J. Indones. Math. Soc.
Vol. 29, No. 03 (2023), pp. 382–407.

FORECASTING DEPENDENT TAIL VALUE-AT-RISK BY ARMA-GJR-GARCH-COPULA METHOD AND ITS APPLICATION IN ENERGY RISK

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Abstract. One widely known risk measure is Tail Value-at-Risk (TVaR), which is the average of the values of random risk that exceed the Value-at-Risk (VaR). This classic risk measure of TVaR does not take into account the excess of another random risk (associated risk) that may have an effect on target risk. Copula function expresses a methodology that represents the dependence structure of random variables and has been used to create a risk measure of Dependent Tail Value-at-Risk (DTVaR). Incorporating copula into the forecast function of the ARMA-GJR-GARCH model, this article argues a novel approach, called ARMA-GJR-GARCH-copula with Monte Carlo method, to calculate the DTVaR of dependent energy risks. This work shows an implementation of the ARMA-GJR-GARCH-copula model in forecasting the DTVaR of energy risks of NYH Gasoline and Heating oil associated with energy risk of WTI Crude oil. The empirical results demonstrate that, the simpler GARCH-Clayton copula is better in forecasting DTVaR of Gasoline energy risk than the MA-GJR-GARCH-Clayton copula. On the other hand, the more complicated MA-GJR-GARCH-Frank copula is better in forecasting DTVaR of Heating oil energy risk than the GARCH-Frank copula. In this context, energy sector market players should invest in Heating oil because the DTVaR forecast of Heating oil is more accurate than that of Gasoline.

Key words and Phrases: ARMA-GJR-GARCH, Copula, DTVaR, energy risk

1. INTRODUCTION

In risk management, a lot of measures of risk have been frequently suggested. Perhaps the two most well-known and frequently used measures are Value-at-Risk (VaR) and its competitor, the Tail Value-at-Risk (TVaR). Various expansions of TVaR have also been promoted. Hürlimann [1] applied multivariate copulas to

2020 Mathematics Subject Classification: 62M10
Received: 26-06-2023, accepted: 02-11-2023.

calculate TVaR of aggregate risk. Landsman *et al.* [2] defined a risk measure called Multivariate TVaR for multivariate risk distribution. Bernard *et al.* [3], Bairakdar *et al.* [4], Wang and Wei [5], as well as Jadhav *et al.* [6] have restricted TVaR by presenting particular upper bound, in place of infinitude, for values larger than the lower bound (VaR). Particularly, Jadhav *et al.* This narrowed measure of risk is called Modified TVaR (MTVaR) by Jadhav *et al.* [6]. In the meantime, Brahim *et al.* [7] recommended a Copula TVaR (CTVaR) which is another expansion of TVaR. Moreover, they introduced the terms target risk and associated risk.

According to Brahim *et al.* [7], dependence has begun to play an important role in the world of risk recently. The increasing complexity of insurance and financial activity products has led to increase actuarial and financial interest in dependent risk modeling. Thus, they estimated the loss[1] of target risk by entangling another dependent or associated risk. Moreover, they claimed that CTVaR will not be smaller than TVaR when both target and associated risks have a positive quadrant dependency.[2] They applied CTVaR to returns of major European stock indices, namely UK FTSE, Germany DAX (Ibis), Switzerland SMI, and France CAC, for a certain period of time. They found that the pair of returns (CAC, FTSE) was the pair with the least risk.

Specifically, when we compute MTVaR forecast, it reduces the amount of loss that is larger than VaR and cause this forecast lower than the TVaR forecast accordingly. This is a nice characteristic in risk management, particularly when dealing with returns that have high variations or data that contains outliers [6]. However, Josaphat *et al.* [9] argued that MTVaR forecast should be escorted by an associated risk because this situation happens in practice.

Motivated by the work of [6] and [7], Josaphat and Syuhada [9] offered another coherent measure of risk which is not solely "allowing a specific upper bound larger than VaR" but also "considering an associated loss", that is named Dependent TVaR (DTVaR). This measure of risk is a copula-based expansion of TVaR. The copula is a function which entirely depicts the dependency structure. It comprises all the information to connect the univariate distributions to their multivariate distribution. Using Sklar's theorem [10], Josaphat and Syuhada [9] built up a bivariate distribution of target and associated risks with arbitrary marginal distributions. Furthermore, Josaphat *et al.* [11] offered a method of optimization for DTVaR by employing a pair of metaheuristic algorithms: particle swarm optimization (PSO) and spiral optimization (SpO).

This article combines ARMA-GJR-GARCH and copula to fit the returns data and to provide a more adequate model in order to substitute the well-known joint multivariate normal distribution. The ARMA-GJR-GARCH-copula model, constructed for calculating the DTVaR of dependent risks, should be more plausible and adequate. Our work analyzes dependent risks of Gasoline and Heating oil of New York Harbor (NYH) and Crude oil of West Texas Intermediate (WTI) with

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[1] We use the terms risk(s) and loss(es) interchangeably.
[2] For more details see [8].

daily returns and then forecasts the one-day ahead DTVaR following the supple model of copula.

This article is related to Huang *et al.* [12] and Han *et al.* [13], in which they talked about the implementation of copula in forecasting the classical VaR of a portfolio; as well as Nikusokhan [14], in which he talked over the application of copula in forecasting the TVaR for portfolio optimization. However, unlike the three literatures, we apply several copula entirely with different marginal distribution and incorporate Monte Carlo method to forecast DTVaR of two different risks, NYH Gasoline and Heating oil returns, which are condidered to be associated with WTI Crude oil returns, respectively. This article shows that the GARCH-Clayton copula model captures the DTVaR of Gasoline returns more succesfully, while the MA-GJR-GARCH-Frank copula and the MA-GJR-GARCH-Gaussian copula models forecast the DTVaR of Heating oil returns more succesfully.

The structure of this article is arranged as follows. Section 2 presents some pertinent definitions and notions used in the subsequent sections. Section 3 presents marginal model which is the ARMA-GJR-GARCH model and its specifications. In addition, we explain the several risk measures for ARMA-GJR-GARCH model. Section 4 presents the procedures of copula estimation and forecasting DTVaR. Section 5 presents the empirical procedure and results, followed by a conclusion in Section 6. Proofs are presented separately in Appendix.

2. Preliminaries

2.1. **Copula concept.** Copula is a method for constructing two or more distributions. Copula was first developed by Abe Sklar in 1959 through a theorem which became known as Sklar's theorem.

Theorem 2.1 ([15]). *Let $F_{X_1,\cdots,X_N}$ denote a joint (multivariate) distribution function (d.f.) with F_{X_i} denote marginal or univariate distribution functions (d.f's.) of X_i, $i = 1, \cdots, N$. Then, there exists a copula C such that for all $(x_1, \cdots, x_N) \in \overline{\mathbb{R}}^N$,*

$$F_{X_1,\cdots,X_N}(x_1, \cdots, x_N) = C(F_{X_1}(x_1), \cdots, F_{X_N}(x_N); \theta), \tag{1}$$

where θ denotes copula parameter (or vector of copula parameters). If all $F_{X_1}, \cdots, F_{X_N}$ all continuous, then C is unique. Alternatively, C is uniquely determined on $\mathrm{Ran}F_{X_1} \times \cdots \times \mathrm{Ran}F_{X_N}$, *where* $\mathrm{Ran}F_{X_i} = F_{X_i}(\overline{\mathbb{R}})$ *for $i = 1, \cdots, N$. Conversely, if $F_{X_1}, \cdots, F_{X_N}$ are d.f's. and C is a copula, then the function F in (1) is a N-dimensional distribution function (d.f.) with marginal d.f's. $F_{X_1}, \cdots, F_{X_N}$.*

Henceforth, we assume the copula under consideration is differentiable. Consequently, when $X_1, \cdots, X_N$ are continuous, Sklar's theorem shows that any multivariate d.f. can be expressed by univariate distributions and a structure of dependency, which is obtained as follows,

$$\begin{aligned} f_{X_1,\cdots,X_N}(x_1,\cdots,x_N) &= \frac{\partial^N F_{X_1,\cdots,X_N}(x_1,\cdots,x_N)}{\partial x_1 \cdots \partial x_N} \\ &= f_{X_1}(x_1)\cdots f_{X_N}(x_N) \times \frac{\partial^N C(F_{X_1}(x_1),\cdots,F_{X_2}(x_N);\theta)}{\partial F_{X_1}(x_1)\cdots \partial F_{X_N}(x_N)} \\ &= f_{X_1}(x_1)\cdots f_{X_N}(x_N)\ c(F_{X_1}(x_1),\cdots,F_{X_N}(x_N);\theta), \end{aligned} \tag{2}$$

where $c(F_{X_1}(x_1), \cdots, F_{X_N}(x_N); \theta)$ denotes the copula density. Especially for $N = 2$, we obtain,

$$f_{X_1,X_2}(x,y) = f_{X_1}(x) f_{X_2}(y)\ c(F_{X_1}(x), F_{X_2}(y); \theta), \tag{3}$$

or

$$C(u,v;\theta) = F_{X_1,X_2}\big(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)\big) \tag{4}$$

where $u, v \in [0,1]$, $F_{X_1}^{-1}$ dan $F_{X_2}^{-1}$ are distribution quantile functions of risks X_1 and X_2.

The copula family used here encompasses copulas of Gaussian, Clayton, Gumbel, and Frank, which are shown in the following:

(1) Gaussian copula
Gaussian copula is defined by,

$$\begin{aligned} C_{\mathrm{N}}(u,v;\rho) &= \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)) \\ &= \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\Phi^{-1}(u)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big(-\frac{s^2 - 2\rho s t + t^2}{2(1-\rho^2)}\Big)\, \mathrm{d}s\ \mathrm{d}t \end{aligned}$$

where Φ^{-1} is the inverse of the d.f. of standard normal distribution and $\rho \in (-1, 1)$. Moreover, the Gaussian copula density is given by,

$$c_{\mathrm{N}}(u,v;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big(\frac{u^2+v^2}{2}\Big) \exp\Big(-\frac{u^2 - 2\rho u v + v^2}{2(1-\rho^2)}\Big)$$

where $u = F_{X_1}(x_1)$ and $v = F_{X_2}(x_2)$.

(2) Clayton copula
Clayton copula can be used to determine the joint d.f. of the bivariate lognormal distribution. This copula is defined by the following,

$$C_{\mathrm{C}}(u,v;\theta) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad \theta \in [-1,\infty)\backslash\{0\}. \tag{5}$$

(3) Gumbel copula
Gumbel copula is defined by the following,

$$C_{\mathrm{G}}(u,v;\theta) = \exp\Big(-\big[(-\ln u)^\theta + (-\ln v)^\theta\big]^{1/\theta}\Big), \quad \theta \in [1,\infty).$$

(4) Frank copula
Frank copula is defined as follows,

$$C_{\mathrm{F}}(u,v;\theta) = -\frac{1}{\theta}\ln\Big(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\Big), \quad \theta \in \mathbb{R}\backslash\{0\}.$$

2.2. **The Dependent Tail Value-at-Risk.** Let X and Y be two random risks that are not always independently and identically distributed (i.i.d.) and have marginal d.f.'s. F_X and F_Y. Given a probability level $\alpha \in (0,1)$, usually close to 1, the Value-at-Risk (VaR) of X at α is the quantile Q_α of F_X that corresponds to α. Then, VaR is mathematically defined by,

$$Q_\alpha = F_X^{-1}(\alpha). \tag{6}$$

While the formulation (7) defines the TVaR as follows,

$$\mathrm{TVaR}_\alpha(X) = E[X|X \geq Q_\alpha(X)] = \frac{1}{1-\alpha}\int_\alpha^1 Q_p(X)\mathrm{d}p. \tag{7}$$

As explained in Section 1, Josaphat and Syuhada [9] offered a different measure of risk as a copula-based expansion of TVaR that does not only consider the magnitude of the risk X between lower and upper quantiles but also notices the excess of any different risk Y which is associated with X. Formula (8) defines the Dependent Tail Value-at-Risk (DTVaR).

$$\mathrm{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y) = E\big[X|Q_\alpha \leq X \leq Q_{\alpha_1}, Q_\delta \leq Y \leq Q_{\delta_1}\big], \tag{8}$$

where $\alpha_1 = \alpha + (1-\alpha)^{1+a}$, $\delta_1 = \delta + (1-\delta)^{1+d}$, and a, $d \geq 0$. Moreover, α and δ express levels of probability and excess, a and d denote contraction parameters, X denotes target risk, while Y associated risk. According to [9], we recall a lemma related to DTVaR.

Lemma 2.2. [9] *Suppose that X and Y represent two random risks with a joint d.f. denoted by a copula C having parameter θ. Suppose that $\alpha, \delta \in (0,1)$ and a, $d \geq 0$ be certain numbers. Then, DTVaR of X provided values larger than its lower quantile up to another specific value and an associated risk Y is given as follows,*

$$\mathrm{DTVaR}_{(\alpha,a)}^{(\delta,d)}(X|Y;C) = \frac{\int_\alpha^{\alpha_1}\int_\delta^{\delta_1} F_X^{-1}(u)\, c(u,v;\theta)\,\mathrm{d}v\,\mathrm{d}u}{C(\alpha_1,\delta_1;\theta)-C(\alpha,\delta_1;\theta)-C(\alpha_1,\delta;\theta)+C(\alpha,\delta;\theta)}, \tag{9}$$

where F_X^{-1} represents the inverse of marginal d.f. of X, $u = F_X(x)$, $v = F_Y(y)$, $\alpha_1 = \alpha + (1-\alpha)^{a+1}$ and $\delta_1 = \delta + (1-\delta)^{d+1}$.

Remark 2.3. *Note that the denominator $C(\alpha_1,\delta_1;\theta) - C(\alpha,\delta_1;\theta) - C(\alpha_1,\delta;\theta) + C(\alpha,\delta;\theta)$ in (9) is the bivariate significance level (b.s.l.) for DTVaR.*

Remark 2.4. *When we presume that X and Y are independent and $a = d = 0$, then DTVaR simplifies into TVaR given by,*

$$\mathrm{TVaR}_\alpha(X) = \frac{\int_\alpha^1 Q_u(X)\,\mathrm{d}u}{1-\alpha}. \tag{10}$$

3. Model for the Univariate Distribution

The GARCH model and its generalizations are very crucial in the analysis of data of time series, in particular in financial applications when the objective is to analyze and to forecast volatility. In general, volatility expresses the level of risk caused by price oscillations. Moreover, the bigger the volatility, the bigger the risk.

There are several empirical properties that can be observed from volatility. According to [16], one of the empirical properties of volatility is asymmetry. This property states that the positive and negative values of returns have different effects on the amount of volatility. Engle and Patton [17] described that the good volatility model is a model that can accommodate the empirical properties of returns and volatility. Apparently, the classical GARCH model has not been able to accommodate asymmetric property.

Mohammadi and Su [18] observed autocorrelation and heavy tails in stock returns. The volatility of stock leans to climb more after a substantial decrease of price than after a price increase of the same size, which is familiar as leverage effect. Extended GARCH models such as EGARCH, TGARCH, GJR-GARCH, IGARCH, and FIGARCH have been proposed for years to catch these *stylish characteristics* of stock returns [19].

Here, we use the model of ARMA$(1,1)$-GJR-GARCH$(1,1)$-normal to forecast the univariate distributions of series of returns. Let the returns of a commodity, in general, be given by $\{Z_t\}$, $t = 1, \cdots, T$. The model of ARMA$(1,1)$-GJR-GARCH$(1,1)$ is as follows,

$$\begin{aligned} &Z_t = \mu + \kappa_1 Z_{t-1} + \eta_1 \xi_{t-1} + \xi_t, \qquad \xi_t = h_t^{1/2} \varepsilon_t, \\ &h_t = \omega_0 + \omega_1 h_{t-1} + \omega_2 \xi_{t-1}^2 + \phi I_{t-1} \xi_{t-1}^2, \quad \varepsilon_t \sim \mathcal{N}(0,1), \end{aligned} \tag{11}$$

where $\omega_0 > 0$, ω_1, $\omega_2 \geq 0$, $\omega_2 + \phi \geq 0$, whilst function I_{t-1} is given by,

$$I_{t-1} = \begin{cases} 1, & \text{if } \xi_{t-1} < 0, \\ 0, & \text{if } \xi_{t-1} \geq 0. \end{cases}$$

The model of ARMA$(1,1)$-GJR-GARCH$(1,1)$ is a combination of two random variables Z_t and ξ_t, where Z_t is modeled with ARMA$(1,1)$ while ξ_t is modeled with GJR-GARCH(1,1). If $\kappa_1 \to 0$ and $\eta_1 \to 0$, then Eq. (11) becomes the model of GJR-GARCH$(1,1)$, which is given by the following,

$$\begin{aligned} &Z_t = \mu + \xi_t, \qquad \xi_t = h_t^{1/2} \varepsilon_t, \\ &h_t = \omega_0 + \omega_1 h_{t-1} + \omega_2 \xi_{t-1}^2 + \phi I_{t-1} \xi_{t-1}^2, \quad \varepsilon_t \sim \mathcal{N}(0,1), \end{aligned} \tag{12}$$

3.1. **The stationarity of GJR-GARCH(1,1).** In the following, we describe the stationarity of the GJR-GARCH$(1,1)$ model. The explanation of the stationarity of this model is used indirectly to explain the stationarity of a more general model, ARMA(1,1)-GJR-GARCH(1,1). In the GJR-GARCH$(1,1)$ model, h_t is given by,

$$\begin{aligned} h_t &= \omega_0 + \omega_1 h_{t-1} + \omega_2 h_{t-1} \varepsilon_{t-1}^2 + \phi I_{t-1} h_{t-1} \varepsilon_{t-1}^2 \\ &= \omega_0 + (\omega_1 + \omega_2 \varepsilon_{t-1}^2 + \phi I_{t-1} \varepsilon_{t-1}^2) h_{t-1} \end{aligned}$$

$$
\begin{aligned}
&= \omega_0 + \omega_0(\omega_1 + \omega_2\varepsilon_{t-1}^2 + \phi I_{t-1}\varepsilon_{t-1}^2) + (\omega_1 + \omega_2\varepsilon_{t-1}^2 + \phi I_{t-1}\varepsilon_{t-1}^2) \\
&\quad \times (\omega_1 + \omega_2\varepsilon_{t-2}^2 + \phi I_{t-2}\varepsilon_{t-2}^2)[\omega_0 + (\omega_1 + \omega_2\varepsilon_{t-3}^2 + \phi I_{t-3}\varepsilon_{t-3}^2)h_{t-3}] \\
&= \omega_0 + \omega_0 \sum_{k=1}^{\infty} \prod_{j=1}^{k} (\omega_1 + \omega_2\varepsilon_{t-j}^2 + \phi I_{t-j}\varepsilon_{t-j}^2).
\end{aligned}
$$

Then, we find the expected value of h_t,

$$
\begin{aligned}
E[h_t] &= \omega_0 + \omega_0 \sum_{k=1}^{\infty} \prod_{j=1}^{k} (\omega_1 + \omega_2 E[\varepsilon_{t-j}^2] + \varphi E[I_{t-j}]E[\varepsilon_{t-j}^2]) \\
&= \omega_0 + \omega_0 \sum_{k=1}^{\infty} \prod_{j=1}^{k} \left(\omega_1 + \omega_2 + \frac{\phi}{2}\right) \\
&= \omega_0 \sum_{k=0}^{\infty} \left(\omega_1 + \omega_2 + \frac{\phi}{2}\right)^k = \frac{\omega_0}{1-\varphi}, \qquad (13)
\end{aligned}
$$

where $\varphi = \omega_1 + \omega_2 + \frac{\phi}{2}$.

Since h_t must be positive, we obtain $0 \le \omega_1 + \omega_2 + \frac{\phi}{2} < 1$. In order to fulfill the stationarity assumption of the GJR-GARCH (1,1) model, the parameters must be bounded, namely,

$$
\begin{aligned}
&\omega_0 > 0, \quad \omega_1, \omega_2 \ge 0, \\
&\omega_2 + \phi \ge 0, \quad \omega_1 + \omega_2 + \frac{\phi}{2} < 1.
\end{aligned} \qquad (14)
$$

3.2. **The estimation of parameters of ARMA(1,1)-GJR-GARCH(1,1).** Recall model (11). Here, $\mu = E(Z_t) = E(E(Z_t|\mathcal{G}_{t-1})) = E(\mu_t) = \mu$ is the unconditional mean of return series (see [12]), $h_t = \mathrm{Var}(Z_t|\mathcal{G}_{t-1})$ is the conditional variance, where $\mathcal{G}_{t-1}$ is the σ-algebra at $t-1$. The parameters estimation method is the maximum likelihood estimation (MLE). Let $\vartheta_{Z_{t+1}} = \left(\mu, \kappa_1, \eta_1, \omega_0, \omega_1, \omega_2, \phi\right)^T$ be a vector of parameters of ARMA$(1,1)$-GJR-GARCH(1,1). Suppose $\mathcal{G}_t = \{z_1, z_2, \cdots, z_t\}$ is a σ-algebra at t. The joint probability function (p.f.) can be written as $f(z_2, \cdots, z_{t+1}) = f(z_{t+1}|\mathcal{G}_t)f(z_t|\mathcal{G}_{t-1})\cdots f(z_2|\mathcal{G}_1)f(z_1)$. Furthermore, the conditional p.f. for ARMA(1,1)-GJR-GARCH(1,1) is given by,

$$
f_{Z_{t+1}|\mathcal{G}_t}(z_{t+1};\vartheta) = \frac{1}{\sqrt{2\pi h_{t+1}}} \exp\left\{-\frac{(z_{t+1}-\mu)^2}{2h_{t+1}}\right\}
$$

Moreover, the likelihood function is given in the following,

$$
\begin{aligned}
L(\vartheta_{Z_{t+1}}) &= \prod_{t=1}^{n-1} f_{Z_{t+1}|\mathcal{G}_t}(z_{t+1};\vartheta) \\
&= \left\{ \prod_{t=1}^{n-1} \left(2\pi(\omega_0 + \omega_1 h_t + \omega_2\xi_t^2 + \phi I_t \xi_t^2)\right)^{-1/2} \right\}
\end{aligned}
$$

$$\times \exp\left\{-\sum_{t=1}^{n-1}\frac{(\kappa_1 z_t+\eta_1\xi_t+\xi_{t+1})^2}{2(\omega_0+\omega_1 h_t+\omega_2\xi_t^2+\phi I_t\xi_t^2)}\right\}.$$

Hence, the function of log-likelihood is given in the following,

$$\begin{aligned}
l(\vartheta_{Z_{t+1}}) &= \ln L(\vartheta_{Z_{t+1}}) \\
&= -\frac{1}{2}\left\{(n-1)\ln 2\pi+\sum_{t=1}^{n-1}\ln(\omega_0+\omega_1 h_t+\omega_2\xi_t^2+\phi I_t\xi_t^2)\right\} \\
&\quad -\sum_{t=1}^{n-1}\frac{(\kappa_1\mu+\kappa_1^2 z_{t-1}+\kappa_1\eta_1\xi_{t-1}+(\kappa_1+\eta_1)\xi_t+\xi_{t+1})^2}{2(\omega_0+\omega_1 h_t+\omega_2\xi_t^2+\phi I_t\xi_t^2)}
\end{aligned} \tag{15}$$

The MLE method estimates the parameter vector $\vartheta_{Z_{t+1}}$ which maximizes the above log-likelihood function using the first partial derivatives, i.e.,

$$\begin{aligned}
&\frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\mu}=0,\quad \frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\kappa_1}=0,\quad \frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\eta_1}=0,\\
&\frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\omega_0}=0,\quad \frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\omega_1}=0,\quad \frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\omega_2}=0,\quad \frac{\partial l(\vartheta_{Z_{t+1}})}{\partial\phi}=0
\end{aligned} \tag{16}$$

Note that it is difficult to obtain the estimate of the parameter vector $\vartheta_{Z_{t+1}} = \big(\mu,\kappa_1,\eta_1,\omega_0,\omega_1,\omega_2,\phi\big)^T$ analytically. As a result, the log-likelihood function (15) is maximized numerically.

3.3. **The moment properties.** Expectation and variance are important aspect of a measure, which are called moments. This subsection also briefly describes the stationarity and kurtosis of the ARMA(1,1)-GJR-GARCH(1,1) model.

ARMA(1,1)-GJR-GARCH(1,1). The unconditional expectation and variance of ARMA(1,1)-GJR-GARCH(1,1) are given by the following proposition. Based on the proposition, we can determine the weak stationarity of ARMA(1,1)-GJR-GARCH(1,1).

Proposition 3.1. *Suppose Z_t follows the model* (11)*. Then, the unconditional expectation and unconditional variance of Z_t are given in the following,*

$$E[Z_t]=\frac{\mu}{1-\kappa_1}, \tag{17}$$

$$\mathrm{Var}[Z_t]=\frac{2\mu^2\kappa_1}{(1-\kappa_1)(1-\kappa_1^2)}+\frac{\eta_1^2+2\kappa_1\eta_1+1}{1-\kappa_1^2}\,\frac{\omega_0}{1-\varphi}. \tag{18}$$

Stationarity. One of the conditions for forecasting is stationarity. According to [20], stationarity is a behavior of data that does not change with time in a time series process. If the time series data are not stationary, it will be difficult to determine the behavior of the data and to forecast a value in the future. Time series data are weakly stationary if the unconditional mean and unconditional variance are fixed. Therefore, based on (18) and (14), the stationarity of the ARMA(1,1)-GJR-GARCH(1,1) model can be obtained if,

$$|\kappa_1| < 1, \quad 0 < \varphi < 1, \quad \omega_0 > 0,$$
$$\omega_1, \omega_2 \geq 0, \quad \omega_2 + \phi \geq 0, \quad \omega_1 + \omega_2 + \frac{\phi}{2} < 1.$$

KURTOSIS. Kurtosis is generally identical to the fourth moment of returns. Data with large kurtosis relative to the normal distribution have sharper peak around the mean and heavier tail, and conversely, data with small kurtosis relative to the normal distribution have a more sloping peak around the mean and thinner tail. Based on the nature of the returns (the kurtosis is greater than 3), it can be said that the returns do not have normal distribution. The heavy tail also indicates that the tail of distribution of the returns is slower to 0 when contrasted to normal distribution. This is due to the extreme values in the returns series. The formulas for unconditional kurtosis and conditional kurtosis for the MA(1)-GJR-GARCH(1,1) model are given by the following proposition.

Proposition 3.2. *Let Z_t follow ARMA(1,1)-GJR-GARCH(1,1) in* (11) *where the innovation ε_t is assumed to have the distribution $\mathcal{N}(0,1)$. Then, the unconditional kurtosis of Z_t is given by,*

$$\varkappa_Z = \frac{\frac{\kappa_1^4\mu^4}{(\kappa_1-1)^4} + \frac{4\kappa_1^2\mu^2(\eta_1^2+1)}{(\kappa_1-1)^2}\bar{h} + 6\eta_1^2\bar{h}^2 + 3(\eta_1^4+1)\upsilon}{\left(\frac{\kappa_1^2\mu^2}{(\kappa_1-1)^2} + (1+\eta_1^2)\bar{h}\right)^2}, \tag{19}$$

where

$$\upsilon = \frac{\omega_0^2 + 2\omega_0\varphi\bar{h}}{1-\gamma}, \qquad \bar{h} = \frac{\omega_0}{1-\varphi}.$$

Remark 3.3. The closed form formula for the conditional kurtosis of Z_{t+v} following ARMA(1,1)-GJR-GARCH(1,1) cannot be obtained because it is complicated to obtain the conditional distribution of Z_{t+v}. This is caused by Z_{t+v} which contains the autoregressive component Z_{t+v-1}.

MA(1)-GJR-GARCH(1,1). This model can be considered a simplification of ARMA(1,1)-GJR-GARCH(1,1), given by,

$$\begin{aligned} Z_t &= \mu + \eta_1\,\xi_{t-1} + \xi_t, \qquad \xi_t = h_t^{1/2}\,\varepsilon_t \\ h_t &= \omega_0 + \omega_1\,h_{t-1} + \omega_2\,\xi_{t-1}^2 + \phi I_{t-1}\,\xi_{t-1}^2. \end{aligned} \tag{20}$$

Based on (11), since ε_t follows a standard normal distribution, the moment generating function (m.g.f.) of ε_t is given as follows,

$$M_{\varepsilon_t}(s) = E(e^{\varepsilon_t\, s}) = e^{\frac{1}{2}s^2}.$$

The m.g.f's. of Z_t and $Z_{t+v}|\mathcal{G}_t$ are given by,

$$M_{Z_t}(s) = E\big[e^{(\mu+\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1}+h_t^{1/2}\varepsilon_t)s}\big] = e^{\mu s}E\big[e^{\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1}s}\big]E\big[e^{h_t^{1/2}\varepsilon_t s}\big]$$

$$
\begin{aligned}
&= e^{\mu s+\frac{1}{2}(\eta_1^2 h_{t-1}+h_t)s^2} \\
M_{Z_{t+v}|\mathcal{G}_t}(s) &= E\big[e^{(\mu+\eta_1 h_{t+v-1}^{1/2}\varepsilon_{t+v-1}+h_{t+v}^{1/2}\varepsilon_{t+v})s|\mathcal{G}_t}\big] \\
&= e^{\mu s}E\Big[e^{\eta_1 h_{t+v-1}^{1/2}\varepsilon_{t+v-1}s}\Big]E\Big[e^{h_{t+v}^{1/2}\varepsilon_{t+v}s}\Big] \\
&= e^{\mu s+\frac{1}{2}(\eta_1^2 h_{t+v-1}+h_{t+v})s^2},
\end{aligned}
$$

for $v = 2, 3, \cdots, n$. Thus, $Z_t \sim \mathcal{N}(\mu, \eta_1^2 h_{t-1} + h_t)$ dan $Z_{t+v}|\mathcal{G}_t \sim \mathcal{N}(\mu, \eta_1^2 h_{t+v-1} + h_{t+v})$.

Expectation and variance

- Unconditional expectation and variance

$$
\begin{aligned}
E[Z_t] &= E\big[\mu + \eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} + h_t^{1/2}\varepsilon_t\big] \\
&= \mu + \eta_1 E\big[h_{t-1}^{1/2}\big]E[\varepsilon_{t-1}] + E\big[h_t^{1/2}\big]E[\varepsilon_t] = \mu, \\
\mathrm{Var}[Z_t] &= \mathrm{Var}\big[\mu + \eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} + h_t^{1/2}\varepsilon_t\big] \\
&= \eta_1^2\big(E\big[\varepsilon_{t-1}^2\big]\mathrm{Var}\big[h_{t-1}^{1/2}\big] + \mathrm{Var}[\varepsilon_{t-1}]E[h_{t-1}]\big) + \big(E[\varepsilon_t^2]\mathrm{Var}\big[h_t^{1/2}\big] \\
&\quad + \mathrm{Var}[\varepsilon_t]E[h_t]\big) \\
&= \eta_1^2 E[h_{t-1}] + E[h_t] = (\eta_1^2+1)\frac{\omega_0}{1-\varphi}.
\end{aligned}
$$

Thus, we obtain $Z_t \sim \mathcal{N}\big(\mu, (\eta_1^2+1)\frac{\omega_0}{1-\varphi}\big)$.

- Conditional expectation and variance
Perhaps the most important use of MA(1)-GJR-GARCH(1,1) is to forecast the future values of $\{Z_t\}$. The following theorem gives conditional moments of returns for MA(1)-GJR-GARCH(1,1).

Theorem 3.4. *The conditional expectation and variance of forward returns that follow model* (20) *are given by,*

$$E[Z_{t+v}|\mathcal{G}_t] = \mu, \tag{21}$$

$$Var[Z_{t+v}|\mathcal{G}_t] = (\eta_1^2+1)\bar{h} + (\eta_1^2\varphi^{v-2} + \varphi^{v-1})(h_{t+1} - \bar{h}), \tag{22}$$

for $v = 2, 3, \cdots, n$.

The conditional expectation $E[Z_{t+v}|\mathcal{G}_t]$ simply states that, with a constant conditional expectation equation, the conditional expectation of v step ahead return over time t equals the constant unconditional expectation μ. The conditional variance of the return $\mathrm{Var[Z_{t+v}|\mathcal{G}_t]}$ indicates that the conditional variance of the v step ahead return over time t equals a multiplication of the variance of steady state $\bar{h}$, added with an exponentially declining correction term that takes into account the difference between the one step ahead variance h_{t+1} and the variance of stabile state $\bar{h}$.

Remark 3.5. Especially for $v = 2$, we obtain,

$$\mathrm{Var}[Z_{t+2}|\mathcal{G}_t] = (\eta_1^2 + 1 - \varphi)\bar{h} + \eta_1^2 h_{t+1}.$$

The limit of $\mathrm{Var}[Z_{t+v}|\mathcal{G}_t]$ is given by the following lemma.

Lemma 3.6. *Suppose Z_t follows the model (20). Let $\varphi \in (0,1)$. Then,*

$$\lim_{v\to\infty} \mathrm{Var}[Z_{t+v}|\mathcal{G}_t] = (\eta_1^2 + 1)\bar{h}. \tag{23}$$

3.4. **Selected risk measures.** In the following, we describe several risk measures associated with the ARMA$(1,1)$-GJR-GARCH(1,1) model.

1. VaR
   The VaR of $Z_{t+v}|\mathcal{G}_t$ at $t+v$ (return from $t+v-\Delta t$ to $t+v$) with probability level α is defined as,
   $$Q_{\alpha,t+v} = Q_\alpha(Z_{t+v}|\mathcal{G}_t) = \inf\{z : F_{Z_{t+v}|\mathcal{G}_t}(z_{t+v}) \geq \alpha\}, \tag{24}$$
   where $F_{Z_{t+v}|\mathcal{G}_t}$ is the conditional d.f. of Z_{t+v} at $t+v$ on $\mathcal{G}_t$, $\mathcal{G}_t$ is $\sigma-$algebra at time t, and we have $\Pr(Z_{t+v} \leq Q_{\alpha,t+v}|\mathcal{G}_t) = \alpha$. It means that we have $100\alpha\%$ confidence that the risk in the period Δt is not greater than VaR.
2. TVaR
   The TVaR is actually a conditional expectation. In this case, the condition is $Z_{t+v} \geq Q_\alpha(Z_{t+v}|\mathcal{G}_t)$ Therefore, TVaR of $Z_{t+v}|\mathcal{G}_t$ is given by,
   $$\mathrm{TVaR}_{\alpha,t+v} = E[Z_{t+v}|Z_{t+v} \geq Q_{\alpha,t+v}]$$

   Note that,
   $$\begin{aligned}\Pr\big((Z_{t+v} \leq z \,|Z_{t+v} \geq Q_{\alpha,t+v})\big|\mathcal{G}_t\big) &= \frac{\Pr\big((Q_{\alpha,t+v} \leq Z_{t+v} \leq z)|\mathcal{G}_t\big)}{\Pr\big((Z_{t+v} \geq Q_{\alpha,t+v})\big|\mathcal{G}_t\big)} \\ &= \frac{1}{1-\alpha}\int_{Q_{\alpha,t+v}}^{z} f_{Z_{t+v}|\mathcal{G}_t}(z_{t+v})\mathrm{d}z_{t+v}.\end{aligned}$$

   Then, $\mathrm{TVaR}_{\alpha,t+v}$ is given by,
   $$\mathrm{TVaR}_{\alpha,t+v} = \frac{1}{1-\alpha}\int_{Q_{\alpha,t+v}}^{\infty} z_{t+v} f_{Z_{t+v}|\mathcal{G}_t}(z_{t+v})\mathrm{d}z_{t+v}.$$
3. DTVaR
   We set $a = d = 0$. Similar to TVaR, the DTVaR of $Z_{t+v}|\mathcal{G}_t$ is also a conditional expectation of $Z_{t+v} \geq Q_\alpha(Z_{t+v}|\mathcal{G}_t)$. However, Z_t in DTVaR is not only conditional on $Z_{t+v} \geq Q_\alpha(Z_{t+v}|\mathcal{G}_t)$ but also conditional over another stochastic process. Thus, the DTVaR of Z_t associated with another stochastic process W_t is given by,
   $$\begin{aligned}&\mathrm{DTVaR}^{(\delta,0)}_{(\alpha,0)}(Z_{t+v}|W_{t+v};C) = E(Z_{t+v}|Z_{t+v} > Q_{\alpha,t+v}, W_{t+v} > Q_{\delta,t+v}) \\ &= \frac{1}{\Pr(Z_{t+v} > Q_{\alpha,t+v}, W_{t+v} > Q_{\delta,t+v}|\mathcal{G}_t)}\end{aligned}$$

$$\times \int_{Q_{\alpha,t+v}}^{\infty} \int_{Q_{\delta,t+v}}^{\infty} z_{t+v}\, c(F(z_{t+v}), F(w_{t+v})|\mathcal{G}_t)$$
$$\times f_{W_{t+v}|\mathcal{G}_t}(w_{t+v})\, \mathrm{d}w_{t+v}\, \mathrm{d}z_{t+v}, \tag{25}$$

where $f_{\cdot|\mathcal{G}_t}(\cdot)$ is the conditional p.f. of return at $t+v$ on $\sigma-$algebra $\mathcal{G}_t$. The denominator of (25) or the bivariate significance level (b.s.l.) is given by,

$$\Pr\left(Z_{t+v} > Q_{\alpha,t+v}, W_{t+v} > Q_{\delta,t+v}|\mathcal{G}_t\right)$$
$$= \int_{Q_{\alpha,t+v}}^{\infty} \int_{Q_{\delta,t+v}}^{\infty} c(F(z_{t+v}), F(w_{t+v})|\mathcal{G}_t) f_{Z_{t+v}|\mathcal{G}_t}(z_{t+v})$$
$$\times f_{W_{t+v}|\mathcal{G}_t}(w_{t+v})\, \mathrm{d}w_{t+v}\, \mathrm{d}z_{t+v},$$

and $Q_{\alpha,t+v}$ and $Q_{\delta,t+v}$ satisfy

$$\Pr(Z_{t+v} \le Q_{\alpha,t+v}|\mathcal{G}_t) = \int_{-\infty}^{Q_{\alpha,t+v}} f_{Z_{t+v}|\mathcal{G}_t}(z_{t+v})\, \mathrm{d}z_{t+v},$$
$$\Pr(W_{t+v} \le Q_{\delta,t+v}|\mathcal{G}_t) = \int_{-\infty}^{Q_{\delta,t+v}} f_{W_{t+v}|\mathcal{G}_t}(w_{t+v})\, \mathrm{d}w_{t+v}.$$

Remark 3.7. If $a, d > 0$, then it is difficult to compare the b.s.l.

$$\Pr\left(Q_{\alpha,t+v} < Z_{t+v} < Q_{\alpha_1,t+v}, Q_{\delta,t+v} < W_{t+v} < Q_{\delta_1,t+v}|\mathcal{G}_t\right) \tag{26}$$

and the number of violations related to DTVaR forecast. The b.s.l. corresponds to the target and associated returns that are smaller than $Q_{\alpha_1,t+v}$ and $Q_{\delta_1,t+v}$, respectively (bounded above). While the number of violations usually corresponds to the number of returns Z_{t+v} that are larger than DTVaR forecast (unbounded above).

4. The Procedures of Estimation and Forecasting

4.1. **Copula estimation method.** To estimate copula parameters, we use the MLE method. Based on (3), we obtain,

$$f_{X_t,Y_t}(x_t, y_t) = c(u_{1t}, u_{2t}; \theta) f_{X_t}(x_t) f_{Y_t}(y_t),$$

where θ denotes copula parameter. Thus, it is clear that the estimated copula parameter relies on the estimate of marginal distribution of each component. The number of parameters that must be estimated is the sum of the parameters of the univariate distributions and the number of copula parameters. Furthermore, the logarithmic function of the joint p.f. is given by,

$$\ln\left(f_{X_t,Y_t}(x_t, y_t)\right) = \ln f_{X_t}(x_t) + \ln f_{Y_t}(y_t) + \ln c(u_{1t}, u_{2t}; \theta).$$

Suppose there are m i.i.d. observations. The log-likelihood function is given in the following,

$$l = \sum_{t=1}^{m} \ln\left(f_{X_t,Y_t}(x_t, y_t)\right)$$

$$= \sum_{t=1}^{m} \ln f_{X_t}(x_t) + \sum_{t=1}^{m} \ln f_{Y_t}(y_t) + \sum_{t=1}^{m} \ln \big(c(u_t, v_t); \theta\big)$$
$$= l_{X_t} + l_{Y_t} + l_C, \tag{27}$$

where l_{X_t} and l_{Y_t} are the log-likelihood functions for marginal distributions, while l_C for copula.

Let $\theta_{X_t} = (\theta_1^X, \cdots, \theta_{m_1}^X)$ and $\theta_{Y_t} = (\theta_1^Y, \cdots, \theta_{m_1}^Y)$ be vectors of the marginal parameters of X_t and Y_t, respectively, to be estimated. The MLE method estimates $\widehat{\theta}_{X_t}$ and $\widehat{\theta}_{Y_t}$ which maximizes l_{X_t} and l_{Y_t} with the first partial derivatives that satisfy,

$$\frac{\partial l_{X_t}}{\partial \theta_{X_t}} = \mathbf{0} \quad \text{and} \quad \frac{\partial l_{Y_t}}{\partial \theta_{Y_t}} = \mathbf{0}.$$

Furthermore, in estimating the copula parameter, numerical calculation is required as a result of the difficulty of maximizing the function l_C.

4.2. **Forecasting DTVaR.** Suppose that returns X_t and Y_t are associated with return Z_t, respectively. Thus, without losing generality, the DTVaR of X_{t+v} associated with Z_{t+v} for $a = d = 0$ is given by,

$$\mathrm{DTVaR}_{(\alpha,0)}^{(\delta,0)}(X_{t+v}|Z_{t+v}; C) = E(X_{t+v}|X_{t+v} > Q_{\alpha,t+v}, Z_{t+v} > Q_{\delta,t+v})$$
$$= \Pr(X_{t+v} > Q_{\alpha,t+v}, Z_{t+v} > Q_{\delta,t+v}|\mathcal{G}_t)$$
$$= \int_{Q_{\alpha,t+v}}^{\infty} \int_{Q_{\delta,t+v}}^{\infty} x_{t+v}\, c(F(x_{t+v}), F(z_{t+v})|\mathcal{G}_t) f_{X_{t+v}|\mathcal{G}_t}(x_{t+v})$$
$$\times f_{Z_{t+v}|\mathcal{G}_t}(z_{t+v})\, \mathrm{d}z_{t+v}\, \mathrm{d}x_{t+v}, \tag{28}$$

In the following, we show the empirical results of DTVaR on the energy market, with commodities being Gasoline, Heating oil, and Crude oil. According to Remark 3.3, the conditional kurtosis of the ARMA(1,1)-GJR-GARCH(1,1) model does not possess a closed form expression. This means that the first to fourth central moments do not have closed form expressions. Therefore, the forecasting of DTVaR is not carried out on ARMA(1,1)-GJR-GARCH(1,1) model but on the MA(1,1)-GJR-GARCH(1,1) model. As a comparison with the MA-GJR-GARCH−copula method, DTVaR was also forecasted using the GARCH−copula and GJR-GARCH−copula methods. Both GARCH and GJR-GARCH are specifications of the MA-GJR-GARCH.

5. Empirical Results

5.1. **Data.** This study intends to examine the accomplishment of the ARMA-GJR-GARCH-copula methodology on energy risk for the period from January 2, 2001 to December 31, 2009 with 2,251 daily observations. Energy risk is the returns of energy commodities. To investigate interactions in the energy market, three important commodities are selected, namely Crude oil from West Texas Intermediate

(WTI), Heating oil and Gasoline from New York Harbor (NYH). The data are obtained from [21]. The market returns of Gasoline, Heating oil, and Crude oil are shown in Fig. 1. All figures and computations in this article are generated using Matlab software.

We give definition for returns Z_t in the following,

$$Z_t = -\ln\Big(\frac{P_t}{P_{t-1}}\Big), \tag{29}$$

where P_t expresses the daily closing price for trading day t. Furthermore, we name the returns Z_t energy risk. In Fig. 1, it is clear that there is volatility clustering in the three commodity returns. Therefore, we examine if the squared returns are serially correlated, which is called the ARCH effects and displayed in Table 1.

TABLE 1. Descriptive statistics and Engle tests.

Statistics	Gasoline		Heating oil		Crude oil	
Sample size	2,252		2,252		2,252	
Mean	4.3595×10^{-4}		3.8029×10^{-4}		4.7553×10^{-4}	
Standard deviation	0.0310		0.0252		0.0263	
Skewness	0.0770		-0.1092		-0.1344	
Excess of kurtosis	3.9962		1.1700		4.1198	
Engle test	Q-statistic-Q	p-value	Q-statistic	p-value	Q-statistic	p-value
LM(4)	423.5103	0.0000	77.4134	0.0000	261.8488	0.0000
LM(6)	425.8648	0.0000	106.6309	0.0000	289.5009	0,0000
LM(8)	429.9366	0.0000	116.4914	0.0000	298.5928	0.0000
LM(10)	430.5990	0.0000	125.9035	0.0000	320.9760	0.0000

Table 2 displays statistical summary of energy risks and statistics tests for the ARCH effect. It can be seen that Gasoline has a positive skewness (0.0770), while Heating oil and Crude oil have a negative skewness (-0.1092 and -0.1344). The LM(K) statistic expressly accounts that the ARCH effects tend to be found in the returns of Gasoline, Heating oil, and Crude Oil. We consider the ARMA-GJR-GARCH model (including the MA-GJR-GARCH and GARCH) introduced on p. 387 to match the data of time series to generate i.i.d. observations to estimate the parameters of copula. In addition, the kurtosis is significant if it is larger than 3. It implies that the empirical observation of returns shows heavier tail than the normal distribution.

5.2. **The marginal distribution.** Since the return series possesses volatility clustering, we need to consider univariate distribution for adapting the empirical return distribution. Therefore, we take into consideration the univariate model presented in Section 3, the ARMA-GJR-GARCH model and its specifications, namely the MA-GJR-GARCH and the classic GARCH. We fit the three models for the return series of the three energy commodities as the initial models with normal distribution.

Tables 2, 3 and 4 show the maximum likelihood results of the model parameters, the Bayesian Information Criterion (BIC) and Akaike Information Criterion

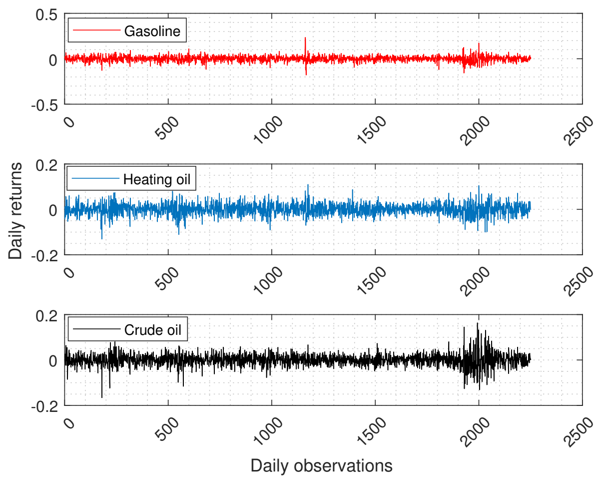


FIGURE 1. Daily returns of Gasoline, Heating oil, and Crude oil.

TABLE 2. Parameter estimates of GARCH model and statistic test.

	Gasoline		Heating oil		Crude oil	
Parameter	Value	Std	Value	Std	Value	Std
μ	0,0012	0,0006	0,0010	0,0005	0,0011	0,0005
ω_0	$5,6101 \times 10^{-5}$	$1,1063 \times 10^{-5}$	$1,2525 \times 10^{-5}$	$3,4717 \times 10^{-6}$	$1,1277 \times 10^{-5}$	$2,9417 \times 10^{-6}$
ω_1	0,8478	0,0168	0,9238	0,0109	0,9219	0,0103
ω_2	0,0914	0,0084	0,0564	0,0078	0,0596	0,0069
FLL	4.753,8		5.180,4		5.238,9	
AIC	-9.499,6		-10.353		-10.470	
BIC	-9.476,8		-10.330		-10.447	
Lag	p-value	Q-statistic	p-value	Q-statistic	p-value	Q-statistic
Uji Ljung-Box						
QW(1)	0,2723	1,2051	0,1892	4,7727	0,1347	8,4172
QW(3)	0,1392	2,1873	0,4489	2,6493	0,5918	3,7103
QW(5)	0,3624	10,8295	0,6354	1,7069	0,4584	4,6628
QW(7)	0,8335	0,0442	0,8079	0,9725	0,6122	3,5741
Uji Engle						
LM(4)	0,0014	17,6520	0,0012	21,9942	0,0026	23,7204
LM(6)	0,2500	5,3850	0,3147	7,0672	0,4924	7,4165
LM(8)	0,2649	5,2258	0,3963	6,2451	0,5708	6,6866
LM(10)	0,0648	8,8564	0,0895	10,9632	0,1576	11,8586

(AIC) for model selection. For Gasoline returns, based on the smallest AIC and BIC values, the most suitable model is GARCH(1,1). As for Heating oil and Crude oil returns, the most suitable model according to the smallest AIC value is the ARMA(1,1)-GJR-GARCH(1,1), but that in accordance with the smallest BIC value is the GARCH(1,1). In Fig. 2, it can be seen that the volatilities of Gasoline and Crude oil returns are greater than that of Heating oil return in the three volatility models.

TABLE 3. Parameter estimates of MA-GJR-GARCH model.

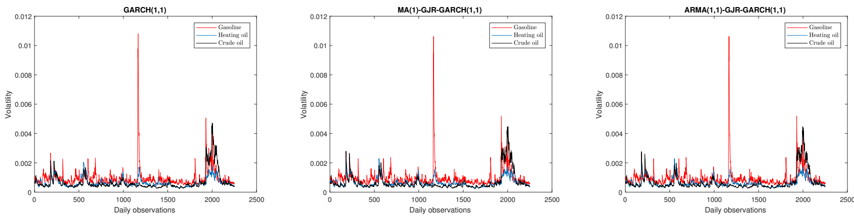
	Gasoline		Heating oil		Crude oil	
Parameter	Value	Std	Value	Std	Value	Std
μ	-0,0012	0,0006	-0,0009	0,0005	-0,0008	0,0005
η_1	0,0200	0,0224	-0,0332	0,02206	-0,0269	0,0217
ω_0	$5,6680 \times 10^{-5}$	$1,2191 \times 10^{-5}$	$1,4185 \times 10^{-5}$	$3,8395 \times 10^{-6}$	$1,1502 \times 10^{-5}$	$2,6801 \times 10^{-6}$
ω_1	0,8465	0,0206	0,9210	0,0115	0,9257	0,0096
ω_2	0,0944	0,0188	0,0667	0,0108	0,0805	0,0093
ϕ	-0,0050	0,0203	-0,0212	0,0141	-0,0524	0,0118
FLL	4.754,2		5.182,6		5.244,7	
AIC	-9.496		-10.353		-10.477	
BIC	-9.462		-10.319		-10.443	

TABLE 4. Parameter estimates of ARMA-GJR-GARCH model.

	Gasoline		Heating oil		Crude oil	
Parameter	Value	Std	Value	Std	Value	Std
μ	-0,0014	0,0014	-0,0015	0,0008	-0,0032	0,0020
κ_1	-0,1720	1,0633	-0,7091	0,1488	-0,9884	0,0066
η_1	0,1918	1,0599	0,6837	0,1542	0,9891	0,0072
ω_0	$5,6637 \times 10^{-5}$	$1,2239 \times 10^{-5}$	$1,4434 \times 10^{-5}$	$3,9142 \times 10^{-6}$	$6,7910 \times 10^{-5}$	$2,0815 \times 10^{-5}$
ω_1	0,8466	0,0207	0,9199	0,0117	0,9209	0,0129
ω_2	0,0944	0,0188	0,0681	0,0111	0,0598	0,0117
ϕ	-0,0051	0,0203	-0,0225	0,0145	-0,0137	0,0134
FLL	4.754,3		5.183,9		5.246,6	
AIC	-9.494,6		-10.354		-10.479	
BIC	-9.454,5		-10.314		-10.439	

Table 2 provides that the Ljung-Box test employed to residuals of the GARCH(1,1) does not refuse the H_0 of autocorrelations at lags 1, 3, 5, and 7, at significance level 5%. The square of series of residuals examined by the Engle test does not reject the null hypothesis of ARCH effects either at lags 6, 8, and 10, at significance level 5%. However, for lag 4, the Engle test refuse the H_0 of ARCH effect. On the other hand, Brooks [22] asserted that "the GARCH(1,1) model is adequate to catch volatility clustering in the data, and it is rare for higher-order models to be estimated in the academic finance literature". Therefore, order 1 in the three GARCH models is still selected. Furthermore, more complex models such as MA-GJR-GARCH(1,1) and ARMA-GJR-GARCH(1,1) are also applied for the three energy risks.

However, in this article, the forecast of DTVaR using the ARMA-GJR-GARCH-copula method is only carried out on the specifications of the method, namely MA-GJR-GARCH-copula and GARCH-copula. This is due to the difficulty of obtaining a conditional distribution of the ARMA-GJR-GARCH model (see Remark 3.3 on p. 390). Next, we divide the data into 2 groups, namely data of sample-in and data of sample-out to verify whether the forecasted DTVaR is sufficient. The data of sample-in comprise the first 1,400 observations, and the remaining 851 observations are data of sample-out for testing. All distributions of univariate models and copula functions are estimated applying the sample-in data containing 1,400 return observations.



(a) (b) (c)

FIGURE 2. Generated volatilities of the returns of Gasoline, Heating oil, and Crude oil in the three volatility models.

5.3. **Copula modeling.** Shortly after estimating the parameters of the univariate distribution F_{X_i}, we then carry out the estimation of the copula parameters as described previously on p. 394. Four Copula functions are applied in this study: Clayton, Gumbel, Frank, and Gaussian. Under the method of MLE, the picked copula functions will be matched to these series of residuals. The results of the Copula modeling are shown in Table 5, where the results are estimated using the method of MLE.

TABLE 5. Copula parameter estimates.

Copula	Return	Parameter	MA-GJR-GARCH	GARCH
Clayton	Gasoline	$\theta_{\mathrm{C}}(X_t, Z_t)$	1.3635	1.2852
	Heating oil	$\theta_{\mathrm{C}}(Y_t, Z_t)$	1.9419	2.1471
Gumbel	Gasoline	$\theta_{\mathrm{G}}(X_t, Z_t)$	1.8355	1.9506
	Heating oil	$\theta_{\mathrm{G}}(Y_t, Z_t)$	2.2503	2.2331
Frank	Gasoline	$\theta_{\mathrm{F}}(X_t, Z_t)$	7.2748	6.8273
	Heating oil	$\theta_{\mathrm{F}}(Y_t, Z_t)$	1.6767	1.7283
Gaussian	Gasoline	$\theta_{\mathrm{N}}(X_t, Z_t)$	0.6938	0.6625
	Heating oil	$\theta_{\mathrm{N}}(Y_t, Z_t)$	0.7815	0.7743

5.4. **Forecasting DTVaR.** We have followed the algorithm developed by [12] for forecasting VaR. Initially, this article applies the sample-in data containing 1,400 return observations to forecast $\mathrm{VaR}_{1,401}$ at time $t = 1,401$, and at every novel observation we re-forecast VaR, because of the level of conditional probability and the VaR forecasting formula. It implies that we forecast $\mathrm{VaR}_{1,402}$ by utilizing observations $t = 2$ to $t = 1,401$ and forecast $\mathrm{VaR}_{1,403}$ by utilizing observations $t = 3$ to $t = 1,402$ till the sample-out observations we renew are applied.

The following is a DTVaR forecasting algorithm using the ARMA-GJR-GARCH$-$copula method wih Monte Carlo method:

(1) Estimate the ARMA-GJR-GARCH model parameters for the energy risks of Gasoline (X_t), Heating oil (Y_t), and Crude oil (Z_t).
(2) Transforms each returns to a Uniform distribution $U(0,1)$ employing the distribution function,

$$u_t = F_X(x_t), \quad v_t = F_Y(y_t), \quad w_t = F_Z(z_t),$$

$t = 1, 2, \cdots, 1,400$.

(3) Estimate copula parameters from paired return data of $\{(u_t, w_t)\}$, and $\{(v_t, w_t)\}$, for $t = 1, \cdots, 1,400$.

(4) Generate paired data $\{(u'_t, w'_t)\}$ and $\{(v'_t, w'_t)\}$, with each component whose value is in the interval $(0, 1)$, based on the Copula parameter estimate in Step 3.

(5) Retransform each element of the paired data in Step 4 to the original distribution using the inverse of the distribution function,

$$x'_t = F_X^{-1}(u'_t), \quad y'_t = F_Y^{-1}(v_t), \quad z'_t = F_Z^{-1}(w'_t).$$

(6) Forecast VaR of X_{t+1} and VaR of Y_{t+1} at probability level α and VaR of Z_{t+1} at excess level δ for $t = 1, \cdots, 1,400$.

(7) Calculate DTVaRs of Gasoline returns (X_t) and Heating oil returns (Y_t) associated with Crude oil returns (Z_t) for $t = 1, \cdots, 1,400$, taking into account: the values of x_{t+1} and z_{t+1} that satisfy $x_{t+1} > \mathrm{VaR}_\alpha(X_{t+1})$ and $z_{t+1} > \mathrm{VaR}_\delta(Z_{t+1})$; y_{t+1} and z_{t+1} that satisfy $y_{t+1} > \mathrm{VaR}_\alpha(Y_{t+1})$ and $z_{t+1} > \mathrm{VaR}_\delta(Z_{t+1})$.

(8) Repeat Steps 1-7 as many times as $m = 100$ (Monte Carlo method).

(9) Calculate the mean of the forecasted values of the DTVaR generated in Step 8.

The numbers of DTVaR forecast violations calculated using four copula functions are shown in Tables 6-9. In this context, the number of violations is the number of sample observations beyond the critical value, which are larger than the DTVaR forecast. Table 6 shows that the DTVaR forecast of Gasoline returns, which are associated with Crude oil returns using MA(1)-GJR-GARCH(1,1)-Clayton copula method, has the lowest distinction between the bivariate significance level (b.s.l.) and percentage of violations (0.67%, 1.37%, 1, 50%, 0.21%). Thus, this shows that the Clayton copula is the most precise copula to describe the bivariate distribution of Gasoline returns and Crude oil returns for the MA(1)-GJR-GARCH(1,1) model. Moreover, based on Table 7, the DTVaR forecast of Heating oil returns, which are associated with Crude oil returns using MA(1)-GJR-GARCH(1,1)-Frank copula method, has the smallest difference between b.s.l. and the percentage of violations (1.30%, 0.57%, 1.51%, 1.04%). This shows that Frank copula is the most precise copula to describe the bivariate distribution of Heating oil returns and Crude oil returns.

Table 8 shows that the DTVaR forecast of Gasoline returns, which are associated with Crude oil returns using the GARCH(1,1)-Clayton copula, has the smallest difference between b.s.l. and the percentage of violations (0.62%, 1, 26%, 1.42%, 0.13%). Hence, this shows that the Clayton copula is the most precise copula to describe the joint distribution of Gasoline returns and Crude oil returns for the GARCH(1,1) model. Meanwhile, based on Table 9, the DTVaR forecast of Heating oil returns, which are associated with Crude oil returns using Frank copula, has the smallest difference between b.s.l. and the percentage of violations (1.33%, 0.62%, 1.55%, 0.97%).

TABLE 6. The bivariate significance level (b.s.l.) and the number of violations of the forecast of $\mathrm{DTVaR}_{(\alpha,0)}^{(\delta,0)}$ of Gasoline returns using MA(1)-GJR-GARCH(1,1)-copula.

Copula	Parameter			
Clayton	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\theta_{\mathrm{C}}}=1.3635$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	2.08	4.42	3.03	3.03
No. violations	12	26	13	24
% violations	1.41	3.05	1.53	2.82
Difference	0.67	1.37	1.50	0.21

Copula	Parameter			
Gumbel	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\theta_{\mathrm{G}}}=1.8355$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	5.75	8.89	6.95	6.95
No. violations	13	26	17	19
% violations	1.53	3.05	2.00	2.23
Difference	4.22	5.84	4.95	4.72

Copula	Parameter			
Frank	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\theta_{\mathrm{F}}}=7.2748$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	4.28	8.01	5.79	5.79
No. violations	8	26	12	16
% violations	0.94	3.05	1.41	1.88
Difference	3.34	4.96	4.38	3.91

Copula	Parameter			
Gaussian	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\rho}=0.6938$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	4.63	7.75	5.89	5.89
No. violations	9	19	11	13
% violations	1.06	2.23	1.29	1.53
Difference[a]	3.57	5.52	4.60	4.36

[a]Difference denotes absolute difference between bivariate significance level (b.s.l) and percentage of violations.

Thus, this shows that Frank copula is the most precise copula to describe the joint distribution of Heating oil and Crude oil returns.

Note that for Gasoline returns, the difference between b.s.l. and the percentage of violations for the GARCH-Clayton copula method is always smaller than the difference for the MA-GJR-GARCH-Clayton copula method. This shows that the simpler GARCH-Clayton copula is better in forecasting DTVaR. An illustration of the DTVaR forecast of Gasoline returns using the GARCH-copula method is presented in Fig. 3. We can see the DTVaR forecast are almost always greater than Gasoline returns.

TABLE 7. The b.s.l. and the number of violations of the forecast of $\mathrm{DTVaR}_{(\alpha,0)}^{(\delta,0)}$ of Heating oil returns using MA(1)-GJR-GARCH(1,1)-copula.

Copula Clayton ($\widehat{\theta_C} = 1.9419$)	Parameter $\alpha=\delta=0.80$ $C_\theta(0.90, 0.90)$	$\alpha=\delta=0.85$ $C_\theta(0.85, 0.85)$	$\alpha=0.80, \delta=0.85$ $C_\theta(0.90, 0.85)$	$\alpha=0.85, \delta=0.80$ $C_\theta(0.85, 0.90)$
b.s.l.	2.47	5.14	3.55	3.55
No. violations	9	28	13	25
% violations	1.06	3.29	1.53	2.93
Difference	1.41	1.85	2.02	0.62

Copula Gumbel ($\widehat{\theta_G} = 2.2503$)	Parameter $\alpha=\delta=0.90$ $C_\theta(0.90, 0.90)$	$\alpha=\delta=0.85$ $C_\theta(0.85, 0.85)$	$\alpha=0.90, \delta=0.85$ $C_\theta(0.90, 0.85)$	$\alpha=0.85, \delta=0.90$ $C_\theta(0.85, 0.90)$
b.s.l.	6.64	10.16	7.92	7.92
No. violations	11	24	19	21
% violations	1.29	2.82	2.23	2.46
Difference	5.35	7.34	5.69	5.46

Copula Frank ($\widehat{\theta_F} = 1.6767$)	Parameter $\alpha=\delta=0.80$ $C_\theta(0.90, 0.90)$	$\alpha=\delta=0.85$ $C_\theta(0.85, 0.85)$	$\alpha=0.80, \delta=0.85$ $C_\theta(0.90, 0.85)$	$\alpha=0.85, \delta=0.80$ $C_\theta(0.85, 0.90)$
b.s.l.	1.77	3.74	2.57	2.57
No. violations	4	27	9	13
% violations	0.47	3.17	1.06	1.53
Difference	1.30	0.57	1.51	1.04

Copula Gaussian ($\widehat{\rho} = 0.7815$)	Parameter $\alpha=\delta=0.90$ $C_\theta(0.90, 0.90)$	$\alpha=\delta=0.85$ $C_\theta(0.85, 0.85)$	$\alpha=0.90, \delta=0.85$ $C_\theta(0.90, 0.85)$	$\alpha=0.85, \delta=0.90$ $C_\theta(0.85, 0.90)$
b.s.l.	5.43	8.86	6.79	6.79
No. violations	9	23	11	20
% violations	1.06	2.70	1.29	2.35
Difference	4.37	6.16	5.50	4.44

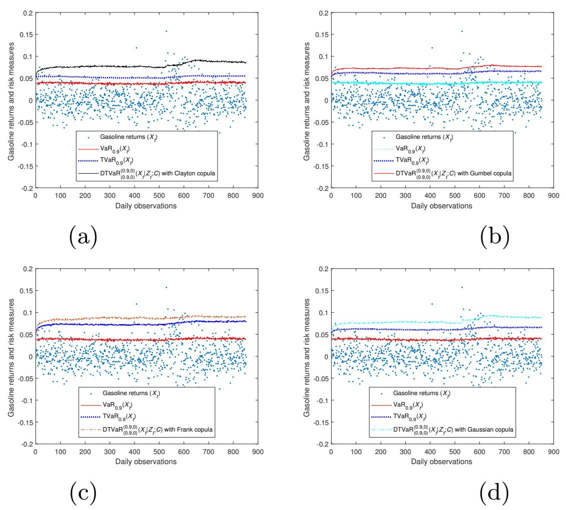


FIGURE 3. DTVaR forecast of Gasoline returns associated with Crude oil returns, using the GARCH(1,1)-copula method, compared with VaR and TVaR forecasts.

TABLE 8. The b.s.l. and the number of violations of the forecast of $\mathrm{DTVaR}_{(\alpha,0)}^{(\delta,0)}$ of Gasoline returns using GARCH(1,1)-copula.

Copula	Parameter			
Clayton	$\alpha=\delta=0,90$	$\alpha=\delta=0,85$	$\alpha=0,90,\delta=0,85$	$\alpha=0,85,\delta=0,90$
$(\widehat{\theta_{\mathrm{C}}}=1,2852)$	$C_\theta(0,90,0,90)$	$C_\theta(0,85,0,85)$	$C_\theta(0,90,0,85)$	$C_\theta(0,85,0,90)$
b.s.l.	2,03	4,31	2,95	2,95
no. violations	12	26	13	24
% violations	1,41	3,05	1,53	2,82
Difference	0,62	1,26	1,42	0,13

Copula	Parameter			
Gumbel	$\alpha=\delta=0,90$	$\alpha=\delta=0,85$	$\alpha=0,90,\delta=0,85$	$\alpha=0,85,\delta=0,90$
$(\widehat{\theta_{\mathrm{G}}}=1,9506)$	$C_\theta(0,90,0,90)$	$C_\theta(0,85,0,85)$	$C_\theta(0,90,0,85)$	$C_\theta(0,85,0,90)$
b.s.l.	6.04	9.31	7.27	7.27
no. violations	14	25	16	19
% violations	1.64	2.93	1.88	2.23
Difference	4.40	6.38	5.39	5.04

Copula	Parameter			
Frank	$\alpha=\delta=0,90$	$\alpha=\delta=0,85$	$\alpha=0,90,\delta=0,85$	$\alpha=0,85,\delta=0,90$
$(\widehat{\theta_{\mathrm{F}}}=6.8273)$	$C_\theta(0,90,0,90)$	$C_\theta(0,85,0,85)$	$C_\theta(0.90,0.85)$	$C_\theta(0,85,0,90)$
b.s.l.	4.12	7.76	5.59	5.59
no. violations	8	26	13	17
% violations	0.94	3.05	1.53	2.00
Difference	3.18	4.71	4.06	3.59

Copula	Parameter			
Gaussian	$\alpha=\delta=0,90$	$\alpha=\delta=0,85$	$\alpha=0,90,\delta=0,85$	$\alpha=0,85,\delta=0,90$
$(\widehat{\rho}=0,6625)$	$C_\theta(0,90,0,90)$	$C_\theta(0,85,0,85)$	$C_\theta(0,90,0,85)$	$C_\theta(0,85,0,90)$
b.s.l.	4,37	7,39	5,60	5,60
no. violations	10	19	12	13
% violations	1,17	2,23	1,41	1,53
Difference	3.20	5.16	4,19	4.07

For Heating oil returns, the difference for the MA-GJR-GARCH-Frank copula method is always smaller than the difference for the GARCH-Frank copula method. This shows that the more complicated MA-GJR-GARCH-Frank copula is better in forecasting DTVaR. An illustration of the DTVaR forecast of Heating oil returns using the MA-GJR-GARCH-copula method is presented in Fig.4.

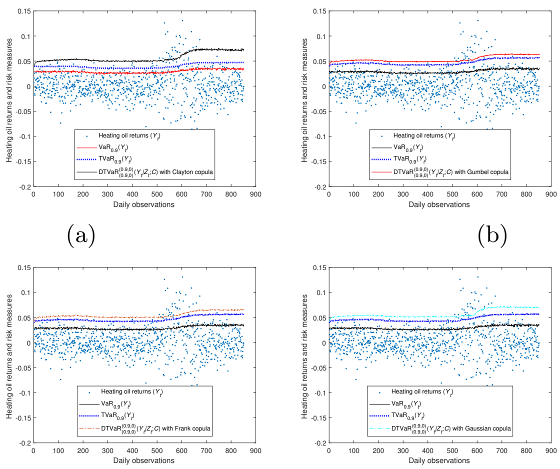
Table 9. The b.s.l. and the number of violations of the forecast of $\mathrm{DTVaR}_{(\alpha,0)}^{(\delta,0)}$ of Heating oil returns using GARCH(1,1)-copula.

Copula	Parameter			
Clayton	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\theta_C}=2.1471$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	2.66	5.48	3.80	3.80
No. violations	8	27	12	25
% violations	0.94	3.17	1.41	2.93
Difference	1.72	2.31	2.39	0.87

Copula	Parameter			
Gumbel	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\theta_G}=2.2331$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	6.61	10.12	7.88	7.88
No. violations	11	26	19	21
% violations	1.29	3.05	2.23	2.46
Difference	5.32	7.07	5.65	5.42

Copula	Parameter			
Frank	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\theta_F}=1.7283$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	1.80	3.79	2.61	2.61
No. violations	4	27	9	14
% violations	0.47	3.17	1.06	1.64
Difference	1.33	0.62	1.55	0.97

Copula	Parameter			
Gaussian	$\alpha=\delta=0.90$	$\alpha=\delta=0.85$	$\alpha=0.90,\delta=0.85$	$\alpha=0.85,\delta=0.90$
($\widehat{\rho}=0.7743$)	$C_\theta(0.90,0.90)$	$C_\theta(0.85,0.85)$	$C_\theta(0.90,0.85)$	$C_\theta(0.85,0.90)$
b.s.l.	5.36	8.76	6.71	6.71
No. violations	9	23	11	21
% violations	1.06	2.70	1.29	2.46
Difference	4.30	6.06	5.42	4.25



(a) (b)

(c) (d)

Figure 4. DTVaR forecast of Heating oil returns associated with Crude oil returns, using the MA(1,1)-GJR-GARCH(1,1)-copula method, compared with VaR and TVaR forecasts.

Furthermore, if we compare the differences between b.s.l. and the percentage of violations of the forecasts of DTVaR of Gasoline returns using GARCH-Clayton copula (1.30%, 0.57%, 1.51%, 1.04%) and those of the forecasts of DTVaR of Heating oil returns using MA-GJR-GARCH-Frank copula (0.62%, $1,26\%$, 1.42%, 0.13%), then we find that the forecasts of DTVaR of Heating oil returns using MA-GJR-GARCH-Frank copula is more accurate than those of Gasoline returns using GARCH-Clayton copula. Consequently, in this context, energy commodities market players should invest in Heating oil. However, different results could occur if, for example, we increase or decrease daily observations.

6. Conclusions

This article explains a model for forecasting a copula-based extension TVaR by the ARMA-GJR-GARCH-conditional copula model combined with Monte Carlo method. Firstly, we have derived analytical expressions for the unconditional moments of the returns that follow the ARMA(1,1)-GJR-GARCH(1,1) model. For the MA(1)-GJR-GARCH model, we obtain the unconditional moments of the returns. Moreover, we provide several risk measures for the ARMA$(1,1)$-GJR-GARCH$(1,1)$ model, such as VaR, TVaR, and DTVaR. Furthermore, the DTVaR forecasting algorithm using the ARMA(1,1)-GJR-GARCH(1,1)- copula method accompanied by Monte Carlo method is also proposed, in which the innovation is assumed to be normally distributed.

Second, we find that this method is quite robust in forecasting DTVaR. The ARMA-GJR-GARCH model allows for a highly supple bivariate distribution by separating the marginal conducts from the dependence relationship. This article estimates several copulas with distinct univariate marginal distributions. We obtain that, for both GARCH(1,1) and MA(1)-GJR-GARCH(1,1) models, the Clayton copula is the most precise copula to describe the joint distribution of the energy risks of Gasoline and Crude oil, while Frank copula is the most precise copula for the energy risks of Heating oil and Crude oil. However, the simpler GARCH-Clayton copula is better in forecasting DTVaR of Gasoline energy risk than the MA-GJR-GARCH-Clayton copula. On the other hand, the more complicated MA-GJR-GARCH-Frank copula is better in forecasting DTVaR of Heating oil energy risk than the GARCH-Frank copula. Furthermore, we find that the forecast of DTVaR of Heating oil returns using MA-GJR-GARCH-Frank copula is more accurate than that of Gasoline returns using GARCH-Clayton copula. Therefore, in this context, energy sector market players should invest in Heating oil rather than in Gasoline.

We recognize that these empirical forecasts may differ using different datasets. Moreover, the results are also likely to be influenced by the assumed distribution of innovation. For future research, we will forecast DTVaR where innovation is assumed to follow heavy tail distributions such as Student's t or two-sided Weibull and apply them to different data of returns. We will also use other copulas, namely the Student's-t copula and the Archimedean copula family such as Rotated-Clayton copula, Plackett copula, and the Rotated-Gumbel copula.

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Appendix

Proof for Proposition 3.1. First, note that,

$$E[Z_t] = E[\mu + \kappa_1 Z_{t-1} + \eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} + h_t^{1/2}\varepsilon_t] = \mu + \kappa_1 E[Z_{t-1}] = \frac{\mu}{1-\kappa_1}.$$

Next, we have $E[Z_t^2]$ given by,

$$\begin{aligned} E[Z_t^2] &= E[(\mu + \kappa_1 Z_{t-1} + \eta_1 \xi_{t-1} + \xi_t)^2] \\ &= E[(\mu^2 + \kappa_1^2 Z_{t-1}^2 + 2\mu\kappa_1 Z_{t-1}) + (\eta_1^2 \xi_{t-1}^2 + \xi_t^2 + 2\eta_1 \xi_{t-1}\xi_t) \\ &\quad + 2\mu\eta_1\xi_{t-1} + 2\mu\xi_t + 2\kappa_1\eta_1 Z_{t-1}\xi_{t-1} + 2\kappa_1 Z_{t-1}\xi_t] \\ &= \mu^2 + \kappa_1^2 E[Z_t^2] + 2\mu\kappa_1 \frac{\mu}{1-\kappa_1} + (\eta_1^2 + 1)E[h_t] + 2\kappa_1\eta_1 E[Z_{t-1}\xi_{t-1}]. \end{aligned}$$

Consider $E[Z_{t-1}\xi_{t-1}]$,

$$E[Z_{t-1}\xi_{t-1}] = E[\mu\xi_{t-1} + \kappa_1 Z_{t-2}\xi_{t-1} + \eta_1\xi_{t-2}\xi_{t-1} + \xi_{t-1}^2] = E[h_t\varepsilon_{t-1}^2] = E[h_t].$$

Hence,

$$\begin{aligned} E[Z_t^2] &= \frac{\mu^2}{(1-\kappa_1)^2} + \frac{2\mu^2\kappa_1}{(1-\kappa_1)(1-\kappa_1^2)} + \frac{\eta_1^2 + 2\kappa_1\eta_1 + 1}{1-\kappa_1^2} E[h_t] \\ &= \frac{\mu^2}{(1-\kappa_1)^2} + \frac{2\mu^2\kappa_1}{(1-\kappa_1)(1-\kappa_1^2)} + \frac{\eta_1^2 + 2\kappa_1\eta_1 + 1}{1-\kappa_1^2} \frac{\omega_0}{1-\varphi}, \end{aligned}$$

where $\varphi = \omega_2 + \frac{\phi}{2} + \omega_1$. Therefore,

$$\mathrm{Var}[Z_t] = \frac{2\mu^2\kappa_1}{(1-\kappa_1)(1-\kappa_1^2)} + \frac{\eta_1^2 + 2\kappa_1\eta_1 + 1}{1-\kappa_1^2} \frac{\omega_0}{1-\varphi}.$$

□

Proof for Proposition 3.2. We know $E[Z_t] = \frac{\mu}{1-\kappa_1}$, so that,

$$\begin{aligned} Z_t - E[Z_t] &= \frac{\kappa_1\mu}{\kappa_1 - 1} + \eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} + h_t^{1/2}\varepsilon_t, \\ (Z_t - E[Z_t])^2 &= \frac{\kappa_1^2\mu^2}{(\kappa_1-1)^2} + \frac{2\kappa_1\mu}{\kappa_1-1}(\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} + h_t^{1/2}\varepsilon_t) \\ &\quad + \eta_1^2 h_{t-1}\varepsilon_{t-1}^2 + h_t\varepsilon_t^2 + 2\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} h_t^{1/2}\varepsilon_t, \\ (Z_t - E[Z_t])^4 &= \frac{\kappa_1^4\mu^4}{(\kappa_1-1)^4} + \frac{4\kappa_1^3\mu^3}{(\kappa_1-1)^3}(\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} + h_t^{1/2}\varepsilon_t) \\ &\quad + \frac{4\kappa_1^2\mu^2}{(\kappa_1-1)^2}(\eta_1^2 h_{t-1}\varepsilon_{t-1}^2 + 2\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1}h_t^{1/2}\varepsilon_t + h_t\varepsilon_t^2) \\ &\quad + \eta_1^4 h_{t-1}^2\varepsilon_{t-1}^4 + 4\eta_1^3 h_{t-1}^{3/2}\varepsilon_{t-1}^3 h_t^{1/2}\varepsilon_t + 4\eta_1^2 h_{t-1}\varepsilon_{t-1}^2 h_t\varepsilon_t^2 \\ &\quad + 2\eta_1^2 h_{t-1}\varepsilon_{t-1}^2 h_t\varepsilon_t^2 + 4\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1} h_t^{3/2}\varepsilon_t^3 + h_t^2\varepsilon_t^4 \\ &\quad + \frac{4\kappa_1\mu}{\kappa_1-1}(\eta_1^2 h_{t-1}\varepsilon_{t-1}^2 h_t^{1/2}\varepsilon_t + h_{t-1}^{3/2}\varepsilon_t^3 + 2\eta_1 h_{t-1}^{1/2}\varepsilon_{t-1}h_t\varepsilon_t^2 \\ &\quad + \eta_1^3 h_{t-1}^{3/2}\varepsilon_{t-1}^3 + \eta_1 h_{t-1}^{1/2}\varepsilon_{t-1}h_t\varepsilon_t^2 + 2\eta_1^2 h_{t-1}\varepsilon_{t-1}^2 h_t^{1/2}\varepsilon_t). \end{aligned}$$

According to [23], we obtain,

$$E[h_t] = \frac{\omega_0}{1-\varphi} = \bar{h}, \quad E[h_t^2] = \frac{\omega_0^2 + 2\omega_0\varphi\bar{h}}{1-\gamma} = \upsilon,$$

where

$$\gamma = \varphi^2 + 2\Big(\omega_2 + \frac{\phi}{2}\Big)^2 + \frac{3}{4}\phi^2, \qquad \bar{h} = \frac{\omega_0}{1-\varphi}.$$

Hence,

$$\begin{aligned} E[(Z_t - E[Z_t])^2] &= \frac{\kappa_1^2\mu^2}{(\kappa_1-1)^2} + (1+\eta_1^2)\bar{h} \\ E[(Z_t - E[Z_t])^4] &= \frac{\kappa_1^4\mu^4}{(\kappa_1-1)^4} + \frac{4\kappa_1^2\mu^2(\eta_1^2+1)}{(\kappa_1-1)^2}\bar{h} + 6\eta_1^2\bar{h}^2 + 3(\eta_1^4+1)\upsilon. \end{aligned}$$

Therefore,

$$\varkappa_Z = \frac{\frac{\kappa_1^4\mu^4}{(\kappa_1-1)^4} + \frac{4\kappa_1^2\mu^2(\eta_1^2+1)}{(\kappa_1-1)^2}\bar{h} + 6\eta_1^2\bar{h}^2 + 3(\eta_1^4+1)\upsilon}{\left(\frac{\kappa_1^2\mu^2}{(\kappa_1-1)^2} + (1+\eta_1^2)\bar{h}\right)^2}.$$

□

Proof for Theorem 3.4. First, consider the return Z_{t+v},

$$\begin{aligned} E[Z_{t+v}|\mathcal{G}_t] &= E[(\mu + \eta_1 h_{t+v-1}^{1/2}\varepsilon_{t+v-1} + h_{t+v}^{1/2}\varepsilon_{t+v})|\mathcal{G}_t] \\ &= \mu + \eta_1 E[h_{t+v-1}^{1/2}|\mathcal{G}_t]E[\varepsilon_{t+v-1}] + E[h_{t+v}^{1/2}|\mathcal{G}_t]E[\varepsilon_{t+v}] \\ &= \mu. \end{aligned}$$

Note that,

$$\begin{aligned} Z_{t+v}^2 &= \mu^2 + 2\mu\eta_1\xi_{t+v-1} + \eta_1^2\xi_{t+v-1}^2 + 2\mu\xi_{t+v} + 2\eta_1\xi_{t+v-1}\xi_{t+v} + \xi_{t+v}^2 \\ &= \mu^2 + 2\mu\eta_1 h_{t+v-1}^{1/2}\varepsilon_{t+v-1} + \eta_1^2 h_{t+v-1}\varepsilon_{t+v-1}^2 + 2\mu h_{t+v}^{1/2}\varepsilon_{t+v} \\ &\quad + 2\eta_1 h_{t+v-1}^{1/2}\varepsilon_{t+v-1}h_{t+v}^{1/2}\varepsilon_{t+v} + h_{t+v}\varepsilon_{t+v}^2, \end{aligned}$$

so that,

$$E[Z_{t+v}^2|\mathcal{G}_t] = \mu^2 + \eta_1^2 E[h_{t+v-1}|h_t^{1/2}] + E[h_{t+v}|h_t^{1/2}].$$

According to Theorem 1 in [23],

$$E[h_{t+v}|h_t^{1/2}] = \bar{h} + \varphi^{v-1}(h_{t+1} - \bar{h}).$$

Hence,

$$\begin{aligned} E[Z_{t+v}^2|\mathcal{G}_t] &= \mu^2 + \eta_1^2(\bar{h} + \varphi^{v-2}(h_{t+1} - \bar{h})) + (\bar{h} + \varphi^{v-1}(h_{t+1} - \bar{h})) \\ &= \mu^2 + (\eta_1^2+1)\bar{h} + (\eta_1^2\varphi^{v-2} + \varphi^{v-1})(h_{t+1} - \bar{h}). \end{aligned}$$

Thus, the conditional variance of the return is given by,

$$\mathrm{Var}[Z_{t+v}|\mathcal{G}_t] = (\eta_1^2+1)\bar{h} + (\eta_1^2\varphi^{v-2} + \varphi^{v-1})(h_{t+1} - \bar{h}).$$

□

Proof for Lemma 3.6. Consider Theorem 3.4. Taking the limit to infinity of v, we obtain (21) and (22). □