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A NEW THREE- STEP DERIVATIVE FREE ITERATIVE METHOD AND ITS DYNAMICS

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Abstract. A new free derivative iterative method is presented in this article. The method is developed by combining Newton's method and Euler's method. Derivatives in this method are approximated by the forward difference, *hyperbola* and divided difference. The order of convergence is proven analytically to be of sixth order. Numerical results exhibit that the new method is comparable to other methods. Basins of attraction are also provided to support the proposed method.

Key words and Phrases: forward difference, divided difference, efficiency index, derivative-free method, order of convergence, basins of attraction

1. Introduction

Nonlinear equation of the form

$$g(x) = 0. \tag{1}$$

has been used to model many real-world problems such as in chemistry, physics, economics and others. Newton's method ([2] and [4]) given by

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \tag{2}$$

where $g'(x_n) \neq 0$ for $n = 0, 1, 2, \ldots$, has quadratic convergence and is one of the well-known and ancient methods to solve (1). Development of this method has been growing exponentially for decades by using many types of approaches. One of the examples is the classical Chebyshev-Halley's method promoted by Gutierrez

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and Hernandez [5] where (2) and Taylor's expansion [1] are employed to get the following method:

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{\eta_g(x_n)}{1-\beta\eta_g(x_n)}\right)\frac{g(x_n)}{g'(x_n)}, \tag{3}$$

where,

$$\eta_g(x_n) = \frac{g''(x_n)g(x_n)}{(g'(x_n))^2},\ \beta \in \mathbb{R},\ n = 0, 1, 2, \ldots, \tag{4}$$

with third-order of convergence [5].

Xiaojian [6] developed Ostrowski's method by modifying (4) and using one point $(x_n, g(y_n))$ hyperbola to approximate $g''(x_n)$, that is

$$g''(x_n) \approx \frac{2(g'(x_n))^2 g(y_n)}{g^2(x_n) - g(x_n)g(y_n)}. \tag{5}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{6}$$

Substituting (5) into (4) we have

$$\eta_f(x_n) \approx \frac{2g(y_n)}{g(x_n) - g(y_n)}. \tag{7}$$

The following method is Euler's method with third order of convergence ([7], [9]).

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}\frac{2}{1+\sqrt{1-2\eta_g(x_n)}}, \tag{8}$$

where $\eta_g(x_n)$ is (4)

Syakir et al. [10] combined Newton, Halley, and Chebyshev methods to develop a three-step method of fourteenth order of convergence where the derivatives are estimated by Taylor polynomial. Amelia et al. [11] developed a two-step iterative method formulated by

$$y_n = x_n - \frac{2\gamma g(x_n)^2}{g(w_n) - g(u_n)}, \tag{9}$$

$$x_{n+1} = x_n - \frac{g(x_n) - 2\beta g(y_n)}{g(x_n) - (1+2\beta)g(y_n)}\frac{2\gamma g(x_n)^2}{g(w_n) - g(u_n)} \tag{10}$$

where $w_n = x_n + \gamma g(x_n)$ and $u_n = x_n - \gamma g(x_n)$. The method described by (9) and (10) is referred to as DFIM onward. DFIM has fourth order of convergence with efficiency index 1.414. Inspired by methods in [10] and [11], we develop a new three-step sixth order iterative method by combining Newton's method (2) and the

Euler's method (8) that is formulated as

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}, \tag{11}$$

$$z_n = y_n - \frac{g(y_n)}{g'(x_n)} \frac{2}{1+\sqrt{1-2\eta_g(x_n)}}, \tag{12}$$

$$x_{n+1} = z_n - \frac{g(z_n)}{g'(z_n)}, \tag{13}$$

where

$$\eta_g(x_n) = \frac{g''(x_n)g(x_n)}{(g'(x_n))^2}. \tag{14}$$

Approximation of $g'(z_n)$ is done by

$$g'(z_n) \approx \frac{g[y_n, z_n]g[w_n, z_n]}{g[w_n, y_n]}, \tag{15}$$

where $g[w_n, y_n], g[y_n, z_n]$ and $g[w_n, z_n]$ are first order divided difference and $w_n = x_n + \gamma g(x_n)$.

The organization of this article is as follows. In the following section, the three-step free derivative iterative method is presented where forward difference and divided difference are used as means of approximation to the first derivative. A theorem is also proposed in this section to show that the method is of order sixth. In the penultimate section, numerical simulations are given to show the performance of the presented method and its comparisons with several iterative methods. The dynamics of the studied methods through their basins of attraction are also discussed in this section. Finally, we draw some conclusions based on the numerical simulations we did in the previous section.

2. The Three-Step Free Derivative Iterative Method by Forward Difference and Divided Difference Approximation

In this section, we present a modification of DFIM to obtain a three-step free derivative iterative method. We modify the method by approximating $g'(x_n)$ in (11) and (12) with one parameter γ forward difference,

$$g'(x_n) \approx \frac{g(x_n + \gamma g(x_n)) - g(x_n)}{g(x_n)}. \tag{16}$$

The next step is approximating $g''(x_n)$ in (14) with (5) and substituting (15) into (13). Hence, we develop a three-step free derivative iterative method as follows:

$$\left.\begin{aligned} y_n &= x_n - \frac{\gamma g(x_n)^2}{g(w_n) - g(x_n)}, \\ z_n &= y_n - \frac{\gamma g(y_n) g(x_n)}{g(w_n) - g(x_n)} \frac{2}{1 + \sqrt{1 - 4R_g(x_n)}}, \\ x_{n+1} &= z_n - \frac{g(z_n) g[w_n, y_n]}{g[y_n, z_n] g[w_n, z_n]}, \end{aligned}\right\} \tag{17}$$

with

$$R_g(x_n) = \frac{g(y_n)}{g(x_n) - g(y_n)}.$$

The method described by (17) is referred to as DFIM2.

The followings are two definitions that will be useful to draw conclusions and comparisons to the proposed method.

Definition 2.1. [2] *Given a sequence $\{x_n | n \geq 0\}$ produced by an iterative method where it converges to a root θ. If there exists an integer $q \geq 1$ and some constant $\xi \geq 0$ such that*

$$\lim_{n \longrightarrow \infty} \frac{x_{n+1} - \theta}{(x_n - \theta)^q} = \xi$$

then q is the order of convergence of the method. The constant ξ is called the asymptotic error constant. If we let $e_n = x_n - \theta$ be the error in the n-th iteration, then

$$e_{n+1} = c e_n^q + \mathcal{O}(e_n^{q+1}) \tag{18}$$

is the error equation of the iterative method.

Definition 2.2. [3] *Efficiency index of an iteration method, defined by $EI(p, m) = p^{1/m}$, where p is the order of convergence of the method and m denotes number of functions evaluations (including some derivatives).*

The following theorem gives the order of convergence of DFIM2.

Theorem 2.3. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ is a continuous differentiable function in an open interval I and the simple root of g is θ. Suppose that x_0 is chosen to be sufficiently close to θ, then the method defined by (17) is of sixth-order convergent.*

Proof. Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function in open interval I and θ be a simple root of $g(x) = 0$. By expanding $g(x)$ about $x = \theta$ through Taylor expansion until sixth order, we obtain

$$\begin{aligned} g(x) = g(\theta) + g'(\theta)(x - \theta) + \frac{1}{2!} g''(x - \theta)^2 + \frac{1}{3!} g'''(x - \theta)^3 + \frac{1}{4!} g^{(4)}(x - \theta)^4 \\ + \frac{1}{5!} g^{(5)}(x - \theta)^5 + \frac{1}{6!} g^{(6)}(x - \theta)^6 + O(x - \theta)^7. \end{aligned} \tag{19}$$

By evaluating $g(x)$ at x_n, we have

$$g(x_n) = g'(\theta)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)\right). \tag{20}$$

with $c_k = \dfrac{g^{(k)}(\theta)}{k! g'(\theta)}$, $k = 2, 3, \ldots$ and $e_n = x_n - \theta$.

To obtain $\gamma g^2(x_n)$, take (20) into the power of two and multiply by γ resulting in

$$\gamma g^2(x_n) = \gamma (g'(\theta))^2 e_n^2 + 2\gamma (g'(\theta))^2 c_2 e_n^3 + \cdots + O(e_n^7). \tag{21}$$

It is trivial to get $w_n = x_n + \gamma g(x_n)$, that is

$$w_n = e_n + \theta + \gamma\Bigg(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6\Bigg) g'(\theta) + O(e_n^7). \tag{22}$$

Substituting w_n into (19) and simpliying, we get

$$\begin{aligned} g(w_n) = {} & \Bigg(\gamma (g'(\theta))^2 + g'(\theta)\Bigg) e_n + \Bigg((g'(\theta))^3 c_2 \gamma^2 + g'(\theta) c_2 + 3\gamma (g'(\theta))^2 c_2\Bigg) e_n^2 \\ & + \cdots + O(e_n^7). \end{aligned} \tag{23}$$

Using (20), (21) and (23) we have

$$\frac{\gamma g^2(x_n)}{g(w_n) - g(x_n)} = e_n + \Bigg(- c_2 - g'(\theta) c_2 \gamma\Bigg) e_n^2 + \cdots + O(e_n^7). \tag{24}$$

Substituting (24) into the first equation of (17) and remembering that $x_n = e_n + \theta$ gives

$$\begin{aligned} y_n = {} & \theta + g'(\theta)\Bigg(- \gamma g'(\theta) c_2 - c_2\Bigg) e_n^2 \\ & + \Bigg(2g'(\theta) c_2^2 \gamma - 3c1 c_3 \gamma + 2c_2^2 - (g'(\theta))^2 c_3 \gamma^2 - 2c_3 + (g'(\theta))^2 c_2^2 \gamma^2\Bigg) e_n^3 \\ & + \cdots + O(e_n^7). \end{aligned} \tag{25}$$

Applying the same technique to get (23) gives

$$g(y_n) = g'(\theta)\Bigg(c_2 + g'(\theta) c_2 \gamma\Bigg) e_n^2 + \cdots + O(e_n^7), \tag{26}$$

Employing (20), (23) and (26) and simplifying, yields

$$\begin{aligned} \frac{\gamma g(y_n) g(x_n)}{g(w_n) - g(x_n)} = {} & \Bigg(\gamma g'(\theta) c_2 + c_2\Bigg) e_n^2 + \Bigg((\gamma g'(\theta) c_2 + c_2) + \cdots + \gamma^2 (g'(\theta))^2 c_3\Bigg) e_n^3 \\ & + \cdots + O(e_n^7). \end{aligned} \tag{27}$$

Again, using (20) and (26) and simplifying, we obtain

$$\frac{2}{1+\sqrt{1-\dfrac{4g(y_n)}{g(x_n)-g(y_n)}}} = 1+\Big(\gamma g'(\theta)c_2+c_2\Big)e_n$$
$$+\Big(2c_3+\cdots+3g'(\theta)c_2^2\gamma\Big)e_n^2+\cdots+O(e_n^7). \quad (28)$$

Substituting (25), (27) and (28) into the second equation of (17) we obtain

$$\begin{aligned} z_n = \theta + \Big(& -(c_2+g'(\theta)c2\gamma)^2 - g'(\theta)c_2^2\gamma - c_2^2 \\ & -(c_2+g'(\theta)c_2\gamma)(-g'(\theta)c_2\gamma-3c_2)\Big)e_n^3 \\ & +\cdots+O(e_n^7). \end{aligned} \quad (29)$$

Evaluating (20) with respect to z_n gives

$$\begin{aligned} g(z_n) = & g'(\theta)\Big(-(c_2+g'(\theta)c_2\gamma)^2 - g'(\theta)c_2^2\gamma - c_2^2 \\ & -(c_2+g'(\theta)c_2\gamma)(-g'(\theta)c_2\gamma-3c_2)\Big)e_n^3+\cdots+O(e_n^7). \end{aligned} \quad (30)$$

Taking (22), (23), (25), (26), (29) and (30) and simplifying, we have

$$\begin{aligned} g[w_n,y_n] = & \frac{-g'(\theta)(1+\gamma g'(\theta))}{(-1-\gamma g'(\theta))} \\ & + \left(\frac{-g'(\theta)(\gamma^2 g'(\theta)^2 c_2 + 2\gamma g'(\theta)c_2)}{(-1-\gamma g'(\theta))} + \frac{g'(\theta)(1+\gamma g'(\theta))c_2}{(-1-\gamma g'(\theta))^2}\right)e_n^2 \\ & +\cdots+O(e_n^7). \end{aligned} \quad (31)$$

$$g[y_n,z_n] = g'(\theta) + \frac{11g'(\theta)^2c_2^3\gamma^2+3g'(\theta)^3c_2^3\gamma^3+\cdots-10c_2c_3}{c_2+g'(\theta)c_2\gamma}e_n^2+\cdots+O(e_n^7). \quad (32)$$

$$\begin{aligned} g[w_n,z_n] = & \left(\frac{g'(\theta)+\gamma g'(\theta)^2}{1+\gamma g'(\theta)}\right) \\ & + \Bigg(-\frac{g'(\theta)+\gamma(g'(\theta))^2\gamma g'(\theta)c_2}{(1+\gamma(g'(\theta))^2} \\ & + \frac{(3\gamma(g'(\theta))^2c_2+g'(\theta)c_2+(g'(\theta))^3c_2\gamma^2)}{(1+\gamma g'(\theta))}\Bigg)e_n+\cdots+O(e_n^7). \end{aligned} \quad (33)$$

Hence,

$$
\begin{aligned}
e_{n+1} =&\Big((-\gamma^3(g'(\theta))^3c_3 + \gamma^3(g'(\theta))^3c_2^2 - 3\gamma^2(g'(\theta))^2c_3 \\
&+ 4\gamma^2(g'(\theta))^2c_2^2 - 3\gamma g'(\theta)c_3 + 5g'(\theta)c_2^2\gamma + 2c_2^2 - c_3)c_2^3\Big)e_n^6 \\
&+ O(e_n^7).
\end{aligned}
$$

From (18), we conclude that (17) is of order sixth. □

3. Numerical Simulations

3.1. Numerical Examples.

In this section, we analyze several root-finding methods such as Newton's method (NM), Ostrowski's method (OM) [13], Liu's method (LiM) [14], Ren method (RM) [15], derivative-free iterative method (DFIM) in (11))-(15) and derivate-free iterative method (DFIM2) given in (17). We set the stopping criteria to be $|x_{n+1} - \theta| \leq 1.0 \times 10^{-15}$ or $|g(x_{n+1})| \leq 1.0 \times 10^{-15}$ and the maximum number of iterations allowed are 300. The following are five nonlinear functions used for the simulations.

(1) $g_1(x) = x^3 + 4x^2 - 10$
(2) $g_2(x) = \cos(x) - x$
(3) $g_3(x) = \sin(x)^2 - x^2 + 1$
(4) $g_4(x) = x^6 - 10x^3 + x^2 - x + 3$
(5) $g_5(x) = (x + 2)e^x - 1$

Table 1. Comparison of the number of iteration of iterative methods

g_i	x_0	Number of iterations							θ
		NM	OM	EM	LiM	RM	DFIM	DFIM2	
g_1	0.7	6	3	3	4	4	3	3	1.36523001341409684576
	0.9	5	3	3	4	4	3	3	
g_2	-0.1	5	3	3	3	3	3	3	0.73908513321516064166
	3.2	12	4	4	3	4	4	3	
g_3	3.8	6	3	4	36	$*$	3	3	1.40449164821534122604
	2.5	6	3	4	8	24	3	3	
g_4	2.0	5	3	3	4	4	3	3	0.65860484711814043676
	1.29	6	3	4	4	4	3	3	
g_5	-1.0	5	3	3	3	4	3	3	0.25753028543986076046
	-0.9	5	3	3	3	4	3	3	

Table 1 displays iterations gained to obtain the approximated root of the methods above. The simulations are done by providing two distinct initial guesses for each technique.

The simulation shows that DFIM2 is relatively favorable in terms of the speed of convergence. For all presented functions with different initial guesses, DFIM2 requires fewer iterations followed by DFIM. In the particular case of the function g_3, for the first initial guess, RM exceeds the tolerated maximum iteration (marked as $*$ in the table); meanwhile, DFIM2 and DFIM outperform the rest of the methods. In addition, NM is unfavorable except for one case of $g_3(x)$ with the first initial guess where LiM generates 36 iterates to converge.

Another comparison that can be pointed out is that the efficiency index of DFIM2 can compete with the rest of the methods. DFIM2 has a higher efficiency index compared to NM, EM, and DFIM (1.414, 1.442, and 1.414, respectively). Although it is not higher than the efficiency index of the rest of the studied methods (OM, LiM, and RM each have 1.587), the fact that our proposed method avoids calculation of derivatives helps solve non-linear equations better.

The following table displays the computational order of convergence of DFIM and DFIM2. It is evident that for the chosen initial guesses, the methods can maintain its computational order of convergence.

Table 2. Computational Order of Convergence of DFIM and DFIM2

	g_1		g_2		g_3		g_4		g_5	
	0.7	0.9	-0.1	3.2	3.8	2.5	2.0	1.29	-1.0	-0.9
DFIM	3.994	3.999	3.996	3.981	3.952	3.985	3.998	0.000	3.999	3.999
DFIM2	5.793	5.999	5.997	5.803	5.989	5.998	5.956	5.937	5.969	5.997

3.2. **Basins of Attraction.**

When comparing iterative methods, one often uses several initial guesses that are close enough to the roots to test the discussed methods on several functions. This results in a limitation on numerical simulations, and observation of the methods' behavior on the tested functions is also narrow. Hence, we cannot fully understand the behavior of the studied methods when tested at random points.

In this section, we discuss the behavior of our method to solve equation $h(z) = 0$ where $h : \mathbf{C} \longrightarrow \mathbf{C}$ is the complex plane through its basins of attraction. A better view of the behavior of the proposed method on a broader scale is provided by the basins of attraction. Many researchers have used basins of attraction as a means of measuring the efficiency of an iterative method, for example, see [16], [18], [17] [19], [20], [21], [22] and references therein. In addition to producing the basins of attraction, we also count a number of points that converge and diverge to better understand the figures.

We compare the method with several methods mentioned above by displaying the number of initial guesses that are convergent and divergent. Four functions are being tested for this simulation, and several colors are assigned for each root of the studied function. The figures of the basins of attraction are constructed by taking a uniform grid of $[-1, 1] \times [-1, 1] \subset \mathbf{C}$ resulting in 1000000 initial guesses to be tested. Each grid point is colored based on the root where it converges. We fix the tolerance error to be 10^{-15} and consider 50 significant digits.

To see the method's sensitivity to the initial guess, we restrict the iteration to 10 iterations only.

Four functions being tested are

(1) $h_1(z) = z^3 - z$, $z = \{-1.000, 0, 1.000\}$.
(2) $h_2(z) = z^3 - 1$, $z = \{-0.5 - 0.866i, -0.5 + 0.866i, 1\}$.
(3) $h_3(z) = z^4 - 1$, $z = \{-1, -i, i, 1\}$.
(4) $h_4(z) = z^5 - 1$, $z = \{-0.8090 - 0.5878i, -0.809 + 0.5878i, 0.309 - 0.951i, -0.309 + 0.951i, 0.309 + 0.951i, 1.000\}$.

TABLE 3. Comparison of number of divergent points of iterative methods in solving $h_1(z) = 0$ through $h_4(z) = 0$ in complex plane.

$h_1(z)$	Methods						
	NM	OM	EM	LiM	RM	DFIM	DFIM2
divergent	41708	108	0	11876	0	464	6068
%	4.17%	0.01%	0.00%	1.19%	0.00%	0.05%	0.61%
$h_2(z)$	Methods						
	NM	OM	EM	LiM	RM	DFIM	DFIM2
divergent	201190	2974	0	139632	152924	14606	106544
%	20.12%	0.003%	0.00%	13.96%	15.29%	1.46%	10.65%
$h_3(z)$	Methods						
	NM	OM	EM	LiM	RM	DFIM	DFIM2
divergent	438616	34168	1436	407282	144744	111884	287938
%	43.86%	3.42%	0.14%	40.73%	14.47%	11.19%	28.79%
$h_4(z)$	Methods						
	NM	OM	EM	LiM	RM	DFIM	DFIM2
divergent	471838	73182	18874	484850	0	182422	394374
%	47.18%	7.32%	1.89%	48.49%	0.00%	18.24%	39.44%

Table 3 displays the convergence of the initial guesses to the roots of tested functions from the studied iterative methods. On the first column of the table, $h_i(z)$ for $i = 1, 2, 3, 4$ denotes the function being tested, divergent marks several non-convergent points, and the "%" denotes the percentage of the non-convergent points.

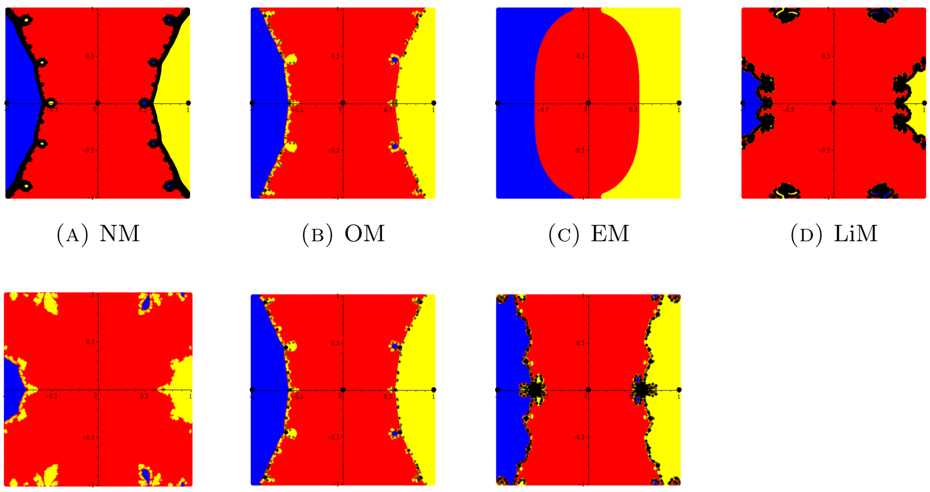
Of all the tested methods, NM is the most sensitive iterative method for all four tested functions. Table 3 shows that for $h_1(z)$, EM and RM successfully send all the initial guesses to converge while the convergence areas of DFIM and DFIM2 are bigger than NM and LiM. The basins of attraction of this function are given in Figure 1. As one can see in Figure 1, DFIM2 gives a better dynamic than NM and LiM, while EM and RM perform the best.

For $h_2(z)$, NM, LiM and RM are more sensitive to initial guesses compared to DFIM2. This can be observed from Table 3. In addition, the basins of attraction of NM, LiM and RM in Figure 2 show a more chaos area than the rest of the studied methods. In this case, EM has the largest convergence area since all of the initial guesses converge.

In the case of $h_3(z)$, Table 3 shows that DFIM2 has a smaller number of divergent points than NM and LiM. The basins of attraction in Figure 3 display the chaos of basins of attraction in NM and LiM. The basins of attraction for $h_4(z)$ in Figure 4, show that the convergent area of DFIM2 is bigger than NM and LiM. It can also be seen that the number of divergent points of DFIM2 is fewer than NM and LiM.

4. Conclusion

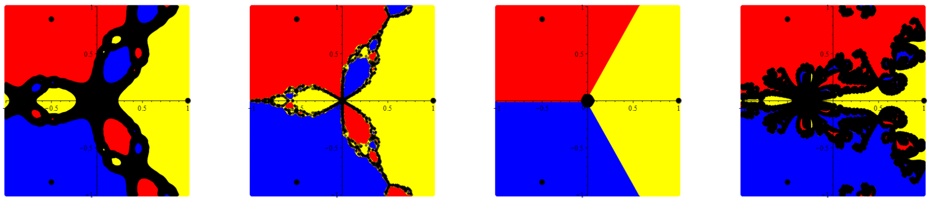
We have developed a new derivative-free three-step iterative method, employing forward difference, hyperbola, and divided difference to approximate the derivatives. Our new method has a higher efficiency index (1.565) than the original method. The convergence of the method has been studied, and its order has been proven. We provided numerical comparisons with several iterative methods as well as dynamic comparisons through the basins of attraction. Numerical simulations have shown that our proposed method is favorable. The dynamics of basins of attraction demonstrate the advantages, where they have been shown to be not as sensitive to initial guesses as other iterative methods.



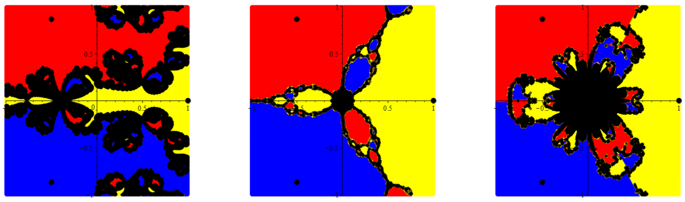
(A) NM (B) OM (C) EM (D) LiM

(E) RM (F) DFIM (G) DFIM2

Figure 1. Basins of attraction of iterative methods for $h_1(z) = z^3 - z$

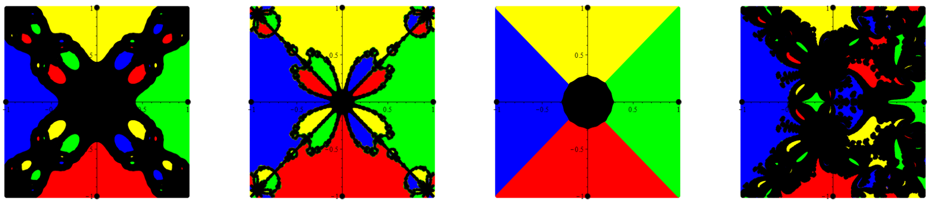


(A) NM (B) OM (C) EM (D) LiM

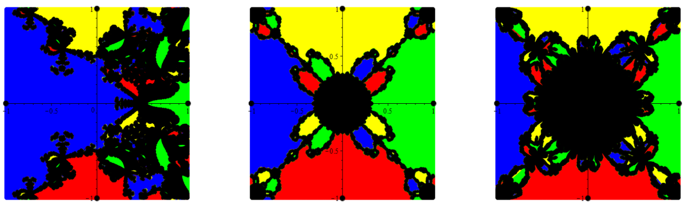


(E) RM (F) DFIM (G) DFIM2

FIGURE 2. Basins of attraction of iterative methods for $h_2(z) = z^3 - 1$

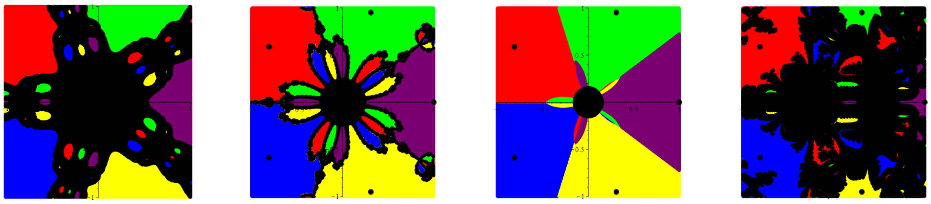


(A) NM (B) OM (C) EM (D) LiM

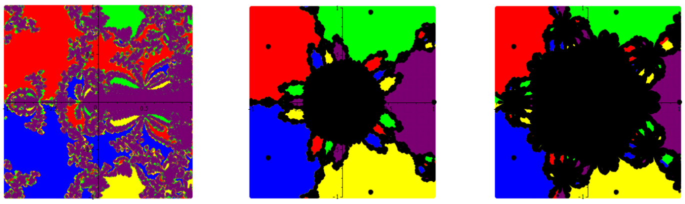


(E) RM (F) DFIM (G) DFIM2

FIGURE 3. Basins of attraction of iterative methods for $h_3(z) = z^4 - 1$



(A) NM (B) OM (C) EM (D) LiM



(E) RM (F) DFIM (G) DFIM2

FIGURE 4. Basins of attraction of iterative methods for $h_4(z) = z^5 - 1$

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