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GLOBAL STABILITY OF SPATIO-TEMPORAL MODEL WITH QUARANTINE AND VACCINATION

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Abstract. In this paper, we suggest a spatio-temporal epidemic model for coronavirus. Our model will be represented by a system of six partial differential nonlinear equations that describe the dynamics of susceptible, exposed, infected, quarantined, removed, and vaccinated individuals. We will start the study of this model by presenting some results of the existence and uniqueness to the solution of our suggested model. By using the method of next-generation matrix, we obtain the basic reproduction number. The model has one disease-free equilibrium point and another endemic steady state. The global stability of these steady states is proved by using some Lyapunouv functions. Finally, different numerical simulations are given to confirm our results given in the theoretical part of the paper.

Key words: SEIQRV model; COVID-19; Reaction-diffusion; Global Stability.

1. INTRODUCTION

The mathematical modeling has a vital role in describing the mechanisms of transmission of several infectious diseases such as coronavirus (COVID-19) [1], hepatitis B virus (HBV) [2], hepatitis C virus (HCV) [3], human immunodeficiency virus (HIV) [4] and many others. Since the COVID-19 pandemic has been known for its dangerous spread around the world [5], many mathematical models have been developed to understand the infectious disease fatality well. As the order of

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magnitude, last year, it was observed more than 756.291.327 individuals have been infected by COVID-19, and around 6.841.640 are dead as a consequence of it [6].

The first susceptible-infected-removed called (SIR) model was suggested by Kermack and McKendrick in [7]. In several cases, the infection takes a long time to appear in the infected person for certain infectious diseases like coronavirus. Therefore, another class for the exposed individual will be added to the classical SIR model for a better description of the dynamic infection. This new part of the model will be abbreviated by $SEIR$, with E designed for the exposed individuals. Some infected diseases such as COVID-19 have been studied by using the $SEIR$ mathematical model [8, 9, 10, 11, 12, 13]. Recently, Meskaf et al. in [14] suggest an $SEIR$ epidemic model. They started the analysis of this model by giving some results of the existence, positivity, and boundedness of all solutions of their suggested model, and they have shown the global stability of all equilibrium points; in order to value their theoretical results, they suggested some numerical simulations in the last part of this work. More recently, Yaagoub et al. [15] proposed an $SEIR$ epidemic model with a quarantine strategy. The authors started the model analysis by giving the well-posedness results of all model solutions. They also discussed the global stability of steady states. To value their theoretical findings. Finally, To show the effect of the quarantine on the infection and to value their theoretical findings, they gave some numerical simulations in the last part of their work.

The vaccination strategy is one the most effective solutions to fight most infectious diseases like COVID-19 [16]. Some work includes this strategy to analyze the dynamics of some infectious diseases; these models are called $SVEIR$, with V representing the vaccinated population. In mathematical literature, several works use these $SVEIR$ models to for well analyzed the dynamics of infections diseases [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Recently, in [27], Baba et al. present a $SVEIR$ model; the authors stated their studies of the suggested model by presenting some theorems of the model solutions' existence, positivity, and boundedness. They gave some theorems on the global stability of the different equilibria. Finally, some numerical simulations were given in their work to value the theoretical findings. More recently, in the same context, Yaagoub and Allali in [28] suggest a three-strain $SEIQRV$ model; the authors established the different results of existence, positivity, and boundedness of all solutions they discussed after the global stability of the equilibria, and they finished this work by giving some numerical simulations. The $SEIQRV$ models are exciting in describing the dynamics of some infectious diseases, but another more interesting model exists in the mathematical literature. These models are called $SEIQRV$, with Q representing the quarantined individuals. Many models study the dynamic of infectious diseases using the $SEIQRV$ models [29, 30, 31, 32, 33]. Recently, in [34], the authors suggest an $SEIQRV$ to describe the spread of coronavirus in the Kingdom of Saudi Arabia; the authors started the studies of these models by giving some results of the existence and well-posedness of all solutions and they calculated the basic reproduction number. Finally, they gave some numerical simulations using actual data from COVID-19 cases. More recently, Malik et al., in [35], suggest a coronavirus model with quarantine and vaccination. The authors started their studies by giving some results of the existence and well-posedness of the solutions, and by using the next-generation technique, they gave the basic reproduction number of the model and gave some numerical simulations in the last part to value their theoretical findings.

All previous models depend on time only. However, we can always define another space variable to describe species distribution better. In the mathematical literature, some models based on partial differential systems exactly are reaction-diffusion systems; they describe how concentration or density is distributed in actual space under the influence of two processes: the interactions of species on the one hand and the diffusion that causes the spread of these species in space, on the other hand, [36]. Some works use the reaction-diffusion models to describe the spread of infectious diseases [37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. Recently, Jinhu Xu suggested an $SVEIR$ epidemic model with reaction-diffusion. The author established the different results of all solutions' existence, positivity, and boundedness. The primary reproduction number is calculated by using the next-generation matrix. The author also gave some results on the global stability of steady states. Motivated by the previous works, we aim to include the quarantine strategy in the earlier works and consider the diffusion in all model compartments except the quarantine compartment since its individuals are assumed not to move outside a specific place. Hence, we will continue the investigation of reaction-diffusion systems by suggesting a new $SEIQRV$ model. The following partial differential equations model will represent our model:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S + \gamma - \alpha S I - (\gamma_1 + \delta) S, \\ \dfrac{\partial E}{\partial t} = d_E \Delta E + \alpha S I - (\gamma_2 + \delta) E, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \gamma_2 E - (\gamma_3 + \gamma_4 + \delta) I, \\ \dfrac{\partial Q}{\partial t} = \gamma_3 I - (\gamma_5 + \delta) Q, \\ \dfrac{\partial R}{\partial t} = d_R \Delta R + \gamma_4 I + \gamma_5 Q - \delta R, \\ \dfrac{\partial V}{\partial t} = d_V \Delta V + \gamma_1 S - \delta V. \end{cases} \tag{1}$$

S represents the susceptible individuals, E the exposed individuals, I the infected individuals, Q the quarantined individuals, R the removed individuals, and V the vaccinated individuals. The model dynamics are illustrated in Figure 1. The model parameters are given in Table 1.

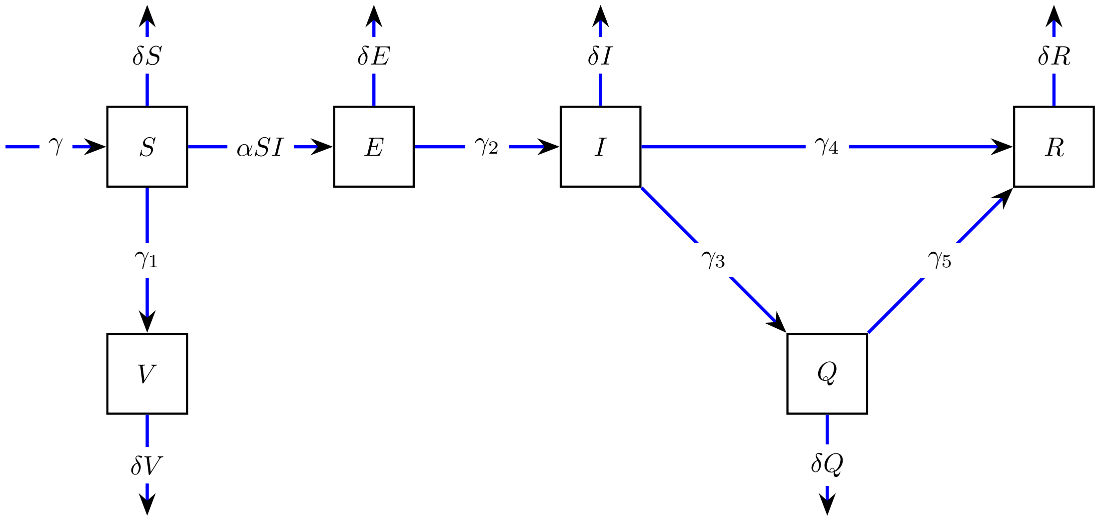

FIGURE 1. The diagram of the $SEIQRV$ model (1)

TABLE 1. Description of the $SEIQRV$ model parameter's

Parameters	Description
d_S	The coefficient of the diffusion of the susceptible individuals
d_E	The coefficient of the diffusion of the exposed individuals
d_I	The coefficient of the diffusion of the infected individuals
d_R	The coefficient of the diffusion of the removed individuals
d_V	The coefficient of the diffusion of of the vaccinated individuals
γ	Recruitment rate
δ	Mortality rate
γ_1	The vaccinated individuals rate
γ_2	Incubation period
γ_3	Quarantined rate
γ_4	Removal rate of infected individuals
γ_5	Removal rate of quarantined individuals

In this work, we consider the following initial conditions:

$$S(x,0) = \varphi_1(x) \geq 0,\ E(x,0) = \varphi_2(x) \geq 0,\ I(x,0) = \varphi_3(x) \geq 0, \tag{2}$$

$$Q(x,0) = \varphi_4(x) \geq 0,\ R(x,0) = \varphi_5(x) \geq 0,\ V(x,0) = \varphi_6(x) \geq 0,\ \forall x \in \bar{\Omega}. \tag{3}$$

With Ω is a bounded set in $\mathbb{R}^n$.
and the Neumann boundary conditions of the system (1) are given by:

$$\frac{\partial S}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial Q}{\partial n} = \frac{\partial R}{\partial n} = \frac{\partial V}{\partial n}, \; on \; \partial\Omega \times [0, +\infty[\,. \tag{4}$$

The initial data φ_i for $i = 1, ...6$ are non-negative functions. $\frac{\partial S}{\partial n}, \frac{\partial E}{\partial n}, \frac{\partial Q}{\partial n}, \frac{\partial R}{\partial n}$ and $\frac{\partial V}{\partial n}$ represent respectively the normal derivatives of the compartments S, E, Q, R and V on $\partial\Omega$.

The present paper is organized as follows. In Section 2, some results of the existence, positivity, and boundedness of solutions will be given. The global stability of all equilibria is provided in section 3. In Section 4, some numerical simulations are suggested to value our theoretical results. The last Section concludes the present work.

2. Existence, positivity, and boundedness of solutions

In this section, we will show the solutions of the system (1) remain bounded and non-negative.

Proposition 2.1. *For any given initial conditions verified the condition* (4)*, the system* (1) *has a unique solution for the system* (1) *defined on* $[0, +\infty[$*. In addition, this solution remains non-negative and bounded for all* $t > 0$*.*

Proof. Let $X = C(\bar{\Omega}) \times C(\bar{\Omega})$ the Banach space. So the system (1) can rewrite in the the form

$$\begin{cases} U'(t) = AU + G(U(t)), \;\; t > 0, \\ U(0) = U_0 \in X, \end{cases}$$

$U = (S, E, I, Q, R, V)^T$, $U_0 = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)^T$,
$AU(t) = (d_S\Delta S, d_E\Delta E, d_I\Delta I, 0, d_R\Delta R, d_V\Delta V)^T$ and

$$G(U(t)) = \begin{pmatrix} \gamma - \alpha SI - (\gamma_1 + \delta)S \\ \alpha SI - (\gamma_2 + \delta)E \\ \gamma_2 E - (\gamma_3 + \gamma_4 + \delta)I \\ \gamma_3 I - (\gamma_5 + \delta)Q \\ \gamma_4 I + \gamma_5 Q - \delta R \\ \gamma_1 S - \delta V \end{pmatrix}.$$

It is easy to see that the function G is locally Lipschitz in the set X, so we can conclude that the system (1) has one local solution on $[0, T_m[$ where T_m is the time maximal for the existence of the system (1) solution. Now we will show that this solution remains non-negative, for that the system (1) can be rewritten in the form

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = G_1(S,\, E,\, I,\, Q,\, R,\, V), \\ \dfrac{\partial E}{\partial t} - d_S \Delta E = G_2(S,\, E,\, I,\, Q,\, R,\, V), \\ \dfrac{\partial I}{\partial t} - d_S \Delta I = G_3(S,\, E,\, I,\, Q,\, R,\, V), \\ \dfrac{\partial Q}{\partial t} = G_4(S,\, E,\, I,\, Q,\, R,\, V), \\ \dfrac{\partial R}{\partial t} - d_R \Delta R = G_5(S,\, E,\, I,\, Q,\, R,\, V), \\ \dfrac{\partial V}{\partial t} - d_V \Delta V = G_6(S,\, E,\, I,\, Q,\, R,\, V). \end{cases} \tag{5}$$

It is clear that the functions $G_i(S,\, E,\, I,\, Q,\, R,\, V) \in C(\bar{\Omega})$ for $i = 1, ..., 6$ and satisfying

$$\begin{cases} G_1(0,\, E,\, I,\, Q,\, R,\, V) = \gamma \geq 0, \\ G_2(S,\, 0,\, I,\, Q,\, R,\, V) = \alpha S I \geq 0, \\ G_3(S,\, E,\, 0,\, Q,\, R,\, V) = \gamma_2 E \geq 0, \\ G_4(S,\, E,\, I,\, Q,\, R,\, V) = \gamma_3 I \geq 0, \\ G_5(S,\, E,\, I,\, Q,\, R,\, V) = \gamma_4 I + \gamma_5 Q \geq 0, \\ G_6(S,\, E,\, I,\, Q,\, R,\, V) = \gamma_1 S \geq 0. \end{cases} \tag{6}$$

Then, we deduce the non-negativity of all solutions. Finally, we show the boundedness of the solution, from the system (1) and (4), we have

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S \leq \gamma - \delta S, \\ \dfrac{\partial S}{\partial n} = 0, \\ S(x,0) = \varphi_1(x) = \parallel \varphi_1 \parallel_\infty = \max_{x\in\bar{\Omega}} \varphi_1(x). \end{cases} \tag{7}$$

According to the principle of comparison as in [47], we will have have

$$S(x,t) \leq S_1(t), \tag{8}$$

with $S_1(t) = \varphi_1(x)e^{-\delta t} + \dfrac{\gamma}{\delta}\left(1 - e^{-\delta t}\right)$ is the solution of the problem:

$$\begin{cases} \dfrac{\partial S_1}{\partial t} \leq \gamma - \delta S_1, \\ S_1(0) = \parallel \varphi_1 \parallel_\infty. \end{cases} \tag{9}$$

Since

$$S_1(t) \leq \max\{\frac{\gamma}{\delta}, \parallel \varphi_1 \parallel_\infty\}, \;\; \forall t \in [0, +\infty[\,, \tag{10}$$

we will have

$$S(x,t) \leq \max\{\frac{\gamma}{\delta}, \parallel \varphi_1 \parallel_\infty\}, \;\; \forall (x,t) \in \bar{\Omega} \times [0, T_m[\,.$$

to show L^∞ uniform boundedness of the other variables, just prove the L^1 uniform boundedness of this variables.
Let the total population

$$N = S + E + I + Q + R + V.$$

Since

$$\frac{\partial S}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial Q}{\partial n} = \frac{\partial R}{\partial n} = \frac{\partial V}{\partial n} = 0,$$

and

$$\frac{\partial N}{\partial t} - \Delta(d_S S + d_E E + d_I I + d_Q Q + d_R R + d_V V) \leq \gamma - \delta N,$$

we will have

$$\frac{\partial}{\partial t}\left(\int_\Omega N dx\right) \leq mes(\Omega)\gamma - \delta \int_\Omega N dx.$$

So

$$\int_\Omega N dx \leq mes(\Omega) \max\{\frac{\gamma}{\delta}, \parallel \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 \parallel_\infty\},$$

which implies

$$\sup_{t \geq 0} \int_\Omega (E(x,t) + I(x,t) + Q(x,t) + R(x,t) + V(x,t))dx \leq M, \tag{11}$$

with

$$M = mes(\Omega) \max\{\frac{\gamma}{\delta}, \parallel \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 \parallel_\infty\}, \tag{12}$$

Then, we deduce that there exists a non-negative constant M^* depends on M and $\parallel \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 \parallel$ such that

$$\sup_{t \geq 0} \parallel E(.,t) + I(.,t) + Q(.,t) + R(.,t) + V(.,t) \parallel_\infty \leq M^*$$

We have shown that $S(x,t)$, $E(x,t)$, $I(x,t)$, $Q(x,t)$, $R(x,t)$ and $V(x,t)$ are L^∞ bounded on $\bar{\Omega} \times [0, T_m[$. Therefore, based on the standard theory of semi-linear parabolic systems [48]. We have $T_m = \infty$. Which ends the demonstration. □

3. Basic reproduction number, equilibria, and global stability

This part will show that the problem has a primary reproduction number given by the next-generation matrix method, like in [49, 50]. We will also prove that the model has a free equilibrium and another endemic equilibrium point. Some global stability results of equilibria will be given in this section.

3.1. **Basic reproduction number.** In the case of the absence of spatial dependence, the system (1) we have:

$$\begin{cases} \dfrac{\partial S}{\partial t} = \gamma - \alpha SI - (\gamma_1 + \delta)S, \\ \dfrac{\partial E}{\partial t} = \alpha SI - (\gamma_2 + \delta)E, \\ \dfrac{\partial I}{\partial t} = \gamma_2 E - (\gamma_3 + \gamma_4 + \delta)I, \\ \dfrac{\partial Q}{\partial t} = \gamma_3 I - (\gamma_5 + \delta)Q, \\ \dfrac{\partial R}{\partial t} = \gamma_4 I + \gamma_5 Q - \delta R, \\ \dfrac{\partial V}{\partial t} = \gamma_1 S - \delta V. \end{cases} \tag{13}$$

Let $E_0 \left(\dfrac{\gamma}{\delta + \gamma_1}, 0, 0, 0, 0, \dfrac{\gamma_1 \gamma}{\delta(\gamma_1 + \delta)} \right)$ the disease free equilibrium of the system (1).
Let

$$F = \begin{pmatrix} 0 & \dfrac{\alpha\gamma}{\delta + \gamma_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

and

$$V = \begin{pmatrix} \gamma_2 + \delta & 0 & 0 \\ -\gamma_2 & \gamma_3 + \gamma_4 + \delta & 0 \\ 0 & -\gamma_3 & \gamma_5 + \delta. \end{pmatrix}. \tag{15}$$

Using the next-generation method to calculate the reproduction number, we will have

$$R_0 = \frac{\gamma\alpha\gamma_2}{(\delta + \gamma_2)(\gamma_1 + \delta)(\delta + \gamma_3 + \gamma_4)}, \tag{16}$$

we denote

$$a = \delta + \gamma_2,\ b = \gamma_1 + \delta,\ c = \delta + \gamma_3 + \gamma_4, \tag{17}$$

then,

$$R_0 = \frac{\gamma\alpha\gamma_2}{abc}. \tag{18}$$

The endemic equilibrium $E_1 = (S^*, E^*, I^*, Q^*, R^*, V^*)$ is given by:

$$S^* = \frac{1}{\gamma_2 a} \left(\gamma_2 - (\delta^2 + \delta c + \gamma_2(c - \delta))I^* \right),\ E^* = \frac{1}{c\gamma_2} I^*, \tag{19}$$

$$I^* = \frac{b}{\alpha} R_0 - \frac{\delta^3 + ((c-\delta) + (b-\delta) + (a-\delta))\delta^2 + ((c - \delta + \gamma_2)\gamma_1 + \gamma_2(c-\delta))\delta + \gamma_1\gamma_2(c-\delta)}{\alpha ac}, \tag{20}$$

$$Q^* = \frac{\gamma_3}{\delta + \gamma_5} I^*,\ R^* = \frac{1}{\delta + \gamma_5} \left(\gamma_5\gamma_3 + \gamma_4(\delta + \gamma_5) \right) I^* \tag{21}$$

$$V^* = \frac{\gamma_1}{\delta\gamma_2 b} \left(\gamma\gamma_2 - \left(\delta^2 + \delta c + \gamma_2(c - \delta) \right) I^* \right). \tag{22}$$

Theorem 3.1. *(i) The disease free equilibrium E_0 is always exists.*

(ii) The endemic steady state E_1 exists in the case $R_0 > 1$, $\delta \leq \frac{c}{2}$ and $\gamma_2 \leq \frac{\delta^2+\delta c}{2\delta - c}$.

3.2. **Global stability.**

Theorem 3.2. *The disease-free equilibrium E_0 is globally asymptotically stable for $R_0 \leq 1$.*

Proof. Let the following Lyapunov functional

$$L_0(t) = S_0\left(\frac{S(t)}{S_0} - \ln\left(\frac{S(t)}{S_0}\right) - 1\right) + E(t) + \frac{a}{\gamma_2}I(t) + Q(t) + V_0\left(\frac{V(t)}{V_0} - \ln\left(\frac{V(t)}{V_0}\right) - 1\right) + R(t). \tag{23}$$

The time derivative of L_0 is given by

$$\frac{dL_0}{dt} = \nabla L_0(u)F(u) \tag{24}$$

$$= \left(1 - \frac{S_0}{S}\right)(\gamma - \alpha SI - bS) + \alpha SI - aE + \frac{a}{\gamma_2}(\gamma_2 E - cI) + \gamma_3 I - (\gamma_5+\delta)Q \tag{25}$$

$$+ + \left(1 - \frac{V_0}{V}\right)(\gamma_1 S - \delta V)\,\gamma_4 I + \gamma_5 Q - \delta R,$$

$$\leq \gamma - bS - \gamma\frac{S_0}{S} + \alpha S_0 I + bS_0 - \frac{ac}{\gamma_2}I - (\gamma_5+\delta)Q + \gamma_1 S - \delta V - \gamma_1 S\frac{V_0}{V} \tag{26}$$

$$+ \delta V_0 + \gamma_3 I + \gamma_4 I + \gamma_5 Q - \delta R, \tag{27}$$

as $S_0 = \frac{\gamma}{\delta}$, therefore

$$\frac{dL_0}{dt} \leq \delta S_0\left(2 - \frac{S}{S_0} - \frac{S_0}{S}\right) + \gamma_1 S_0\left(3 - \frac{S_0}{S} - \frac{V}{V_0} - \frac{S}{S_0}\frac{V_0}{V}\right) + \frac{ac}{\gamma_2}I\,(R_0 - 1). \tag{28}$$

Since the geometric mean is lesser than or equal to the arithmetic mean, we will have

$$2 - \frac{S}{S_0} - \frac{S_0}{S} \leq 0, \tag{29}$$

and

$$3 - \frac{S_0}{S} - \frac{V}{V_0} - \frac{S}{S_0}\frac{V_0}{V} \leq 0. \tag{30}$$

If $R_0 \leq 1$, then $\dot{L}_0$, therefore E_0 is globally asymptotically stable. Then the Lyapunov function for diffusion system (1) at E_0 is given by:

$$\mathcal{L}_0 = \int_\Omega L_0(x,t)dx. \tag{31}$$

We have

$$d_S \int_\Omega \nabla S.\nabla \left(\frac{\partial L_0}{\partial S}\right) dx = d_S S_0 \int_\Omega \frac{|\nabla S|^2}{S^2} dx \geq 0, \tag{32}$$

$$d_E \int_\Omega \nabla E.\nabla \left(\frac{\partial L_0}{\partial E}\right) dx = 0 \geq 0, \tag{33}$$

$$d_I \int_\Omega \nabla I.\nabla \left(\frac{\partial L_0}{\partial I}\right) dx = 0 \geq 0, \tag{34}$$

$$0 \int_\Omega \nabla Q.\nabla \left(\frac{\partial L_0}{\partial Q}\right) dx = 0 \geq 0, \tag{35}$$

$$d_V \int_\Omega \nabla V.\nabla \left(\frac{\partial L_0}{\partial V}\right) dx = d_V V_0 \int_\Omega \frac{|\nabla V|^2}{V^2} dx \geq 0, \tag{36}$$

and

$$d_R \int_\Omega \nabla R.\nabla \left(\frac{\partial L_0}{\partial R}\right) dx = 0 \geq 0. \tag{37}$$

From Theorem 1 (i) in [51], we can deduce that $\mathcal{L}_0$ is a Lyapunov functional for the diffusion system (1) at E_0 when $R_0 \leq 1$. □

For the global stability of E_1, we assume that the point satisfies the conditions:

$$\left(\frac{I}{I^*} - 1\right)\left(\frac{I^*}{I} - \frac{R^*}{R}\right) \leq 0, \forall I,\, R \geq 0 \quad (H_1), \tag{38}$$

$$\left(\frac{R^*}{R} - 1\right)\left(\frac{R}{R^*} - \frac{Q}{Q^*}\right) \leq 0, \forall R,\, Q \geq 0 \quad (H_2), \tag{39}$$

and

$$\left(\frac{Q^*}{Q} - 1\right)\left(\frac{Q}{Q^*} - \frac{I}{I^*}\right) \leq 0, \forall R,\, Q \geq 0 \quad (H_3). \tag{40}$$

Theorem 3.3. *The endemic equilibrium E_1 is globally asymptotically stable for $R_0 > 1$.*

Proof. Let the following Lyapunov functional

$$\begin{aligned} L_1(t) = {} & S^*\left(\frac{S}{S^*} - \ln\left(\frac{S}{S^*}\right) - 1\right) + E^*\left(\frac{E}{E^*} - \ln\left(\frac{E}{E^*}\right) - 1\right) + \frac{a}{\gamma_2} I^*\left(\frac{I}{I^*} - \ln\left(\frac{I}{I^*}\right) - 1\right) \\ & + Q^*\left(\frac{Q}{Q^*} - \ln\left(\frac{Q}{Q^*}\right) - 1\right) + V^*\left(\frac{V}{V^*} - \ln\left(\frac{V}{V^*}\right) - 1\right) + R^*\left(\frac{R}{R^*} - \ln\left(\frac{R}{R^*}\right) - 1\right). \end{aligned} \tag{41}$$

The time derivative of L_1 is given by

$$\frac{dL_1}{dt} = \nabla L_1(u)F(u) \tag{42}$$

$$= \left(1-\frac{S^*}{S}\right)(\gamma - \alpha SI - bS) + \left(1-\frac{E^*}{E}\right)(\alpha SI - aE) + \frac{a}{\gamma_2}\left(1-\frac{I^*}{I}\right)(\gamma_2 E - cI)$$
$$+ \left(1-\frac{Q^*}{Q}\right)(\gamma_3 I - (\gamma_5+\delta)Q) + \left(1-\frac{V^*}{V}\right)(\gamma_1 S - \delta V) + \left(1-\frac{R^*}{R}\right)(\gamma_4 I + \gamma_5 Q - \delta R), \tag{43}$$

$$\leq \gamma - bS - \gamma\frac{S^*}{S} + \alpha S^* I + bS^* - \alpha SI\frac{E^*}{E} + aE^* - \frac{ac}{\gamma_2}I - aE\frac{I^*}{I} + \frac{ac}{\gamma_2}I^* + \gamma_4 I - \delta Q$$
$$- \gamma_3 I\frac{Q^*}{Q} + (\gamma_5+\delta)Q^* + \gamma_1 S - \delta V - \gamma_1 S\frac{V^*}{V} + \delta V^* + \gamma_4 I - \delta R - \gamma_4 I\frac{R^*}{R} - \gamma_5\frac{R^*}{R} + \delta R^*. \tag{44}$$

We have

$$\begin{cases} \gamma = \alpha S^* I^* + bS^* = aE^* + bS^*, \\ \dfrac{I^*}{E^*} = \dfrac{\gamma_2}{c},\ \dfrac{Q^*}{I^*} = \dfrac{\gamma_3}{\gamma_5+\delta}, \\ \delta R^* = \gamma_4 I^* + \gamma_5 Q^*,\ \gamma_1 S^* = \delta V^*. \end{cases} \tag{45}$$

Therefore

$$\frac{dL_1}{dt} \leq \delta S^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \gamma_1 S^*\left(3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{S}{S^*}\frac{V^*}{V}\right) + aE^*\left(3 - \frac{S^*}{S} - \frac{SIE^*}{S^*I^*E} - \frac{EI^*}{E^*I}\right)$$
$$+ \gamma_4 I^*\left(\frac{I}{I^*}-1\right)\left(\frac{I^*}{I} - \frac{R^*}{R}\right) + \gamma_5 Q^*\left(\frac{R^*}{R}-1\right)\left(\frac{R}{R^*} - \frac{Q}{Q^*}\right) + \gamma_4 I^*\left(\frac{Q^*}{Q}-1\right)\left(\frac{Q}{Q^*} - \frac{I}{I^*}\right)$$
$$- \frac{ac}{\gamma_2}I. \tag{46}$$

Since the geometric mean is lesser than or equal to the arithmetic mean, we will have

$$2 - \frac{S}{S^*} - \frac{S^*}{S} \leq 0, \tag{47}$$

$$3 - \frac{S^*}{S} - \frac{V}{V^*} - \frac{S}{S^*}\frac{V^*}{V} \leq 0, \tag{48}$$

and

$$3 - \frac{S^*}{S} - \frac{SIE^*}{S^*I^*E} - \frac{EI^*}{E^*I} \leq 0. \tag{49}$$

And as that the point E_1 satisfies the conditions (H_1), (H_2) and (H_3), we deduce if $R_0 > 1$, then $\dot{L}_1 \leq 0$, therefore E_1 is globally asymptotically stable. Then the Lyapunov function for diffusion system (1) at E_1 is given by:

$$\mathcal{L}_1 = \int_\Omega L_1(x,t)dx. \tag{50}$$

We have

$$d_S\int_\Omega \nabla S.\nabla\left(\frac{\partial L_1}{\partial S}\right)dx = d_S S^*\int_\Omega \frac{\mid \nabla S \mid^2}{S^2}dx \geq 0, \tag{51}$$

$$d_E \int_\Omega \nabla E.\nabla \left(\frac{\partial L_1}{\partial E}\right) dx = d_E E^* \int_\Omega \frac{\mid \nabla E \mid^2}{E^2} dx \geq 0, \tag{52}$$

$$d_I \int_\Omega \nabla I.\nabla \left(\frac{\partial L_1}{\partial I}\right) dx = \frac{a}{\gamma_2} d_I I^* \int_\Omega \frac{\mid \nabla I \mid^2}{I^2} dx \geq 0, \tag{53}$$

$$0 \int_\Omega \nabla Q.\nabla \left(\frac{\partial L_1}{\partial Q}\right) dx = 0 \geq 0, \tag{54}$$

$$d_V \int_\Omega \nabla V.\nabla \left(\frac{\partial L_1}{\partial V}\right) dx = d_V V^* \int_\Omega \frac{\mid \nabla V \mid^2}{V^2} dx \geq 0, \tag{55}$$

and

$$d_R \int_\Omega \nabla R.\nabla \left(\frac{\partial L_1}{\partial R}\right) dx = d_R R^* \int_\Omega \frac{\mid \nabla R \mid^2}{R^2} dx \geq 0. \tag{56}$$

For the Theorem 1 (i) in [51], we can deduce that $\mathcal{L}_1$ is a Lyapunov functional for the diffusion system (1) at E_1 when $R_0 > 1$. □

4. Numerical simulations

In this section, we suggest to perform numerical simulations to value our theoretical results. These numerical simulations are performed using MATLAB software. Our problem (1) is solved by using the following numerical scheme:

$$\begin{aligned} S_{i,j+1} = S_{i,j} &+ ((h_t \times dS)/(h_x \times h_x)) \times (S(i+1,j) - 2 \times S(i,j) + S(i-1,j)) \\ &+ \gamma \times h_t - h_t \times \alpha \times S(i,j) \times I(i,j) - h_t \times (\gamma_1 + \delta) \times S(i,j), \end{aligned}$$

$$\begin{aligned} E_{i,j+1} = E_{i,j} &+ ((h_t \times dE)/(h_x \times h_x)) \times (E(i+1,j) - 2 \times E(i,j) + E(i-1,j)) \\ &+ \alpha \times S(i,j) \times I(i,j) \times h_t - h_t \times (\gamma_2 + \delta) \times E(i,j), \end{aligned}$$

$$\begin{aligned} I_{i,j+1} = I_{i,j} &+ ((h_t \times dI)/(h_x \times h_x)) \times (I(i+1,j) - 2 \times I(i,j) + I(i-1,j)) \\ &\gamma_2 \times E(i,j) \times h_t - h_t \times (\gamma_3 + \gamma_4 + \delta) \times I(i,j), \end{aligned}$$

$$Q_{i,j+1} = Q_{i,j} + \gamma_3 \times Q(i,j) \times h_t - h_t \times (\gamma_5 + \delta) \times Q(i,j),$$

$$\begin{aligned} R_{i,j+1} = R_{i,j} &+ ((h_t \times dR)/(h_x \times h_x)) \times (R(i+1,j) - 2 \times R(i,j) + R(i-1,j)) \\ &\gamma_4 \times I(i,j) \times h_t - h_t \times \gamma_5 \times Q(i,j) - h_t \times \delta \times R(i,j), \end{aligned}$$

and

$$\begin{aligned} V_{i,j+1} = V_{i,j} &+ ((h_t \times dV)/(h_x \times h_x)) \times (V(i+1,j) - 2 \times V(i,j) + V(i-1,j)) \\ &\gamma_1 \times S(i,j) \times h_t - h_t \times \delta \times V(i,j). \end{aligned}$$

This problem is solved in $[0,4] \times [0,100]$. Where $f_{i,j} = f(x_i, t_j)$. f designates a variable of the problem (1) with $t_j = t(j-1) + h_t$, for $j = 1, ..., M'$ and $x_i = x(i-1) + h_x$, for $i = 1, ..., N'$. h_x and h_t are the space and time steps. The initial conditions are: $S_0 = 4.5$; $E_0 = 1$; $I_0 = 1$; $Q_0 = 1$; $R_0 = 1$; $V_0 = 6.5$.

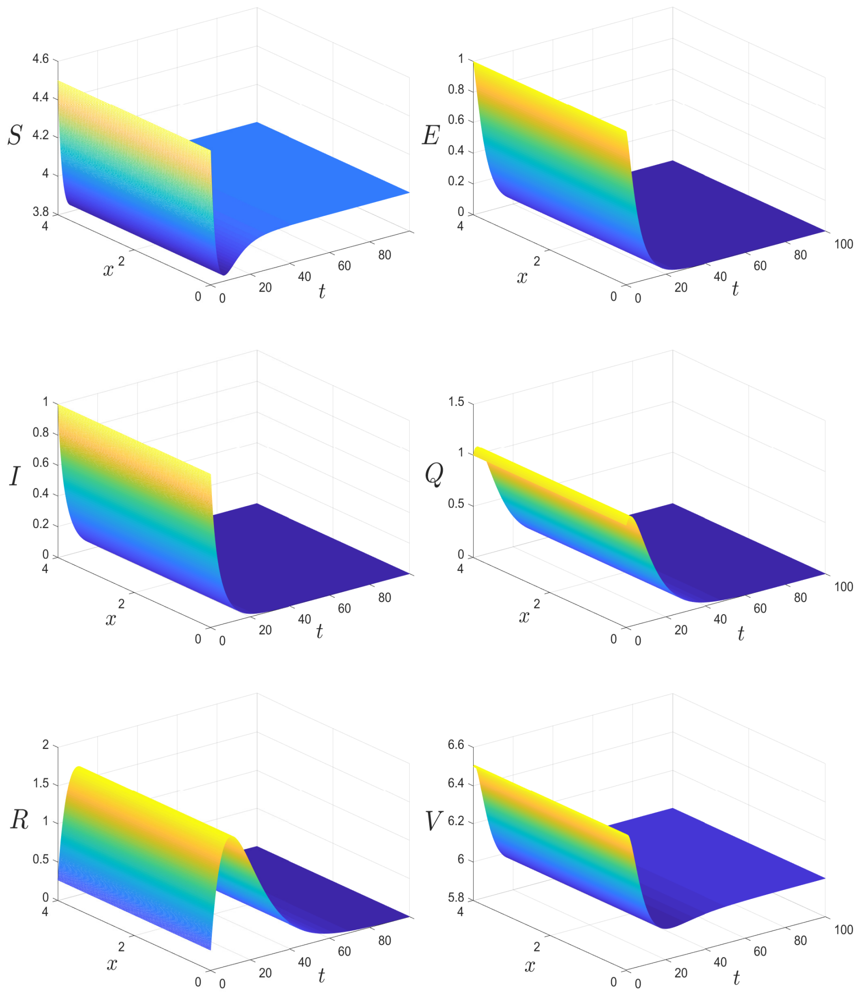

FIGURE 2. Stability of the free equilibrium point E_0

Figure 2 shows the evolution of the infection of the different compartments for the following parameters $d_S = 0.01$; $d_E = 0.01$; $d_I = 0.01$; $d_R = 0.01$; $d_V = 0.01$; $\gamma = 1$; $\alpha = 0.05$; $\gamma_1 = 0.15$; $\delta = 0.1$; $\gamma_2 = 0.2$; $\gamma_3 = 0.2$; $\gamma_4 = 0.2$; $\gamma_5 = 0.01$. We can observe that all curves represent the compartments S, E, I, Q, R and V converge to the point $(4, 0, 0, 0, 0, 6)$. The basic reproduction number for this equilibrium point with these chosen parameters is $R_0 = 0.2667$, which is less than unity; this basic reproduction number value satisfies the condition of Theorem 3.2 in regards to the global stability of the disease-free equilibrium point. These numerical results in this figure coincide with the theoretical results in the analysis part.

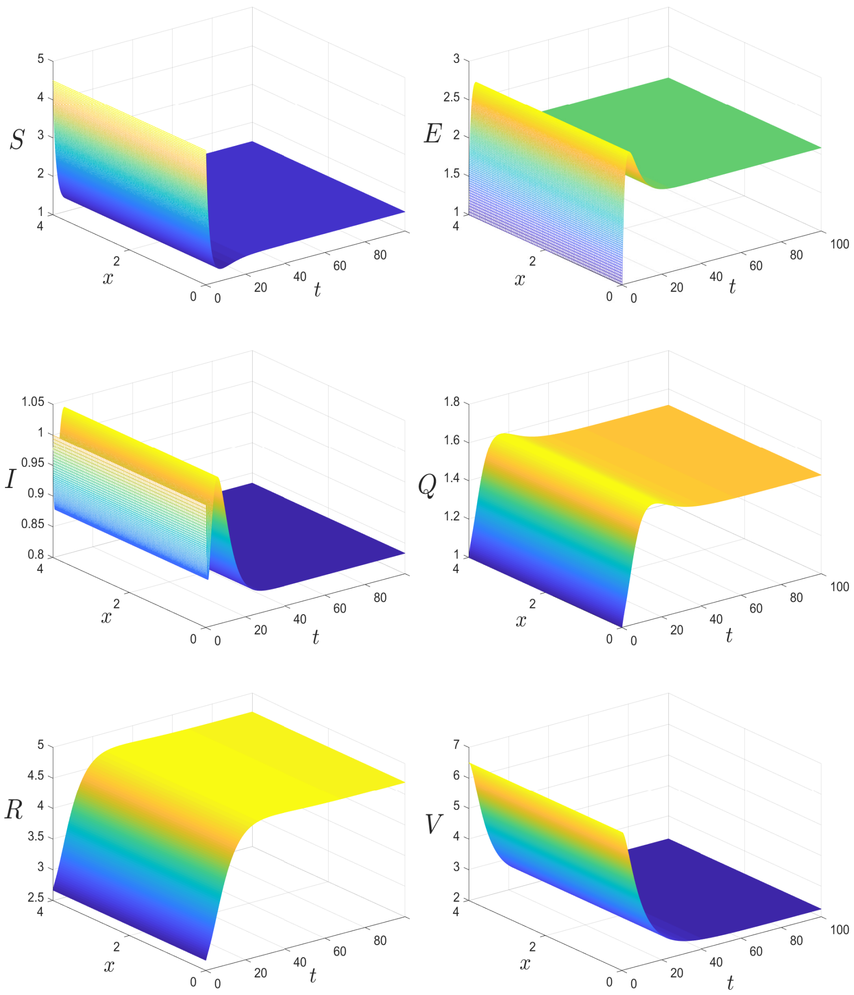

FIGURE 3. Stability of the endemic equilibrium point E_1

The second Figure 3 depicts the dynamics of the infection for the following parameters $dS = 0.01$; $dE = 0.01$; $dI = 0.01$; $dR = 0.01$; $dV = 0.01$; $\gamma = 1$; $\alpha = 0.5$; $\gamma_1 = 0.15$; $\delta = 0.1$; $\gamma_2 = 0.2$; $\gamma_3 = 0.2$; $\gamma_4 = 0.2$; $\gamma_5 = 0.01$. The basic reproduction number within these parameters is $R_0 = 2.6667$, greater than unity in these figures. We can also observe that all curves converge to $(1.5000, 2.0833, 0.8333, 1.5152, 4.6973, 2.2502)$. This numerical result confirms the theoretical finding given in Theorem 3.3 concerning the global stability of the endemic equilibrium E_1.

5. Conclusion

In this work, we have suggested a mathematical spatiotemporal epidemic model. Our model has presented a system of six partial differential equation systems to describe the dynamics of susceptible, exposed, infected, quarantined, removed, and vaccinated individuals evolution in space and time. We have shown that the model has one disease-free and endemic equilibrium point. Then, we have proved the global stability of the two equilibrium points; this stability depends on the value of the primary reproduction number by using some Lyapunoouv functions, we have found that if $R_0 \leq 1$, then the disease-free equilibrium point is globally asymptotically stable and if $R_0 > 1$, then the endemic equilibrium point is globally asymptotically stable. Some numerical simulations are performed, with the help of MATLAB software, are given to value our theoretical results. As the future direction of this present work, we suggest comparing the clinical data with spatial diffusion to the SEIQRV mathematical model results.

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