderivation with $h(e) = 0$. Then the following hold:*
(i) If R is commutative and δ-torsion free, then h is a homoderivation.
(ii) If R is prime, δ-torsion free and h is such that $h + Id_R$ is an onto mapping that is not an anti-homomorphism, then h is a homoderivation.
(iii) If R is semi-prime, δ-torsion free and h is such that $h + Id_R$ is an onto mapping, then there exists an essential ideal U of R such that the restriction of h to U is a direct sum $h_1 \oplus g_2$, where h_1 is a homoderivation of U into R and g_2 is an anti-homomorphism of U into R.

They conclude the paper with the following conjectures:

Conjecture 12.12. *Let R be a (semi)-prime ring with suitable torsion restrictions. Is any Jordan homoderivation of R a homoderivation?*

Conjecture 12.13. *Let $n \geq 2$ be a fixed integer, let R be a (semi)-prime ring with suitable torsion restrictions and let h be an n-Jordan homoderivation. Then there exists an essential ideal U of R such that the restriction of h to U is a direct sum $h_1 \oplus g_2$, where h_1 is a homoderivation of U into R and g_2 is an anti-homomorphism of U into R.*

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