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Generalized $(k, t)-$Narayana Sequence

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Abstract. In this paper, we introduces Narayana sequence in two parameters, namely, (k, t)-Narayana sequence, which is generalization of classical Narayana sequence and provide some identities and matrix expressions. Further, we find relations between (k, t)-Narayana numbers and determinants and permanents of some Hessenberg matrices. We study recurrence relations and the sum of the first n terms of this sequence. We obtain some properties from matrices. Additionally, we define $(k, t)-$Narayana sequence for negative subscripts and derive some relations.

Key words and Phrases: Binet's formula; Fibonacci sequence; Hessenberg matrix; Narayana sequence; Permanent.

1. INTRODUCTION

Narayana sequence was introduced in the 14^{th} century by the Indian mathematician Narayana Pandita. He defined this sequence as the solution to the problem of *herd of cows and calves*:"Every year a cow produces one calf. When its fourth year begins, at the beginning of each year each calf produces one calf. The problem is How many cows are there together after, for example, 20 years" [1].
This problem is solved in a similar manner as Fibonacci solved *Rabbit problem* [2]. As a solution to this problem, we get Narayana numbers, which are defined as

$$N_n = N_{n-1} + N_{n-3} \quad \text{for} \quad \text{all } n \geq 3$$

with the initial conditions

$$N_0 = 0, \ N_1 = 1, \ N_2 = 1.$$

Some of the terms are:

$$0, \ 1, \ 1, \ 1, \ 2, \ 3, \ 4, \ 6, \ 9, \ 13, \ 19, ...$$

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Binet's formula [3] for Narayana sequence is

$$N_n = \frac{\tilde{\alpha}^{n+1}}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})} + \frac{\tilde{\beta}^{n+1}}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})} + \frac{\tilde{\gamma}^{n+1}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})} \tag{1}$$

where $\tilde{\alpha},\ \tilde{\beta}\ \textit{and}\ \tilde{\gamma}$ are the roots of char. equation $x^3 - x^2 - 1 = 0$.

The Narayana sequence is defined by a third order recurrence relation, whereas Fibonacci and Lucas sequences are defined by second order recurrence relations respectively as follows:

$$F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad \text{all } n \geq 2$$

with the initial conditions

$$F_0 = 0,\ F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad \text{all } n \geq 2$$

with the initial conditions

$$L_0 = 2,\ L_1 = 1.$$

Fibonacci and Lucas numbers have applications [4, 5] in many fields. Narayana numbers are also used in fields [6] such as coding theory, cryptography, and communication systems. In [7] Kirthi et al. gave coding algorithms named Narayana universal codes using Narayana numbers. In [8] Das et al. gave second order Narayana universal codes which is extension of codes given by Kirthi. They strengthen the security in sending information due to the formation of straight lines.
Jose et al. [9] defined k-Narayana sequence as

$$b_{k,n} = kb_{k,n-1} + b_{k,n-3} \quad \text{for} \quad \text{all } n \geq 3$$

with the initial conditions

$$b_{k,0} = 0,\ b_{k,1} = 1,\ b_{k,2} = k.$$

The authors [9] studied recurrence relations, Binet's formula and some other properties of this sequence. Binet's formula is

$$b_{k,n} = \frac{\tilde{\alpha}_k^{n+1}}{(\tilde{\alpha}_k-\tilde{\beta}_k)(\tilde{\alpha}_k-\tilde{\gamma}_k)} + \frac{\tilde{\beta}_k^{n+1}}{(\tilde{\beta}_k-\tilde{\alpha}_k)(\tilde{\beta}_k-\tilde{\gamma}_k)} + \frac{\tilde{\gamma}_k^{n+1}}{(\tilde{\gamma}_k-\tilde{\alpha}_k)(\tilde{\gamma}_k-\tilde{\beta}_k)} \tag{2}$$

where $\tilde{\alpha}_k,\ \tilde{\beta}_k$ and $\tilde{\gamma}_k$ are the roots of the equation $z^3 - kz^2 - 1 = 0$.
They related this sequence to determinants of Hessenberg matrices.

Goy [10] studied Toeplitz-Hessenberg determinants with entries for Fibonacci-Narayana numbers or Narayana numbers. He established connections between determinants of Toeplitz-Hessenberg matrices with entries of Fibonacci-Narayana numbers and Fibonacci, Tribonacci numbers.

In [13, 14, 15, 16] Mishra and Bala studied Fibonacci sequence in circulant matrices, in Diophantine equations and matrix form of classical Narayana sequence.
The present paper is organized in a total of 7 sections.

2. $(k,t)-$Narayana sequence

In this section, we introduces a generalization of Narayana sequence in two parameters, i.e. $(k,t)-$ Narayana sequence. Here k and t are non zero real numbers. We find sum of first n terms of Narayana and k-Narayana sequence and also sum of n terms having subscripts of the form $3m,\ 3m+1,\ 3m+2$.

Definition 2.1. **($(k,t)-$Narayana Sequence):** *(k,t)-Narayana sequence is defined as*

$$N_{n+1}(k,t) = kN_n(k,t) + tN_{n-2}(k,t) \quad for \quad all\ n \geq 2 \tag{3}$$

with the initial conditions

$$N_0(k,t) = 0,\ N_1(k,t) = 1,\ N_2(k,t) = k, \tag{4}$$

where k and t are non zero real numbers.

Some of the initial terms of the sequence are given as

$$N_0(k,t) = 0,\ N_1(k,t) = 1,\ N_2(k,t) = k,\ N_3(k,t) = k^2,\ N_4(k,t) = k^3 + t,$$
$$N_5(k,t) = k^4 + 2kt,\ N_6(k,t) = k^5 + 3k^2t,\ N_7(k,t) = k^6 + 4k^3t + t^2,\ ...$$

In particular, if we take $t = 1$, then (k,t)-Narayana sequence becomes k-Narayana sequence

$$\{b_{k,n}\}_{n=0}^{\infty} = \left\{0,\ 1,\ k,\ k^2,\ k^3+1,\ k^4+2k,\ k^5+3k^2,\ k^6+4k^3+1,\ ...\right\}.$$

If we take $k = 1, t = 1$, we get classical Narayana sequence

$$\{N_n\}_{n=0}^{\infty} = \{0,\ 1,\ 1,\ 1,\ 2,\ 3,\ 4,\ 6, 9,\ 13,\ 19,\ 28,\ ...\}.$$

Theorem 2.2. *Sum of n terms of Narayana sequence i.e.*

$$N_1 + N_2 + N_3 + ... + N_n = N_{n+3} - 1.$$

Proof. $N_{n+1} = N_n + N_{n-2}$ for all $n \geq 2$
$N_n = N_{n+1} - N_{n-2}$
By substituting $n = 2, 3, 4, ...$

$$\begin{aligned} N_2 &= N_3 - N_0 \\ N_3 &= N_4 - N_1 \\ N_4 &= N_5 - N_2 \\ N_5 &= N_6 - N_3 \end{aligned}$$

$$\begin{aligned} &\vdots \\ N_{n-3} &= N_{n-2} - N_{n-5} \\ N_{n-2} &= N_{n-1} - N_{n-4} \\ N_{n-1} &= N_n - N_{n-3} \\ N_n &= N_{n+1} - N_{n-2} \end{aligned}$$

Adding all the above inequalities, we get

$$\begin{aligned} N_2 + N_3 + \ldots + N_n &= N_{n+3} - 2 \\ N_1 + N_2 + N_3 + \ldots + N_n &= N_{n+3} - 1 \end{aligned}$$

□

Theorem 2.3. *Prove the following:*

(i) $N_3 + N_6 + N_9 + ... + N_{3n} = N_{3n+1} - N_1$
(ii) $N_4 + N_7 + N_{10} + ... + N_{3n+1} = N_{3n+2} - N_2$
(iii) $N_5 + N_8 + N_{11} + ... + N_{3n+2} = N_{3n+3} - N_3$.

Proof. (i) By using equation (1),

$$\begin{aligned} N_3 + N_6 + N_9 + \ldots + N_{3n} = &\frac{1}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})}(\tilde{\alpha}^4 + \tilde{\alpha}^7 + \tilde{\alpha}^{10} + \ldots + \tilde{\alpha}^{3n+1}) \\ &+ \frac{1}{(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}-\tilde{\gamma})}(\tilde{\beta}^4 + \tilde{\beta}^7 + \tilde{\beta}^{10} + \ldots + \tilde{\beta}^{3n+1}) \\ &+ \frac{1}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}(\tilde{\gamma}^4 + \tilde{\gamma}^7 + \tilde{\gamma}^{10} + \ldots + \tilde{\gamma}^{3n+1}) \\ = &\frac{\tilde{\alpha}^4(\tilde{\alpha}^{3n}-1)}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})(\tilde{\alpha}^3-1)} + \frac{\tilde{\beta}^4(\tilde{\beta}^{3n}-1)}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}^3-1)} + \frac{\tilde{\beta}^4(\tilde{\beta}^{3n}-1)}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})} \\ = &\frac{1}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})}\left(\tilde{\alpha}^{3n+2} - \tilde{\alpha}^2\right) + \frac{1}{(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}-\tilde{\gamma})}\left(\tilde{\beta}^{3n+2} - \tilde{\beta}^2\right) \\ &+ \frac{1}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}\left(\tilde{\gamma}^{3n+2} - \tilde{\gamma}^2\right) \\ &\Big(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \text{ are the roots of equation } x^3 - x^2 - 1 = 0 \\ &\text{so } \tilde{\alpha}^3 - 1 = \tilde{\alpha}^2, \tilde{\beta}^3 - 1 = \tilde{\beta}^2, \tilde{\gamma}^3 - 1 = \tilde{\gamma}^2\Big) \\ &= N_{3n+1} - N_1. \end{aligned}$$

Similarly other two equalities can be proved. □

Theorem 2.4. *Sum of n terms of k-Narayana sequence i.e.*

$$b_{k,1}+b_{k,2}+b_{k,3}+b_{k,4}+...+b_{k,n} = \frac{1}{k}\left(b_{k,n-1} + b_{k,n} + b_{k,n+1} - b_{k,0} - b_{k,1} - b_{k,2}\right)+b_{k,1}.$$

Proof. Since $b_{k,n} = kb_{k,n-1} + b_{k,n-3}$ for all $n \geq 3$,
so $b_{k,n-1} = \frac{1}{k}\left(b_{k,n} - b_{k,n-3}\right)$ for all $n \geq 3$
By substituting $n = 3, 4, 5, ...$

$$\begin{aligned}
b_{k,2} &= \frac{1}{k}\left(b_{k,3} - b_{k,0}\right)\\
b_{k,3} &= \frac{1}{k}\left(b_{k,4} - b_{k,1}\right)\\
b_{k,4} &= \frac{1}{k}\left(b_{k,5} - b_{k,2}\right)\\
b_{k,5} &= \frac{1}{k}\left(b_{k,6} - b_{k,3}\right)\\
b_{k,6} &= \frac{1}{k}\left(b_{k,7} - b_{k,4}\right)\\
&\vdots\\
b_{k,n-3} &= \frac{1}{k}\left(b_{k,n-2} - b_{k,n-5}\right)\\
b_{k,n-2} &= \frac{1}{k}\left(b_{k,n-1} - b_{k,n-4}\right)\\
b_{k,n-1} &= \frac{1}{k}\left(b_{k,n} - b_{k,n-3}\right)\\
b_{k,n} &= \frac{1}{k}\left(b_{k,n+1} - b_{k,n-2}\right)
\end{aligned}$$

Adding all above equalities, we get

$$\begin{aligned}
b_{k,2} + b_{k,3} + b_{k,4} + ... + b_{k,n} &= \frac{1}{k}\left(b_{k,n-1} + b_{k,n} + b_{k,n+1} - b_{k,0} - b_{k,1} - b_{k,2}\right)\\
b_{k,1} + b_{k,2} + b_{k,3} + b_{k,4} + ... + b_{k,n} &= \frac{1}{k}\left(b_{k,n-1} + b_{k,n} + b_{k,n+1} - b_{k,0} - b_{k,1} - b_{k,2}\right) + b_{k,1}
\end{aligned}$$

□

Theorem 2.5. *Prove the following:*

(i) $b_{k,4} + b_{k,7} + b_{k,10} + ... + b_{k,3n+1} = \frac{1}{k}\left(b_{k,3n+2} - b_{k,2}\right)$

(ii) $b_{k,5} + b_{k,8} + b_{k,11} + ... + b_{k,3n+2} = \frac{1}{k}\left(b_{k,3n+3} - b_{k,3}\right)$

(iii) $b_{k,3} + b_{k,6} + b_{k,9} + ... + b_{k,3n} = \frac{1}{k}\left(b_{k,3n+1} - b_{k,1}\right).$

Proof. (i) By using equation (2),

$$b_{k,4} + b_{k,7} + b_{k,10} + \ldots + b_{k,3n+1} = \frac{\tilde{\alpha}^5 + \tilde{\alpha}^8 + \tilde{\alpha}^{11} + \ldots + \tilde{\alpha}^{3n+2}}{(\tilde{\alpha} - \tilde{\beta})(\tilde{\alpha} - \tilde{\gamma})}$$

$$+\frac{\tilde{\beta}^5+\tilde{\beta}^8+\tilde{\beta}^{11}+\ldots+\tilde{\beta}^{3n+2}}{(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}-\tilde{\gamma})}+\frac{\tilde{\gamma}^5+\tilde{\gamma}^8+\tilde{\gamma}^{11}+\ldots+\tilde{\gamma}^{3n+2}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}$$

$$=\frac{\tilde{\alpha}^5(\tilde{\alpha}^{3n}-1)}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})(\tilde{\alpha}^3-1)}+\frac{\tilde{\beta}^5(\tilde{\beta}^{3n}-1)}{(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}^3-1)}$$

$$+\frac{\tilde{\gamma}^5(\tilde{\gamma}^{3n}-1)}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})(\tilde{\gamma}^3-1)}$$

$$=\frac{\tilde{\alpha}^5(\tilde{\alpha}^{3n}-1)}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})k\tilde{\alpha}^2}+\frac{\tilde{\beta}^5(\tilde{\beta}^{3n}-1)}{(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}-\tilde{\gamma})k\tilde{\beta}^2}+\frac{\tilde{\gamma}^5(\tilde{\gamma}^{3n}-1)}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})k\tilde{\gamma}^2}$$

$$\Big(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\ \textit{are the roots of equation}\ x^3-kx^2-1=0$$

$$\textit{so}\ \tilde{\alpha}^3-1=k\tilde{\alpha}^2,\tilde{\beta}^3-1=k\tilde{\beta}^2,\tilde{\gamma}^3-1=k\tilde{\gamma}^2\Big)$$

$$=\frac{1}{k}\left\{\frac{\tilde{\alpha}^{3n+3}-\tilde{\alpha}^3}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})}+\frac{\tilde{\beta}^{3n+3}-\tilde{\beta}^3}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})}+\frac{\tilde{\gamma}^{3n+3}-\tilde{\gamma}^3}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}\right\}$$

$$=\frac{1}{k}\left(b_{k,3n+2}-b_{k,2}\right)$$

Similarly other two equalities can be proved. □

3. Matrix representation

This section gives matrix expression of (k,t)-Narayana sequence and some identities, characteristic equation and Binet's formula.

$(k,t)-$ Narayana sequence can be expressed in matrix form as

$$\begin{pmatrix}N_{n+1}(k,t)\\N_n(k,t)\\N_{n-1}(k,t)\end{pmatrix}=\begin{pmatrix}s&0&t\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}N_n(k,t)\\N_{n-1}(k,t)\\N_{n-2}(k,t)\end{pmatrix}$$

$$\begin{pmatrix}N_{n+1}(k,t)\\N_n(k,t)\\N_{n-1}(k,t)\end{pmatrix}=P\begin{pmatrix}N_n(k,t)\\N_{n-1}(k,t)\\N_{n-2}(k,t)\end{pmatrix}$$

where $P=\begin{pmatrix}s&0&t\\1&0&0\\0&1&0\end{pmatrix}$ and $\det P=t$.

By successive substitutions, we can obtain

$$\begin{pmatrix}N_{n+1}(k,t)\\N_n(k,t)\\N_{n-1}(k,t)\end{pmatrix}=P^{n-1}\begin{pmatrix}N_2(k,t)\\N_1(k,t)\\N_0(k,t)\end{pmatrix}$$

Theorem 3.1.

$$P^n = \begin{pmatrix} N_{n+1}(k,t) & tN_{n-1}(k,t) & tN_n(k,t) \\ N_n(k,t) & tN_{n-2}(k,t) & tN_{n-1}(k,t) \\ N_{n-1}(k,t) & tN_{n-3}(k,t) & tN_{n-2}(k,t) \end{pmatrix} \; for\ all\ n \geq 3. \tag{5}$$

Proof. The proof can be done by induction on n

For n=3, we will prove that $P^3 = \begin{pmatrix} N_4(k,t) & tN_2(k,t) & tN_3(k,t) \\ N_3(k,t) & tN_1(k,t) & tN_2(k,t) \\ N_2(k,t) & tN_0(k,t) & tN_1(k,t) \end{pmatrix}$

Since $P = \begin{pmatrix} k & 0 & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

So $P^3 = \begin{pmatrix} k^3+t & kt & k^2t \\ k^2 & t & kt \\ k & 0 & t \end{pmatrix} = \begin{pmatrix} N_4(k,t) & tN_2(k,t) & tN_3(k,t) \\ N_3(k,t) & tN_1(k,t) & tN_2(k,t) \\ N_2(k,t) & tN_0(k,t) & tN_1(k,t) \end{pmatrix}$

Now suppose equation (5) holds for $n = k$

i.e. $P^k = \begin{pmatrix} N_{k+1}(k,t) & tN_{k-1}(k,t) & tN_k(k,t) \\ N_k(k,t) & tN_{k-2}(k,t) & tN_{k-1}(k,t) \\ N_{k-1}(k,t) & tN_{k-3}(k,t) & tN_{k-2}(k,t) \end{pmatrix}$

We will prove for $n = k+1$

i.e. $P^{k+1} = \begin{pmatrix} N_{k+2}(k,t) & tN_k(k,t) & tN_{k+1}(k,t) \\ N_{k+1}(k,t) & tN_{k-1}(k,t) & tN_k(k,t) \\ N_k(k,t) & tN_{k-2}(k,t) & tN_{k-1}(k,t) \end{pmatrix}$

Now

$$\begin{aligned} P^{k+1} &= P^k P \\ &= \begin{pmatrix} N_{k+1}(k,t) & tN_{k-1}(k,t) & tN_k(k,t) \\ N_k(k,t) & tN_{k-2}(k,t) & tN_{k-1}(k,t) \\ N_{k-1}(k,t) & tN_{k-3}(k,t) & tN_{k-2}(k,t) \end{pmatrix} \begin{pmatrix} k & 0 & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} kN_{k+1}(k,t) + tN_{k-1}(k,t) & tN_k(k,t) & tN_{k+1}(k,t) \\ kN_k(k,t) + tN_{k-2}(k,t) & tN_{k-1}(k,t) & tN_k(k,t) \\ kN_{k-1}(k,t) + tN_{k-3}(k,t) & tN_{k-2}(k,t) & tN_{k-1}(k,t) \end{pmatrix} \\ &= \begin{pmatrix} N_{k+2}(k,t) & tN_k(k,t) & tN_{k+1}(k,t) \\ N_{k+1}(k,t) & tN_{k-1}(k,t) & tN_k(k,t) \\ N_k(k,t) & tN_{k-2}(k,t) & tN_{k-1}(k,t) \end{pmatrix} \end{aligned}$$

So by induction equation (5) holds for all $n \geq 3$. □

Theorem 3.2. $N_{n-1}^3(k,t) + N_{n+1}(k,t)N_{n-2}^2(k,t) - N_{n+1}(k,t)N_{n-1}(k,t)N_{n-3}(k,t) + N_n^2(k,t)N_{n-3}(k,t) - 2N_n(k,t)N_{n-1}(k,t)N_{n-2}(k,t) = t^{n-2}$.

Proof. From equation (5), we get

$$P^n = \begin{pmatrix} N_{n+1}(k,t) & tN_{n-1}(k,t) & tN_n(k,t) \\ N_n(k,t) & tN_{n-2}(k,t) & tN_{n-1}(k,t) \\ N_{n-1}(k,t) & tN_{n-3}(k,t) & tN_{n-2}(k,t) \end{pmatrix} \; for\ all\ n \geq 3 \tag{6}$$

Now taking determinant on both sides,

$$\begin{aligned} t^n &= N_{n+1}(k,t)(t^2N_{n-2}^2(k,t) - t^2N_{n-1}(k,t)N_{n-3}(k,t) - tN_{n-1}(tN_nN_{n-2}(k,t) \\ &\quad - tN_{n-1}^2(k,t)) + tN_n(k,t)(tN_n(k,t)N_{n-3}(k,t) - tN_{n-1}(k,t)N_{n-2}(k,t)) \\ t^n &= t^2(N_{n-1}^3(k,t) + N_{n+1}(k,t)N_{n-2}^2(k,t) - N_{n+1}(k,t)N_{n-1}(k,t)N_{n-3}(k,t) \\ &\quad + N_n^2(k,t)N_{n-3}(k,t) - 2N_n(k,t)N_{n-1}(k,t)N_{n-2}(k,t)) \\ t^{n-2} &= N_{n-1}^3(k,t) + N_{n+1}(k,t)N_{n-2}^2(k,t) - N_{n+1}(k,t)N_{n-1}(k,t)N_{n-3}(k,t) \\ &\quad + N_n^2(k,t)N_{n-3}(k,t) - 2N_n(k,t)N_{n-1}(k,t)N_{n-2}(k,t) \end{aligned}$$

□

Theorem 3.3.
$N_n(k,t) = N_{m+1}(k,t)N_{n-m}(k,t) + tN_{m-1}(k,t)N_{n-m-1}(k,t) + tN_m(k,t)N_{n-m-2}(k,t)$
$N_n(k,t) = N_m(k,t)N_{n-m+1}(k,t) + tN_{m-2}(k,t)N_{n-m}(k,t) + tN_{m-1}(k,t)N_{n-m-1}(k,t)$.

Proof. $P^n = P^mP^{n-m}$

$$\begin{pmatrix} N_{n+1}(k,t) & tN_{n-1}(k,t) & tN_n(k,t) \\ N_n(k,t) & tN_{n-2}(k,t) & tN_{n-1}(k,t) \\ N_{n-1}(k,t) & tN_{n-3}(k,t) & tN_{n-2}(k,t) \end{pmatrix} =$$

$$\begin{pmatrix} N_{m+1}(k,t) & tN_{m-1}(k,t) & tN_m(k,t) \\ N_m(k,t) & tN_{m-2}(k,t) & tN_{m-1}(k,t) \\ N_{m-1}(k,t) & tN_{m-3}(k,t) & tN_{m-2}(k,t) \end{pmatrix} \begin{pmatrix} N_{n-m+1}(k,t) & tN_{n-m-1}(k,t) & tN_{n-m}(k,t) \\ N_{n-m}(k,t) & tN_{n-m-2}(k,t) & tN_{n-m-1}(k,t) \\ N_{n-m-1}(k,t) & tN_{n-m-3}(k,t) & tN_{n-m-2}(k,t) \end{pmatrix}$$

By comparing the $(1,3)^{th}$ and $(2,1)^{th}$ elements, we obtain
$N_n(k,t) = N_{m+1}(k,t)N_{n-m}(k,t) + tN_{m-1}(k,t)N_{n-m-1}(k,t) + tN_m(k,t)N_{n-m-2}(k,t)$
$N_n(k,t) = N_m(k,t)N_{n-m+1}(k,t) + tN_{m-2}(k,t)N_{n-m}(k,t) + tN_{m-1}(k,t)N_{n-m-1}(k,t)$

□

Characteristic equation:
Characteristic equation of the sequence is given by $z^3 - kz^2 - t = 0$.

Theorem 3.4. *(Binet's formula):*

$$N_n(k,t) = \frac{\tilde{\alpha}^{n+1}}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})} + \frac{\tilde{\beta}^{n+1}}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})} + \frac{\tilde{\gamma}^{n+1}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}$$

where $\tilde{\alpha}$, $\tilde{\beta}$ *and* $\tilde{\gamma}$ *are the roots of characteristic equation* $x^3 - kx^2 - t$.

Proof. Take

$$N_n = A\tilde{\alpha}^n + B\tilde{\beta}^n + C\tilde{\gamma}^n \tag{7}$$

where $\tilde{\alpha},\ \tilde{\beta}$ and $\tilde{\gamma}$ are the roots of Characteristic equation x^3-kx^2-t.
Now by using initial conditions from equation (4) and operations of linear algebra, the values of A, B and C can be found as

$$A=\frac{\tilde{\alpha}}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})}$$

$$B=\frac{\tilde{\beta}}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})}$$

$$C=\frac{\tilde{\gamma}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}$$

Now from Equation (7),
we get $N_n=\dfrac{\tilde{\alpha}^{n+1}}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})}+\dfrac{\tilde{\beta}^{n+1}}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})}+\dfrac{\tilde{\gamma}^{n+1}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}$
where $\tilde{\alpha},\ \tilde{\beta}$ and $\tilde{\gamma}$ are the roots of Characteristic equation x^3-kx^2-t

$$\tilde{\alpha}=\frac{1}{3}\left(k+\sqrt[3]{\frac{2k^3+27t+3\sqrt{81t^2+12k^3t}}{2}}+k^2\sqrt[3]{\frac{2}{2k^3+27t+3\sqrt{81t^2+12k^3t}}}\right)$$

$$\tilde{\beta}=\frac{1}{3}\left(k+\omega\sqrt[3]{\frac{2k^3+27t+3\sqrt{81t^2+12k^3t}}{2}}+\omega^2k^2\sqrt[3]{\frac{2}{2k^3+27t+3\sqrt{81t^2+12k^3t}}}\right)$$

$$\tilde{\gamma}=\frac{1}{3}\left(k+\omega^2\sqrt[3]{\frac{2k^3+27t+3\sqrt{81t^2+12k^3t}}{2}}+\omega k^2\sqrt[3]{\frac{2}{2k^3+27t+3\sqrt{81t^2+12k^3t}}}\right)$$

where ω is cube root of unity. □

Alternatively we can prove Binet's formula with diagonalization of generating matrix.

Proof. Generating matrix for $(k,t)-$Narayana sequence is $P=\begin{pmatrix}k & 0 & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$
Eigen values of P are given by the characteristic equation $det(P-xI)=0$ i.e.

$$\det\begin{pmatrix}k-x & o & t\\ 1 & -x & 0\\ 0 & 1 & -x\end{pmatrix}=0$$

$$x^3-kx^2-t=0$$

Let its roots be $\tilde{\alpha},\ \tilde{\beta},\ and\ \tilde{\gamma}$.
Now eigen vector $\begin{pmatrix}u & v & w\end{pmatrix}^T$ corresponding to eigenvalue $\tilde{\alpha}$ is given by solution of

$$\begin{pmatrix} k-\tilde{\alpha} & 0 & t \\ 1 & -\tilde{\alpha} & 0 \\ 0 & 1 & -\tilde{\alpha} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

That is

$$\begin{aligned} (k-\tilde{\alpha})u + tw &= 0 \\ u - \tilde{\alpha}v &= 0 \\ v - \tilde{\alpha}w &= 0 \end{aligned}$$

Take $w = c$. Then we get $v = c\tilde{\alpha},\ u = c\tilde{\alpha}^2$.
In particular, if we consider $c = 1$, then $w = 1,\ v = \tilde{\alpha},\ u = \tilde{\alpha}^2$.

So eigen vectors corresponding to eigen values are $\begin{pmatrix} \tilde{\alpha}^2 \\ \tilde{\alpha} \\ 1 \end{pmatrix}$, $\begin{pmatrix} \tilde{\beta}^2 \\ \tilde{\beta} \\ 1 \end{pmatrix}$, $\begin{pmatrix} \tilde{\gamma}^2 \\ \tilde{\gamma} \\ 1 \end{pmatrix}$

Let $Q = \begin{pmatrix} \tilde{\alpha}^2 & \tilde{\beta}^2 & \tilde{\gamma}^2 \\ \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \tilde{\alpha} & 0 & 0 \\ 0 & \tilde{\beta} & 0 \\ 0 & 0 & \tilde{\gamma} \end{pmatrix}$

$$Q^{-1} = \frac{1}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})(\tilde{\beta}-\tilde{\gamma})} \begin{pmatrix} \tilde{\beta}-\tilde{\gamma} & \tilde{\gamma}^2-\tilde{\beta}^2 & \tilde{\beta}\tilde{\gamma}(\tilde{\beta}-\tilde{\gamma}) \\ \tilde{\gamma}-\tilde{\alpha} & \tilde{\alpha}^2-\tilde{\gamma}^2 & \tilde{\alpha}\tilde{\gamma}(\tilde{\gamma}-\tilde{\alpha}) \\ \tilde{\alpha}-\tilde{\beta} & \tilde{\alpha}^2-\tilde{\beta}^2 & \tilde{\alpha}\tilde{\beta}(\tilde{\alpha}-\tilde{\beta}) \end{pmatrix}$$

$$P = QDQ^{-1} \implies P^n = QD^nQ^{-1}$$

$$P^n = \frac{1}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})(\tilde{\beta}-\tilde{\gamma})} \begin{pmatrix} \tilde{\alpha}^2 & \tilde{\beta}^2 & \tilde{\gamma}^2 \\ \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} & 0 & 0 \\ 0 & \tilde{\beta} & 0 \\ 0 & 0 & \tilde{\gamma} \end{pmatrix} \begin{pmatrix} \tilde{\beta}-\tilde{\gamma} & \tilde{\gamma}^2-\tilde{\beta}^2 & \tilde{\beta}\tilde{\gamma}(\tilde{\beta}-\tilde{\gamma}) \\ \tilde{\gamma}-\tilde{\alpha} & \tilde{\alpha}^2-\tilde{\gamma}^2 & \tilde{\alpha}\tilde{\gamma}(\tilde{\gamma}-\tilde{\alpha}) \\ \tilde{\alpha}-\tilde{\beta} & \tilde{\alpha}^2-\tilde{\beta}^2 & \tilde{\alpha}\tilde{\beta}(\tilde{\alpha}-\tilde{\beta}) \end{pmatrix}$$

By using theorem equation (6),

$$\begin{pmatrix} N_{n+1}(k,t) & tN_{n-1}(k,t) & tN_n(k,t) \\ N_n(k,t) & tN_{n-2}(k,t) & tN_{n-1}(k,t) \\ N_{n-1}(k,t) & tN_{n-3}(k,t) & tN_{n-2}(k,t) \end{pmatrix}$$

$$= \frac{1}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})(\tilde{\beta}-\tilde{\gamma})} \begin{pmatrix} \tilde{\alpha}^{n+2} & \tilde{\beta}^{n+2} & \tilde{\gamma}^{n+2} \\ \tilde{\alpha}^{n+1} & \tilde{\beta}^{n+1} & \tilde{\gamma}^{n+1} \\ \tilde{\alpha}^{n} & \tilde{\beta}^{n} & \tilde{\gamma}^{n} \end{pmatrix} \begin{pmatrix} \tilde{\beta}-\tilde{\gamma} & \tilde{\gamma}^2-\tilde{\beta}^2 & \tilde{\beta}\tilde{\gamma}(\tilde{\beta}-\tilde{\gamma}) \\ \tilde{\gamma}-\tilde{\alpha} & \tilde{\alpha}^2-\tilde{\gamma}^2 & \tilde{\alpha}\tilde{\gamma}(\tilde{\gamma}-\tilde{\alpha}) \\ \tilde{\alpha}-\tilde{\beta} & \tilde{\alpha}^2-\tilde{\beta}^2 & \tilde{\alpha}\tilde{\beta}(\tilde{\alpha}-\tilde{\beta}) \end{pmatrix}$$

Comapring $(2,1)^{th}$ element from both sides, we get

$$N_n(k,t) = \frac{\tilde{\alpha}^{n+1}}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})} + \frac{\tilde{\beta}^{n+1}}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})} + \frac{\tilde{\gamma}^{n+1}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})} \qquad \square$$

4. Hessenberg matrices and $(k,t)-$Narayana numbers

The notions of Hessenberg matrix, permanent of a matrix and contraction of matrix are introduced in this section. Then, these notions gives elegant relation between the n^{th} (k,t)-Narayana number and permanent of a Hessenberg type matrix.

Herein, we introduce some of important definitions as Hessenberg matrix, Permanent of a matrix and Contraction of matrix [11, 12].

Definition 4.1. *(Hessenberg matrix): Define matrix* $A = [\tilde{a}_{ij}]$ *of order n, where* $\tilde{a}_{ij} = 0$ *when* $i - j > 1$ *or* $j - i > 1$, *then A is known as Hessenberg matrix.*

$$i.e.\ A \ = \begin{pmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \tilde{a}_{1,3} & \dots & \tilde{a}_{1,n-1} & \tilde{a}_{1,n} \\ \tilde{a}_{2,1} & \tilde{a}_{2,2} & \tilde{a}_{2,3} & \dots & \tilde{a}_{2,n-1} & \tilde{a}_{2,n} \\ & \tilde{a}_{3,2} & \tilde{a}_{3,3} & \dots & \vdots & \tilde{a}_{3,n} \\ & & \tilde{a}_{4,3} & \dots & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ 0 & & & & \tilde{a}_{n,n-1} & \tilde{a}_{n,n} \end{pmatrix}$$

Hessenberg matrices of order n are defined as

$$A(k,t) = \begin{pmatrix} k^2 & t & kt & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n\times n} \tag{8}$$

$$B(k,t) = \begin{pmatrix} k & 0 & t & & & & 0 \\ -1 & k & 0 & t & & & \\ & -1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & k & 0 & t \\ & & & & -1 & k & 0 \\ 0 & & & & & -1 & k \end{pmatrix}_{n\times n} \tag{9}$$

$$C(k,t) = \begin{pmatrix} k & 0 & t & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n\times n} \tag{10}$$

Definition 4.2. *(Permanent of a matrix): Permanent of a square matrix* $A = [a_{ij}]$ *is defined as*
$per A = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$, *where the summation extends over all permutations* σ *of the symmetric group* S_n.

Definition 4.3. *(Contraction of matrix): Let* $A = [a_{ij}]$ *be a square matrix of order n and its column k contains exactly 2 non zero entries* $a_{ik} \neq 0$ *and* $a_{jk} \neq 0$ *and* $i \neq j$. *Then the matrix of order* $n - 1$ *obtained from matrix A by replacing row*

i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called contraction of A on column k relative to row i and row j. Similarly matrix can be contracted on row k. Also if A is non negative matrix and B is a contraction of A then $perA = perB$.

Theorem 4.4. *Let $A(k,t)$ be a square matrix of order n defined as in equation (8). Then $per(A(k,t)) = N_{n+2}(k,t)$, where $N_{n+2}(k,t)$ is the $(n+2)^{th}$ (k,t)-Narayana number.*

Proof. Let $A_r(k,t)$ be the r^{th} contraction of matrix $A(k,t)$.
Here $A_r(k,t)$ is of order $(n-r \times n-r)$
After contracting $A(k,t)$ on first column

$$A_1(k,t) = \begin{pmatrix} k^3+t & kt & k^2t & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n-1\times n-1}$$

Again contracting $A_1(k,t)$ on its first column

$$A_2(k,t) = \begin{pmatrix} k^4+2kt & k^2t & k^3t+t^2 & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n-2\times n-2}$$

After contracting $A_2(k,t)$ on first column

$$A_3(k,t) = \begin{pmatrix} k^5+3k^2t & k^3t+t^2 & k^4t+2kt^2 & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n-3\times n-3}$$

$$= \begin{pmatrix} N_6(k,t) & tN_4(k,t) & tN_5(k,t) & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n-3\times n-3}$$

Continuing in this way, r^{th} contraction of $A(k,t)$ is

$$A_r(k,t) = \begin{pmatrix} N_{r+3}(k,t) & tN_{r+1}(k,t) & tN_{r+2}(k,t) & & & & 0 \\ 1 & k & 0 & t & & & \\ & 1 & k & 0 & t & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & t \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{pmatrix}_{n-r\times n-r}$$

Hence $(n-3)^{th}$ contraction is

$$A_{n-3}(k,t) = \begin{pmatrix} N_n(k,t) & tN_{n-2}(k,t) & tN_{n-1}(k,t) \\ 1 & k & 0 \\ 0 & 1 & k \end{pmatrix}$$

$A_{n-3}(k,t)$ can be contracted on column first

$$A_{n-2}(k,t) = \begin{pmatrix} N_{n+1}(k,t) & tN_{n-1}(k,t) \\ 1 & k \end{pmatrix}$$

So $per(A(k,t)) = kN_{n+1}(k,t) + tN_{n-1}(k,t) = N_{n+2}(k,t)$ $\square$

Theorem 4.5. *Let $C(k,t)$ be a square matrix of order n as defined in (10) Then $per(C(k,t)) = N_{n+1}(k,t)$, where $N_{n+1}(k,t)$ is the $(n+1)^{th}$ (k,t)-Narayana number.*

Proof. Proof is similar to Theorem 4.4. $\square$

Theorem 4.6. *Let $B(k,t)$ be a square matrix of order n as defined in (9) Then $per(B(k,t)) = N_{n+1}(k,t)$, where $N_n(k,t)$ is the n^{th} (k,t)-Narayana number.*

5. Sum of n terms of (k,t) Narayana numbers

In this section a new sequence $\mathcal{S}_n(k,t)$ sum of the first n terms of the (k,t)-Narayana sequence is introduced. Its recurrence relation, generating function and matrix relations are given.

Let $\mathcal{S}_n(k,t) = \sum_{i=0}^{n} N_i(k,t)$ for all $n \geq 1$, where $N_i(k,t)$ are given by equations

(3) and (4)
So $\mathcal{S}_0(k,t)=0,\ \mathcal{S}_1(k,t)=1,\ \mathcal{S}_2(k,t)=k+1$

Theorem 5.1. *$\mathcal{S}_n(k,t)$ satisfies the recurrence relation*

$$\mathcal{S}_n(k,t)=k\mathcal{S}_{n-1}(k,t)+t\mathcal{S}_{n-3}(k,t)+1 \ for\ all\ n\geq 3 \tag{11}$$

with the initial conditions

$$\mathcal{S}_0(k,t)=0,\ \mathcal{S}_1(k,t)=1,\ \mathcal{S}_2(k,t)=k+1,\ \mathcal{S}_3(k,t)=k^2+k+1. \tag{12}$$

Proof. The proof is given by induction on n.
For $n=3$,

$$\begin{aligned} s\mathcal{S}_2(k,t)+t\mathcal{S}_0(k,t)+1 &= k(k+1)+0+1\\ &= k^2+k+1\\ &= N_3(k,t)+N_2(k,t)+N_1(k,t)+N_0(k,t)\\ &= \mathcal{S}_3(k,t)\end{aligned}$$

Now suppose equation (11) holds for $n=m$
i.e. $\mathcal{S}_m(k,t)=k\mathcal{S}_{m-1}(k,t)+t\mathcal{S}_{m-3}(k,t)+1$
Now Theorem will be proved for $n=m+1$
i.e. we prove $\mathcal{S}_{m+1}(k,t)=k\mathcal{S}_m(k,t)+t\mathcal{S}_{m-2}(k,t)+1$
$\mathcal{S}_{m+1}(k,t)=\mathcal{S}_m(k,t)+N_{m+1}(k,t)$
By using induction hypothesis, we get

$$\begin{aligned} \mathcal{S}_{m+1}(k,t) &= k\mathcal{S}_{m-1}(k,t)+t\mathcal{S}_{m-3}(k,t)+1+N_{m+1}(k,t)\\ &= k\mathcal{S}_{m-1}(k,t)+t\mathcal{S}_{m-3}(k,t)+1+kN_m(k,t)+tN_{m-2}(k,t)\\ &= k\mathcal{S}_m(k,t)+t\mathcal{S}_{m-2}(k,t)+1\end{aligned}$$

So the equation (11) holds for all $n\geq 3$. □

Theorem 5.2. *(Generating function):* $\sum_{n=0}^{\infty}\mathcal{S}_n(k,t)=\dfrac{z}{(1-kz-tz^3)(1-z)}$

Matrices relations of $\mathcal{S}_n(k,t)$

$$\begin{pmatrix}\mathcal{S}_n(k,t)\\ \mathcal{S}_{n-1}(k,t)\\ \mathcal{S}_{n-2}(k,t)\\ 1\end{pmatrix}=\begin{pmatrix}k&0&t&1\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\mathcal{S}_{n-1}(k,t)\\ \mathcal{S}_{n-2}(k,t)\\ \mathcal{S}_{n-3}(k,t)\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\mathcal{S}_n(k,t)\\ \mathcal{S}_{n-1}(k,t)\\ \mathcal{S}_{n-2}(k,t)\\ 1\end{pmatrix}=Q\begin{pmatrix}\mathcal{S}_{n-1}(k,t)\\ \mathcal{S}_{n-2}(k,t)\\ \mathcal{S}_{n-3}(k,t)\\ 1\end{pmatrix},$$

where $Q = \begin{pmatrix} k & 0 & t & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

By successive substitutions, we get

$$\begin{pmatrix} \mathcal{S}_n(k,t) \\ \mathcal{S}_{n-1}(k,t) \\ \mathcal{S}_{n-2}(k,t) \\ 1 \end{pmatrix} = Q^{n-2} \begin{pmatrix} \mathcal{S}_2(k,t) \\ \mathcal{S}_1(k,t) \\ \mathcal{S}_0(k,t) \\ 1 \end{pmatrix}$$

Now $Q^2 = \begin{pmatrix} k^2 & t & st & s+1 \\ s & 0 & t & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$Q^3 = \begin{pmatrix} k^3+t & kt & k^2t & k^2+k+1 \\ k^2 & t & kt & k+1 \\ k & 0 & t & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

$$Q^4 = \begin{pmatrix} k^4+2kt & k^2t & k^3t+t^2 & k^4+t+k^2+k+1 \\ k^3+t & kt & k^2t & k^2+k+1 \\ k^2 & t & kt & k+1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} N_5(k,t) & tN_3(k,t) & tN_4(k,t) & \mathcal{S}_4(k,t) \\ N_4(k,t) & tN_2(k,t) & tN_3(k,t) & \mathcal{S}_3(k,t) \\ N_3(k,t) & tN_1(k,t) & tN_2(k,t) & \mathcal{S}_2(k,t) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 5.3.

$$Q^n = \begin{pmatrix} N_{n+1}(k,t) & tN_{n-1}(k,t) & tN_n(k,t) & \mathcal{S}_n(k,t) \\ N_n(k,t) & tN_{n-2}(k,t) & tN_{n-1}(k,t) & \mathcal{S}_{n-1}(k,t) \\ N_{n-1}(k,t) & tN_{n-2}(k,t) & tN_{n-3}(k,t) & \mathcal{S}_{n-2}(k,t) \\ 0 & 0 & 0 & 1 \end{pmatrix} \; for\ all\ n \geq 3. \tag{14}$$

Proof. Proof is done by induction on n.
For $n = 3$, from equation (13),

$$Q^3 = \begin{pmatrix} k^3+t & kt & k^2t & k^2+k+1 \\ k^2 & t & kt & k+1 \\ k & 0 & t & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} N_4(k,t) & tN_2(k,t) & tN_3(k,t) & \mathcal{S}_3(k,t) \\ N_3(k,t) & tN_1(k,t) & tN_2(k,t) & \mathcal{S}_2(k,t) \\ N_2(k,t) & tN_0(k,t) & tN_1(k,t) & \mathcal{S}_1(k,t) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Suppose equation (14) holds for $n = m$
Now for $n = m + 1$,

$$
\begin{aligned}
Q^{m+1} &= Q^m Q \\
&= \begin{pmatrix} N_{m+1}(k,t) & tN_{m-1}(k,t) & tN_m(k,t) & \mathcal{S}_m(k,t) \\ N_m(k,t) & tN_{m-2}(k,t) & tN_{m-1}(k,t) & \mathcal{S}_{m-1}(k,t) \\ N_{m-1}(k,t) & tN_{m-2}(k,t) & tN_{m-3}(k,t) & \mathcal{S}_{m-2}(k,t) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 & t & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} kN_{m+1}(k,t) + tN_{m-1}(k,t) & tN_m(k,t) & tN_{m+1}(k,t) & N_{m+1}(k,t) + \mathcal{S}_m(k,t) \\ kN_m(k,t) + tN_{m-2}(k,t) & tN_{m-1}(k,t) & tN_m(k,t) & N_m(k,t) + \mathcal{S}_{m-1}(k,t) \\ kN_{m-1}(k,t) + tN_{m-3}(k,t) & tN_{m-2}(k,t) & tN_{m-1}(k,t) & N_{m-1}(k,t) + \mathcal{S}_{m-2}(k,t) \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} N_{m+2}(k,t) & tN_m(k,t) & tN_{m+1}(k,t) & \mathcal{S}_{m+1}(k,t) \\ N_{m+1}(k,t) & tN_{m-1}(k,t) & tN_m(k,t) & \mathcal{S}_m(k,t) \\ N_m(k,t) & tN_{m-1}(k,t) & tN_{m-2}(k,t) & \mathcal{S}_{m-1}(k,t) \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Hence by mathematical induction equation (14) holds for all $n \geq 3$. □

Corollary 5.4. $\det(P^n) = \det(Q^n)$.

6. (k,t)-Narayana sequence for negative subscripts

We have to calculate terms backwards for negative subscripts. So, this section introduces (k,t)-Narayana sequence for negative subscripts, their matrix expression and Binet's formula.

$(k,t)-$Narayana sequence for negative subscripts is defined as
$N_{-n}(k,t) = \dfrac{N_{-n+3}(k,t) - kN_{-n+2}(k,t)}{t}$ for all $n \geq 3$
with the initial conditions $N_0(k,t) = 0,\ N_{-1}(k,t) = 0,\ N_{-2}(k,t) = t^{-1}$
$\{N_{-n}(k,t)\}_{n=0}^{\infty} = \left\{0,\ 0,\ t^{-1},\ 0,\ -kt^{-2},\ t^{-2},\ k^2t^{-3},\ -2kt^{-3},\ \ldots\right\}$

Matrix relations

$$
\begin{pmatrix} N_{-n} \\ N_{-n+1} \\ N_{-n+2} \end{pmatrix} = \begin{pmatrix} 0 & -kt^{-1} & t^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} N_{-n+1} \\ N_{-n+2} \\ N_{-n+3} \end{pmatrix}
$$

,
where $R = \begin{pmatrix} 0 & -kt^{-1} & t^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\det R = t^{-1}$.

By successive substitutions, we obtain
$$\begin{pmatrix} N_{-n} \\ N_{-n+1} \\ N_{-n+2} \end{pmatrix} = R^{n-2} \begin{pmatrix} N_{-2} \\ N_{-1} \\ N_0 \end{pmatrix}.$$

Theorem 6.1. $R^n = \begin{pmatrix} tN_{-n-2} & tN_{-n-3} & N_{-n-1} \\ tN_{-n-1} & tN_{-n-2} & N_{-n} \\ tN_{-n} & tN_{-n-1} & N_{-n+1} \end{pmatrix}$ *for all* $n \geq 1$.

Proof. We will prove by induction on n.

For $n = 1$, $R = \begin{pmatrix} 0 & -kt^{-1} & t^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} tN_{-3} & tN_{-4} & N_{-2} \\ tN_{-2} & tN_{-3} & N_{-1} \\ tN_{-1} & tN_{-2} & N_0 \end{pmatrix}$
Assume result is true for $n = m$.

Now we prove for $n = m + 1$. That is to prove
$$R^{m+1} = \begin{pmatrix} tN_{-m-3} & tN_{-m-4} & N_{-m-2} \\ tN_{-m-2} & tN_{-m-3} & N_{-m-1} \\ tN_{-m-1} & tN_{-m-2} & N_{-m} \end{pmatrix}$$
$$\begin{aligned} R^{m+1} &= R^m R \\ &= \begin{pmatrix} tN_{-m-2} & tN_{-m-3} & N_{-m-1} \\ tN_{-m-1} & tN_{-m-2} & N_{-m} \\ tN_{-m} & tN_{-m-1} & N_{-m+1} \end{pmatrix} \begin{pmatrix} 0 & -kt^{-1} & t^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} tN_{-m-3} & -sN_{-m-2} + N_{-m-1} & N_{-m-2} \\ tN_{-m-2} & -sN_{-m-1} + N_{-m} & N_{-m-1} \\ tN_{-m-1} & -sN_{-m} + N_{-m+1} & N_{-m} \end{pmatrix} \\ &= \begin{pmatrix} tN_{-m-3} & tN_{-m-4} & N_{-m-2} \\ tN_{-m-2} & tN_{-m-3} & N_{-m-1} \\ tN_{-m-1} & tN_{-m-2} & N_{-m} \end{pmatrix} \end{aligned}$$
So theorem holds for all $n \geq 1$. □

Theorem 6.2. *(Binet's formula):*
$$N_{-n} = \frac{\tilde{\alpha}^{-n+1}}{(\tilde{\alpha} - \tilde{\beta})(\tilde{\alpha} - \tilde{\gamma})} + \frac{\tilde{\beta}^{-n+1}}{(\tilde{\beta} - \tilde{\gamma})(\tilde{\beta} - \tilde{\alpha})} + \frac{\tilde{\gamma}^{-n+1}}{(\tilde{\gamma} - \tilde{\alpha})(\tilde{\gamma} - \tilde{\beta})}$$
where $\tilde{\alpha}$, $\tilde{\beta}$ *and* $\tilde{\gamma}$ *are the roots of the equation* $x^3 - kx^2 - t = 0$.

Proof. Proof is similar to Theorem 3.4. □

7. Conclusion

This paper is devoted to the study of the (k, t)-Narayana sequence a generalization of both classical Narayana sequence and k-Narayana sequence and also, provides a lot of identities. Relations between terms of this sequence and Hessenberg matrices are given. Hessenberg matrices are used in inverse iteration, which is a method to compute eigen vectors. In future, this sequence can be used in applications areas like cryptography and in solution of Diophantine equations.

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