J. Indones. Math. Soc.
Vol. 30, No. 01 (2024), pp. 63–76.

The Characterization of Almost Prime Submodule on the Finitely Generated Module over Principal Ideal Domain

I Gede Adhitya Wisnu Wardhana[1], Pudji Astuti[2], and Intan Muchtadi-Alamsyah[3]

[1]Faculty of Mathematics and Natural Sciences, Universitas Mataram, Mataram, Indonesia, adhitya.wardhana@unram.ac.id
[2]Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia, pudji.astuti@itb.ac.id
[3]Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia, ntan@itb.ac.id

Abstract. In most cases, almost prime submodules are equivalent to prime submodules, but in a finitely generated module, it is not necessarily equivalent. Based on the fact that a finitely generated module over a principal ideal domain can be decomposed into a free part and a torsion part, we give a new approach to the characteristic of almost prime submodules in the finitely generated module, especially we point out the cases when the submodules are almost prime but not prime.

Key words and Phrases: almost prime submodule; free module; finitely generated module; prime submodules; torsion module

1. INTRODUCTION

Khashan introduced a generalization of prime submodules called almost prime submodules and give some of its characteristics in multiplicative modules [1]. A prime submodule by definition is an almost prime submodule, the converse is not always true. In some cases, we found that an almost prime submodule is a prime submodule, such as in a cyclic module over principal ideal domain or in a CSM module over a principal ideal domain [2]. Even in a free module over a principal ideal domain, when the rank of its submodule is less than its module, the almost prime submodule is a prime submodule [3]. This makes the study of the almost prime submodule are looking for the module where an almost prime is not a prime.

2000 Mathematics Subject Classification:13C05,16D10,16D70
Received: 24-03-2023, accepted: 27-01-2024.

In a finitely generated module over a principal ideal domain, we give a new approach to the characterization of an almost prime submodule that is not a prime submodule using some module decomposition, such as primary decomposition and cyclic decomposition [4]. The main results of this study, whenever the module can be decomposed into the free part and the torsion part, then the almost prime submodule must be the direct sum of submodules on each part with some conditions.

Definition 1.1. *Let S be a proper submodule of M over a comutative ring R. The set $(S:M) := \{r \in R | rM \subseteq S\}$ is called fraction of submodule S by its module M.*

(1) Submodule S is a prime submodule if for any $r \in R$ and $m \in M$ such that $rm \in S$, then either $r \in (S:M)$ or $m \in S$.
(2) Submodule S is an almost prime submodule if for any $r \in R$ and $m \in M$ such that $rm \in S - (S:M)N$, then either $r \in (S:M)$ or $m \in S$.

A prime submodule must be almost prime, whereas the converse is not always true. For example $\langle \bar{9} \rangle$ is almost prime in $\mathbb{Z}$-module $\mathbb{Z}_{36}$, but $\langle \bar{9} \rangle$ is not prime.

The prime submodules and the almost prime submodules are stacked submodules. The definition of stacked submodules is given below:

Definition 1.2. *Let N be a submodule of finitely generated module M over a principal ideal domain R. Submodule N is stacked if there exists $\{b_1, \ldots, b_n\} \subset M$ such that*

$$M = \oplus_{i=1}^{n} \langle \langle b_i \rangle \rangle$$

and

$$N = \oplus_{i=1}^{k} \langle \langle r_i b_i \rangle \rangle$$

for nonzero $r_i \in R, i = 1, 2, \ldots, k$ and $1 \leq k \leq n$.

2. PRIME SUBMODULES

It is well known that every finitely generated torsion module can be decomposed to its primary submodules [4], therefore we will consider describing almost prime submodules in three cases. The first case is when the module is a primary module, the second when the module is a torsion module, and the last when the module is finitely generated.

First we will give the characterization of fraction of submodules.

Lemma 2.1. *Let M be R-module and S its submodule. If*

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

and

$$S = S_1 \oplus S_2 \oplus \ldots \oplus S_n,$$

with $S_i \subset M_i$ for $i = 1, 2, ..., n$, then we have

$$(S : M) = \bigcap_{i=1}^{n} (S_i : M_i).$$

Proof. Given R-modul

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_n \tag{1}$$

and nonzero $x \in R$. Because (1) is direct product, we have

$$xM = xM_1 \oplus xM_2 \oplus \ldots \oplus xM_n.$$

Now let $x \in \bigcap_{i=1}^{n} (S_i : M_i)$ nonzero. Since $x \in (S_i : M_i)$ for $i = 1, 2, \ldots, n$, by definition we have $xM_i \subset S_i$ for $i = 1, 2, \ldots, n$. Then

$$xM = xM_1 \oplus xM_2 \oplus ... \oplus xM_n \subset S_1 \oplus S_2 \oplus ... \oplus S_n = S.$$

So we have $x \in (S : M)$, hence $\bigcap_{i=1}^{n} (S_i : M_i) \subset (S : M)$.

Conversely, let $x \in (S : M)$. We will show $x \in (S_i : M_i)$ for $1 = 1, 2, ..., n$. For any $i \in \langle 1, 2, ..., n \rangle$ take $a \in xM_i$. Since $x \in (S : M)$ we have $a \in S$. Hence

$$a = s_1 + ... + s_i + ... + s_n$$

with $s_i \in S_i$ for $i = 1, 2, ..., n$. Since $a \in M_i$ then

$$s_1 = ... = s_{i-1} = s_{i+1} = ... = s_n = 0 \text{ and } a = s_i$$

We have $a \in S_i$, hence $xM_i \subset S_i$. Since it is for all i we conclude that $(S : M) \subset \bigcap_{i=1}^{n} (S_i : M_i)$. Then we have $(S : M) = \bigcap_{i=1}^{n} (S_i : M_i)$.

First, we will show that prime submodules are stacked.

Lemma 2.2. *Let M be a module over a principal ideal domain R and N its submodule. If $N \subseteq M$, is prime then N is stacked in M.*

Proof. We need only to show that $p^m(N \cap p^r M) = p^m N \cap p^{m+r} M$ (see [4]). It is obvious that $p^m(N \cap p^r M) \subset p^m N \cap p^{m+r} M$.

Let $x \in p^m N \cap p^{m+r} M$, then $x = p^m N$ and $x = p^{m+r} M$. We have $x = p^m y = p^{r+m} z$ for some $y \in N$ and $z \in M$. And $p^m(y - p^r z) = 0 \in N$ implies $y - p^r z \in N$ or $p^m M \subseteq N$. If $y - p^r z \in N$ then $p^r z \in N$ which result in $x = p^m(p^r z) \in p^m(N \cap p^r M)$. Suppose $p^m M \subseteq N$. This implies $p^m z \in N$. Applying N being prime we have $pM \subseteq N$ or $z \in N$. For both case we have $p^r z \in N$ resulting $x = p^m(p^r z) \in p^m(N \cap p^r M)$.

By Lemma 2.2, to characterize prime submodules of a module, it is enough to investigate its stacked submodules.

The fact that primary modules can be decomposed to their cyclic submodules [4] is essential for the next lemma. Note that in this lemma we investigate $(N : M)$ only for a stacked submodule of M.

Lemma 2.3. *Let M be a primary module over a PID with order p^e having direct sum:*

$$M = \langle\langle v_1 \rangle\rangle \oplus ... \oplus \langle\langle v_n \rangle\rangle$$

of cyclic submodule with annihilators $ann(\langle\langle v_i \rangle\rangle) = \langle p^{e_i} \rangle$ which

$$e = e_1 \geq e_2 \geq ... \geq e_n.$$

Let a submodule of M,

$$N = \left\langle \left\langle p^{f_1} v_1 \right\rangle \right\rangle \oplus ... \oplus \left\langle \left\langle p^{f_n} v_n \right\rangle \right\rangle$$

which $f_i \leq e_i$ for all i. If $f = max\{f_1, f_2, ..., f_n\}$, then

$$(N : M) = p^f.$$

PROOF. Let $r \in \langle p^f \rangle$, then $r = \beta p^f$. We have $rv_i = \beta p^f v_i \in N$ for all $i \in \{1, 2, ..., n\}$. Then for all $x \in M$ with $x = \sum_{i=1}^{n} \alpha_i v_i$ for $\alpha_i \in R, i = 1, 2, ..., n$, we have

$$rx = \sum_{i=1}^{n} \alpha_i r v_i = \sum_{i=1}^{n} \alpha_1 \beta p^f v_i \in N,$$

since f is maximal, so $r \in (N : M)$. Therefore $\langle p^f \rangle \subseteq (N : M)$.

Conversely, let $r \in (N : M)$, then $rx \in N$ for all $x \in M$. Note that $N \bigcap \langle\langle v_i \rangle\rangle = \left\langle \left\langle p^{f_i} v_i \right\rangle \right\rangle$ for all $i \in \{1, 2, ..., n\}$. Then we have $rv_i \in N \bigcap \langle\langle v_i \rangle\rangle = \left\langle \left\langle p^{f_i} v_i \right\rangle \right\rangle$, therefore $rv_i = \alpha_i p^{f_i} v_i$ for $\alpha_i \in R$. So we have $r - \alpha_i p^{f_i} \in ann(\langle\langle v_i \rangle\rangle) = \langle\langle p^{e_i} \rangle\rangle$, and we can write $r - \alpha_i p^{f_i} = \beta_i p^{e_i}$. Therefore

$$r = \alpha_i p^{f_i} + \beta_i p^{e_i} = (\alpha_i + \beta_i p^{e_i - f_i}) p^{f_i}$$

then $p^{f_i} | r$ for all $i = \{1, 2, ..., n\}$, hence $lcm \left\{ p^{f_i}, ..., p f_n \right\} | r$. Then we have $r \in \left\langle \left\langle p^f \right\rangle \right\rangle$. Therefore $(N : M) \subseteq \left\langle \left\langle p^f \right\rangle \right\rangle$.

As an example, let M be a $\mathbb{Z}$-module and N be a submodule of M with primary module decomposition $M = \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$ and $N = 2^3\mathbb{Z}_{16} \oplus 2^2\mathbb{Z}_8 \oplus 2^0\mathbb{Z}_4$. Then $(N : M) = \langle 2^3 \rangle$.

Lemma 2.4. *Let M be R-primary module. Submodule $N = \{0\}$ is a prime submodule if and only if the order M is prime.*

PROOF. Let p be the order of M, p prime in R. We will show that N prime submodule. Let $r \in R$ and $v \in M$ with $rv = 0 \in N$. Clearly for $v = 0$ we have $v \in N$. Now let $v \in M$ nonzero. Supposed $r \notin \langle\langle p \rangle\rangle$, since p prime, we have $(r, p) = 1$, hence $1 = ar + bp$ for $a, b \in R$. Since $rv = 0$, we have $arv = 0$, hence $0 = arv = (1 - bp)v = v - bpv = v$ (contradiction). So, we conclude that $r \in \langle\langle p \rangle\rangle$. Hence N is prime submodule.

Conversely let N be a prime submodule. We have $(N : M) = \langle\langle p \rangle\rangle$ for $p \in R$, hence order of M is p. Supposed p is not prime then there exist non unit $a, b \in R$ such that $p = ab$. We have b is not the order of M. Choose nonzero $v \in M$ with $bv \neq 0$. Then we have $a(bv) = pv = 0 \in N$ with $a \notin (N : M)$ and $bv \notin N$. This contradicts the fact that N is a prime submodule, hence p must be prime in R.

Now we characterize the prime submodule of primary module M by investigating its stacked submodules.

Theorem 2.5. *Let M be a primary module over a PID. Module M has the order p^e and can be decomposed into a direct sum:*

$$M = \langle\langle v_1 \rangle\rangle \oplus ... \oplus \langle\langle v_n \rangle\rangle$$

of cyclic submodules with annihilators $ann(\langle\langle v_i \rangle\rangle) = \langle p^{e_i} \rangle$ which

$$e = e_1 \geq e_2 \geq ... \geq e_n.$$

A nonzero submodule

$$N = \left\langle \left\langle p^{f_1} v_1 \right\rangle \right\rangle \oplus ... \oplus \left\langle \left\langle p^{f_n} v_n \right\rangle \right\rangle$$

with $0 \leq f_i \leq e_i$ for all $i \in \{1, 2, ..., n\}$ is prime if and only if $f_i \leq 1$ for all $i \in \{1, 2, ..., n\}$ and there is j such that $f_j = 1$.

Proof. Let $f_i \leq 1$ for all $i \in \{1, 2, ..., n\}$ and there exist $j \in \{1, 2, ..., n\}$ such that $f_j = 1$. By Lemma 2.3 we have $(N : M) = \langle p \rangle$. Let $rm \in N$ then we have $r\alpha_1 v_1 + ... + r\alpha_n v_n = \beta_1 p^{f_1} v_1 + ... + \beta_n p^{f_n} v_n$. Since $f_j = 1$, we have $r\alpha_j v_j = \beta_j p v_j$, then $p|r\alpha_j$. Therefore $p|r$ or $p|\alpha_j$. Since $(N : M) = \langle p \rangle$, we conclude that $r \in (N : M)$ or $m \in N$. Therefore N is prime submodule.

Conversely, let N is a prime submodule. Since N is nonzero, then exist $j \in \{1, 2, ..., n\}$ such that $f_j < e_j$. Assume there exists k such that $f_k > 1$. By Lemma 2.3 we have $(N : M) = \left\langle p^f \right\rangle$ such that $f > 1$. Choose $r = p$ and $m = p^{f_k - 1} v_k$, then $rm = p^{f_k} v_k \in N$. But $m \notin N$ and $r \in (N : M)$, it contradicts to N a prime submodule, therefore $f_i \leq 1$ for all $i \in \{1, 2, ..., n\}$.

The first case gives us an idea that in the general case when the module is finitely generated, the description of an almost prime submodule is correlated to the decomposition of its primary submodules. But first, we must characterize the $(N : M)$, and by primary cyclic decomposition [5, p.154] and Lemma 2.3 we have this corollary:

Corollary 2.6. *Let M be a torsion module over a principal ideal domain D. If M has order*

$$\mu = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$$

where the p_i's are distinct nonassociate primes in D, then M can be written as a direct sum of cyclic submodules, so that

$$M = \left[\langle\langle v_{1,1} \rangle\rangle) \oplus ... \oplus \langle\langle v_{1,n_1} \rangle\rangle)\right] \oplus ... \oplus \left[\langle\langle v_{k,1} \rangle\rangle) \oplus ... \oplus \langle\langle v_{k,n_k} \rangle\rangle)\right],$$

where $ann(\langle\langle v_{i,j} \rangle\rangle) = \left\langle p_i^{f_{i,j}} \right\rangle$ and the terms in each cyclic decompotion can be arranged so that,

$$e_i = e_{i,1} \geq e_{i,2} \geq ... \geq e_{i,n_i}.$$

If

$$N = [\langle\langle p_1^{f_{1,1}} v_{1,1} \rangle\rangle) \oplus ... \oplus \langle\langle p_1^{f_{1,n_1}} v_{1,n_1} \rangle\rangle)] \oplus ... \oplus [\langle\langle p_k^{f_{k,1}} v_{k,1} \rangle\rangle) \oplus ... \oplus \langle\langle p_k^{f_{k,n_k}} v_{k,n_k} \rangle\rangle)]$$

then

$$(N : M) = \langle p_1^{f_1}...p_k^{f_k} \rangle$$

with $f_i = max\{f_{i,1}, ..., f_{i,n_i}\}$.

By Corollary 2.6 we have this theorem.

Theorem 2.7. *Let M be a torsion module over a principal ideal domain D, with its primary decomposition*

$$M = M_{p_1} \oplus ... \oplus M_{p_k}$$

and

$$N = N_1 \oplus ... \oplus N_k$$

such that $N_i = \langle\langle p_i^{f_{i,1}} v_{i,1} \rangle\rangle \oplus ... \oplus \langle\langle p_i^{f_{i,n_i}} v_{i,n_i} \rangle\rangle$. *Submodule N is prime if and only if there exist a unique m such that*

$$N = M_{p_1} \oplus ... \oplus N_m \oplus ... \oplus M_{p_k}$$

with N_m is prim submodule of M_{p_m}.

Proof. Let $N = N_1 \oplus ... \oplus N_k$ be a prime submodule of M. Assume that N_i not prime and $N_i \neq M_{p_i}$, then $\exists r \in R$, $m_i \in M_{p_i}$ suuch that $rm_i \in N_i \subset N$ but $r \notin (N_i : M_{p_i}) \subset (N : M)$ and $m_i \notin N_i \subset N$. It contradicts to N a prime submodule of M. Therefore N_i must be prime or $N_i = Mp_i$.

Now assume there exist $j, l \in \{1, 2, ..., k\}$ with $j \neq l$ such that N_j, N_l is prime. Then we have $(N_j : M_{p_j}) = \langle p_j^{f_j} \rangle$ and $(N_l : M_{p_l}) = \langle p_l^{f_l} \rangle$. By definition of $f_i = max\{f_{i,1}, ..., f_{i,n_i}\}$ then $f_j = f_{j,i_j}$ and $f_l = f_{l,i_l}$ for some i_j, i_l. Choose $r = p_j$ and $m = p_j^{f_{j,i_j}-1} v_{j,i_j} + p_l^{f_{l,i_l}} v_{l,i_l}$, then $rm = p_j^{f_{j,i_j}} v_{j,i_j} + p_j p_l^{f_{l,i_l}} v_{l,i_l} \in N$ but $r \notin (N : M)$ and $m \notin N$. This contradicts N a prime submodule, therefore there exist unique $m \in \{1, 2, ..., k\}$ such that $N = M_{p_1} \oplus ... \oplus N_m \oplus ... \oplus M_{p_k}$.

Conversely, let $N = M_{p_1} \oplus ... \oplus N_m \oplus ... \oplus M_{p_k}$ with N_m is prime submodule of M_{p_m}. According to Corollary 2.6 we have $(N_m : M_{p_m}) = \langle p \rangle$, and by Lemma 2.3 $(N : M) = \langle p \rangle$. Let $rw = r(w_1 + ... + w_k) \in N$, then we have $rw_m \in N_m$. Since N_m prime submodule of M_{p_m} then $r \in (N_m : M_{p_m})$ or $w_m \in N_m$. Therefore $r \in (N : M)$ or $w \in N$. Hence N is prime.

The finitely generated modules over PID have a torsion part and free part, the characterization of prime submodules of a free module given by Wardhana et all [3]. To characterize prime submodules of a finitely generated module we need to find the fraction first.

Lemma 2.8. *Let $M = M_F \oplus M_T$ is finitely generated module over D with M_F is free part and M_T is torsion part. If $N = N_F \oplus N_T$ is submodule of M with $(N_F : M_F) = p$ and $(N_T : M_T) = q$, then $(N : M) = lcd(p, q)$.*

Proof. Let $N = N_F \oplus N_T$ be submodule of M with $(N_F : M_F) = p$ and $(N_T : M_T) = q$. If $m = m_f + m_t \in M$ with $m_f \in M_F$ and $m_t \in M_T$ then $lc(p, q)m = lcd(p, q)m_f + lcd(p, q)m_t \in N$. Therefore $(N : M) \supseteq lcd(p, q)$.

Conversely, let $r \in (N : M)$. For any $x_f \in M_F$ and for any $rx_t \in M_T$ we have $r(x_f + x_t) \in N$. Therefore $fx_f \in N_F$ for any $x_f \in M_F$ and $rx_t \in N_T$ for any $x_t \in M_T$. We conclude that $r \in (N_F : M_F) = \langle p \rangle$ and $r \in (N_F : M_F) = \langle q \rangle$, hence $pq|r$. Hence $(N : M) = lcd(p, q)$.

The characterization of the prime submodule of a finitely generated module is given by this theorem.

Theorem 2.9. *Let $M = M_F \oplus M_T$ is finitely generated module over D with M_F is free part and M_T is torsion part. Let $N = N_F \oplus N_T$ is submodule of M with $(N_F : M_F) = p$ and $(N_T : M_T) = q$.*
N is prime if and only is submodule N is one of this:

(1) $N = N_F \oplus M_T$, N_F is a prime submodule of M_F
(2) $N = M_F \oplus N_T$, N_T is a prime submodule of M_T
(3) $N = N_F \oplus N_T$, N_F and N_T is prime submodule of M_F and M_T with $(N_F : M_F) = (N_T : M_T)$.

Proof. First if $N = N_F \oplus M_T$, N_F is prime submodule of M_F, then $(N : M) = (N_F : M_F)$. Let $rm = r(m_f + m_r) \in N$, then we have $rm_f \in N_F$. Since N_F is prime then we have $r \in (N_F : M_F)$ or $m_f \in N_F$. Therefore $R \in (N : M)$ or $m \in N$, hence N is prime.

Second if $N = M_F \oplus N_T$, N_T is prime submodule of M_T, then $(N : M) = (N_T : M_T)$. Let $rm = r(m_f + m_r) \in N$, then we have $rm_t \in N_T$. Since N_T is prime then we have $r \in (N_T : M_T)$ or $m_t \in N_T$. Therefore $r \in (N : M)$ or $m \in N$, hence N is prime.

Third if $N = N_F \oplus N_T$, N_F and N_T is prime submodule of M_F and M_T with $(N_F : M_F) = (N_T : M_T)$, then $(N : M) = (N_T : M_T) = (N_F : M_F)$. Let $rm = r(m_f + m_r) \in N$, then we have $rm_f \in N_F$ and $rm_t \in N_T$. Since N_F and N_T is prime then we have $r \in (N_F : M_F)$ or $m_f \in N_F$ and $r \in (N_T : M_T)$ or $m_t \in N_T$. Therefore $r \in (N : M)$ or $m \in N$, hence N is prime.

Conversely, let N be prime. Assume that N_F not prime and $N_F \neq M_F$, then there exist $r \notin (N_F : M_F)$ and $m_f \notin N_F$ such that $rm_f \in N_F$. By Lemma 2.8 we have $r \notin (N : M)$ and $m_f \notin N$ such that $rm_f \in N$. Therefore it contradict to N prime submodule, hence N_F is prime or $N_F = M_F$. In the same way we can show that N_T is prime or $N_T = M_T$.

Write $(N_F : M_F) = \langle p \rangle$ and $(N_T : M_T) = \langle q \rangle$. Wardhana et all [3] prove that since N_F is prime submodule then p is prime or 0 [6, Theorem 1], hence $lcd(p, q) = q$ if $p|q$ or $lcd(p, q) = pq$ if $p \not| q$.

Assume that $p \not| q$, then we have $(N : M) = \langle pq \rangle$, hence p and q can not be both zero. If $p \neq 0$, we can choose $r = p$ and $m = m_f$, with m_f one of the bases of M_F such that pm_f is one of bases N_F (see [3]). Therefore $rm \in N$ but $r \notin (N : M)$ and $m \in N$, it contradict to N is prime submodule of M. If $q \neq 0$ then we can choose $r = q$ and $m = m_t$, m_t is one generator of cyclic submodule of M_T which is not in N_T. Therefore $rm \in N$ but $r \notin (N : M)$ and $m \notin N$, it contradict to N is prime submodule. Then we can conclude that $p|q$.

If $q \not| p$ then since $p|q$ we have $q = xp$ for x not an unit. Then we can choose $r = x$ and $m = pm_t$ with $pm_t \in M_T$ but $pm_t \notin N$. Therefore $rm = pxm_t = qm_t \in N$ with $r \notin (N : M)$ and $m \notin N$. It contadict to N is prime, therefore x is unit then we conclude that $(N_F : M_F) = (N_T : M_T)$.

3. Almost Prime Submodules

Like prime submodules, almost prime submodules are *stacked* submodules.

Theorem 3.1. *Let N be a submodule of primary module M over a principal ideal domain R. If N is an almost prime submodules then N is a stacked submodules.*

Proof. We only need to show that $p^m(N \cap p^r M) = p^m N \cap p^{m+r} M$ (see[4]). It is obvious that $p^m(N \cap p^r M) \subseteq p^m N \cap p^{m+r} M$. We will proved it by contraposition.

Let $(N : M) = \langle\langle p^e \rangle\rangle$, with p is prime and $e \geq 2$. Since N not stacked submodules, then there are exist $m > 0$, $r > 0$ so $p^m(N \cap p^r M) \subsetneq p^m N \cap p^{m+r} M$. Let $x \in p^m N \cap p^{m+r} M$ so $x \notin p^m(N \cap p^r M)$. We have $x = p^m y = p^{m+r} z$ with $y \in N$, $z \in M$. Since $x \notin p^m(N \cap p^r M)$ then $p^r z \notin N$. Then we have $p^r \notin (N : M)$, hence $r < e$. Noted that $p^m(p^r z) \in N$ but $p^r z \notin N$. Let $1 \leq a \leq m$ so $p^a(p^r z) \in N$ and $p^{a-1}(p^r z) \notin N$. We have $u = p^{a-1} z$, then $pu \in N$ but $u \notin N$. We will show that $pu \notin (N : M)N$. Supposed $pu \in (N : M)N$, we have $pu = p^e w$ for $w \in N$. Since $p^{a-1+r} z \notin N$ we have $p^{a-1+r} \notin (N : M)$. So $a - 1 + r < e$, then we have $a \leq e - r$. Then we have $pu = p^{a+r} z \in (N : M)N$ with $pu = p^e w$ for $e \in N$, hence $p^{a+r} z = p^e w = p^{a+r} p^{e-(a+r)} w$ then $x = p^{m+r} z = p^{m+r} p^{e-(a+r)} w \in p^m(N \cap p^r M)$ (contradiction). Then $pu \notin (N : M)N$. So $pu \in N - (N : M)N$ and $p \notin (N : M)$, hence N is not an almost prime submodule.

Therefore all almost prime submodules are stacked submodules. Then to characterize almost prime submodules, we only need to investigate the collection of stacked submodules. Noted that the converse of the Theorem above is not necessarily true. Consider that $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$, submodule $\mathbb{Z}_2 \oplus 4\mathbb{Z}_8$ is a stacked submodule but not almost prime.

Theorem 3.2. *Let M be a primary module over a PID, M has order p^e and can be decomposed into:*

$$M = \langle\langle v_1 \rangle\rangle \oplus ... \oplus \langle\langle v_n \rangle\rangle$$

of cyclic submodules with annihilators $ann(\langle\langle v_i \rangle\rangle) = \langle p^{e_i}$ which

$$e = e_1 \geq e_2 \geq ... \geq e_m > e_{m+1} = ... = e_n = 1.$$

A nonzero submodule

$$N = \langle\langle p^{f_1} v_1 \rangle\rangle \oplus ... \oplus \langle\langle p^{f_n} v_n \rangle\rangle$$

is an almost prime submodule of M if and only if $f_i \leq 1$ for all $i \in \{1, 2, ..., n\}$ or $f_i = e_i$ for all $i \in \{1, 2, ..., m\}$ and $f_i \leq 1$ for all $i \in \{m+1, m+2, ..., n\}$.

Proof. First case, if $f_i \leq 1$ for all $i \in \{1, 2, ..., n\}$, we will show that N is prime submodules. Let $rm \in N$ then we have $r\alpha_1 v_1 + ... + r\alpha_n v_n = \beta_1 p^{f_1} v_1 + ... + \beta_n p^{f_n} v_n$. We only consider j such that $f_j = 1$, then $r\alpha_j v_j = \beta_j p v_j$, then $p|r\alpha_j$. Therefore $p|r$ or $p|\alpha_j$. Since Lemma 2.3 give $(N : M) = \langle p \rangle$, we conclude that $r \in (N : M)$ or $m \in N$. Therefore N is prime submodule and by default is almost prime submodule.

Second case, if $f_i = e_i$ for all $i \in \{1, 2, ..., m\}$ and $f_i \leq 1$ for all $i \in \{m+1, m+2, ..., n\}$. Let $J = \{m+1, m+2, ..., n\}$, since N nonzero, there is a $k \in J$ such that $f_k = 0$. We consider only for $e > 1$, since for $e = 1$ it is the first case. We have $N = \sum_{j \in J}^{f_j=0} \langle\langle v_j \rangle\rangle$. Lemma 2.3 give us $(N : M) = \langle p^e \rangle$ and $N - (N : M)N = N - \{0\}$. Let $rm \in N - \{0\}$, then we have $r\alpha_1 v_1 + ... + r\alpha_n v_n = \sum_{j \in J}^{f_j=0} \langle\langle v_j \rangle\rangle$ and $r\alpha_1 v_1 + ... + r\alpha_n v_n \neq 0$. Therefore $p^{e_i}|r\alpha_i$ for all $i \in \{1, 2, ..., m\}$, $p|(r\alpha_i)$ for all $i \in J$ such that $f_i = 1$ and $p \not| r\alpha_k$. Since $p \not| r\alpha_k$ we have $p \not| r$, then $p^{e_i}|\alpha_i$ for all $i \in \{1, 2, ..., m\}$. And for all $i \in J$ such that $f_i = 1$ we also have $p|\alpha_i$. Therefore $m \in N$, and N is an almost prime submodules.

Conversely, let N be an almost prime submodule. Assume there is $j \in \{1, 2, ..., n\}$ such that $f_i > 1$, we will show that $f_i = e_i$ for all $i \in \{1, 2, ..., m\}$ and $f_i \leq 1$ for all $i \in \{m+1, m+2, ..., n\}$. Since there is exist j such that $f_j > 1$, we have $(N : M) = \langle p^x \rangle$ with $1 < x \leq e$. Noted that $j \notin J$ since $f_j \leq 1$ for all $j \in \{m+1, m+2, ..., n\}$. Now let $i \in \{1, 2, ..., m\}$ arbitrary and assume that $f_i < e_i$, then we have $pp^{f_i-1}v_i = p^{f_i}v_i \in N - (N : M)N$ since $(N : M)N = \langle\langle p^x p^{f_1} v_1 \rangle\rangle \oplus ... \oplus \langle\langle p^x p^{f_n} v_n \rangle\rangle$ and $f_i < e_i$. But $p \notin (N : M)$ and $p^{f_i-1}v_i \notin N$, it contradict the fact that N is almost prime. Therefore for all $i \in \{1, 2, ..., n-1\}$ we have $f_i = e_i$ for all $i \in \{1, 2, ..., m\}$. And since $e_i = 1$ for all $i \in \{m+1, m+2, ..., n\}$, we have $f_i \leq 1$ for all $i \in \{m+1, m+2, ..., n\}$.

Almost prime submodules must be prime submodules but not the other way, next theorem will show in what case almost prime submodules are equivalent to prime submodules.

Theorem 3.3. *Let M primary modules over principal ideal domain R, and the order of M p^e with primary decomposition:*

$$M = \langle\langle x_1 \rangle\rangle \oplus ... \oplus \langle\langle x_n \rangle\rangle$$

of its cyclic submodules with annihilator $ann(\langle\langle x_i \rangle\rangle) = \langle\langle p^{e_i} \rangle\rangle$ with

$$e = e_1 \geq e_2 \geq ... \geq e_n > 1.$$

Submodule N is an almost prime submodules if and only if N is a prime submodule.

Proof. We only need to prove that almost prime submodule is prime. And we investigate it for N stacked submodules. Without loss of generality, let

$$N = \langle\langle p^{f_1} x_1 \rangle\rangle \oplus ... \oplus \langle\langle p^{f_n} x_n \rangle\rangle, \tag{2}$$

with $f_i \leq e_i$ for all $i \in \{1, 2, ..., n\}$ are nonzero submodules.

Let N in (2) be an almost prime submodule. According to Theorem 2.5, to show that N is prime, we only need to show that

$$f_i \leq 1 \text{ for all } i \in \{1, 2, ..., n\}. \tag{3}$$

Supposed there exists $\mu \in \{1, 2, ..., n\}$ that $f_\mu > 1$. Then we have $(N : M) = \langle\langle p^e \rangle\rangle$ with $e > 1$, hence $(N : M)N = p^e N$. We split into two cases, and for each case we will show a contradiction.

First case for $f_n \leq 1$. Choose $r = p \notin (N : M)$ and $m = p_\mu^{f_\mu - 1} x_\mu + x_n \notin N$, we have

$$rm = p_\mu^{f_\mu} x_\mu + p x_n \in N.$$

But since $e_n > 1$ we also have

$$rm = p_\mu^{f_\mu} x_\mu + p x_n \notin p^e N = (N : M)N.$$

So

$$rm = p_\mu^{f_\mu} x_\mu + p x_n \in N - (N : M)N.$$

This contradicts to N an almost prime submodules.

Second case for $f_n > 1$. Since N is a proper submodule, then there exists $\rho \in \{1, 2, ..., n\}$ so $f_\rho < e_\rho$. Choose $r = p \notin (N : M)$ and $m = p^{f_\mu - 1} x_\mu + ... + p^{f_\rho - 1} x_\rho + p^{f_n - 1} v_n \notin N$, then we have

$$rm = p^{f_\mu} x_\mu + ... + p^{f_\rho} x_\rho + p^{f_n} v_n \in N.$$

Since $e_n > 1$ then we also have

$$rm = p^{f_\mu} x_\mu + ... + p^{f_\rho} x_\rho + p^{f_n} v_n \notin p^e N = (N : M)N.$$

Hence

$$rm = p^{f_\mu} x_\mu + ... + p^{f_\rho} x_\rho + p^{f_n} v_n \in N - (N : M)N.$$

This contradicts to N an almost prime submodule.

Based on those cases, we conclude that almost prime submodules must be prime submodules.

Finally, the following conditions give an almost prime submodule that is not prime given by the next theorem.

Theorem 3.4. *Let M be a primary module over a principal ideal domain R and the order of M is p^e with primary decomposition:*

$$M = \langle\langle x_1 \rangle\rangle \oplus ... \oplus \langle\langle x_n \rangle\rangle$$

of its cyclic submodules with annihilator $ann(\langle\langle x_i \rangle\rangle) = \langle\langle p^{e_i} \rangle\rangle$ with

$$e = e_1 \geq e_2 \geq ... \geq e_m \geq e_{m+1} = ... = e_n = 1.$$

Let

$$N = \langle\langle p^{f_1} x_1 \rangle\rangle \oplus ... \oplus \langle\langle p^{f_n} x_n \rangle\rangle$$

a nonzero submodules of M.
Submodule N is an almost prime submodules of M that is not prime if and only if $f_i = e_i$ for all $i \in \{1, 2, ..., m\}$ and $f_i \leq 1$ for all $i \in \{m + 1, ..., n\}$.

PROOF. Let $f_i = e_i$ for all $i \in \{1, 2, ..., m\}$ and $f_i \leq 1$ for all $i \in \{m+1, m+2, ..., n\}$. We will show that N is an almost prime submodule which is not prime. Since N is nonzero, there exist $\mu \in \{m+1, ..., n\}$ so $f_\mu = 0$. Hence we can reweite N as

$$N = \sum_{j \in \{m+1,...,n\}}^{f_j=0} \langle\langle v_j \rangle\rangle.$$

Theorem 2.4 give us $(N : M) = \langle\langle p^e \rangle\rangle$ then $N - (N : M)N = N - \{0\}$. Let $r \in R$ and $x = \alpha_1 x_1 + ... + \alpha_n x_n \in M$ with $rx \in N - \{0\}$, we have

$$\sum_{i=1}^{n} r\alpha_i x_i \in \oplus_{j \in \{m+1,...,n\}}^{f_j=0} \langle\langle x_j \rangle\rangle \tag{4}$$

and

$$\sum_{i=1}^{n} \alpha_i x_i \neq 0. \tag{5}$$

Condition (4) give us two things,

$$p^{e_i} | r\alpha_i \text{ for all } i \in \{1, 2, ..., m\} \tag{6}$$

$$p|(r\alpha_i) \text{ for all } i \in \{m+1, ..., n\} \text{ with } f_i = 1 \tag{7}$$

and condition (5) give

$$p \not| r\alpha_i \text{ for all } i \in \{m+1, ..., n\} \text{ with } f_i = 0. \tag{8}$$

Condition (8) give $p \not| r$. Then from condition (6) we have $p^{e_i}|\alpha_i$ for all $i \in \{1, 2, ..., m\}$. And from condition (7) we also have $p|\alpha_i$ for all $i \in \{m+1, ..., n\}$ with $f_i = 1$. Hence

$$x = \sum_{j \in \{m+1,...,n\}}^{f_j=0} x_j \in N. \tag{9}$$

then N is an almost prime submodule. From initial assumption N is not prime submodule. Hence N is an almost prime submodules which is not prime.

Conversely, let N be an almost prime submodule of M which is not prime. We will show that

$$f_i = e_i \text{ for all } i \in \{1, 2, ..., m\} \tag{10}$$

and

$$f_i \leq 1 \text{ for all } i \in \{m+1, ..., n\}. \tag{11}$$

Since N is not prime, condition $f_i \leq 1$ for all $i \in \{1, 2, ..., n\}$ is impossible. Hence there exist $\rho \in \{1, ..., n\}$ so $f_\rho > 1$, then we have

$$(N : M) = \langle\langle p^x \rangle\rangle \tag{12}$$

with $1 < x \leq e$. Noted that $\rho \notin \{m+1, ..., n\}$ because $f_j \leq 1$ for all $j \in \{m+1, ..., n\}$.

Condition (12) gives

$$(N : M)N = \langle\langle p^x p^{f_1} x_1 \rangle\rangle \oplus ... \oplus \langle\langle p^x p^{f_n} x_n \rangle\rangle. \tag{13}$$

Now let $i \in \{1, 2, ..., m\}$, we will show condition (10) true. Supposed $f_i < e_i$, we will show some contradiction.

Choose $r = p \notin (N : M)$ and $m = p^{f_i - 1} x_i \notin N$, then we have

$$rm = pp^{f_i-1} x_i = p^{f_1} x_i \in N.$$

Since $f_i < e_i$ we also have

$$rm = p^{f_i} x_i \notin \langle\langle p^x p^{f_1} x_1 \rangle\rangle \oplus ... \oplus \langle\langle p^x p^{f_n} x_n \rangle\rangle = (N : M)N.$$

Hence

$$rm = p^{f_i} x_i \notin (N : M)N.$$

This contradicts to N an almost prime submodule. Then we have condition (10) and (11) true. The theorem is proved.

Theorem 3.5. *Let M be a torsion module over a principal ideal domain R, with its primary decomposition*

$$M = M_{p_1} \oplus ... \oplus M_{p_k}$$

and its submodule

$$N = N_1 \oplus ... \oplus N_k.$$

Submodule N is almost prime if and only if submodule N is one of the following:

(1) $N = N_1 \oplus ... \oplus N_k$ with $N_i = 0$ or $N_i = M_{p_i}$, or
(2) $N = N_1 \oplus ... \oplus N_j \oplus ... \oplus N_k$ for N_j is nonzero almost prime submodules of M_{p_j} and $N_i = 0$ or $N_i = M_{p_i}$ for $i \neq j$.

Proof. Let $j \in \{1, 2, ..., k\}$ such that $N = N_1 \oplus ... \oplus N_j \oplus ... \oplus N_k$ with N_j is nonzero prime submodule of M_{p_j}, then we have

$$(N : M) = \langle p_1^{f_1} ... p_j^{f_j} ... p_k^{f_k} \rangle$$

with $f_i = 0$ or $f_i = e_i$ if $i \neq j$. If

$$rm = r(m_1, ..., m_j, ..., m_k) \in N - (N : M)N$$

then we have

$$rm_j \in N_j - (N : M)N_j.$$

Since N_j is almost prime submodule of M_j and

$$(N : M)N_j = p_1^{f_1} ... p_j^{f_j} ... p_k^{f_k} N_j = p_j^{f_j} N_j = (N_j : M_j)N_j,$$

then we have

$$r \in (N : M) \text{ or } m_j \in N_j.$$

Therefore

$$r \in (N : M) \text{ or } m \in N,$$

hence N is an almost prime submodule of M.

Conversely, let N be an almost prime submodule. Assume there exist $j, l \in \{1, 2, ..., k\}$, $j \neq l$, such that

$$0 \neq N_j \neq M_{p_j} \text{ and } 0 \neq N_l \neq M_{p_l}$$

is almost prime. There exist i_j, i_l such that

$$0 < f_{j,i_j} < e_{j,i_j} \text{ and } 0 < f_{l,i_l} < e_{l,i_l}.$$

Choose $r = p_l$ and $m = p_j^{f_{j,i_j}} m_{j,i_j} + p_l^{f_{l,i_l}-1} m_{l,i_l}$, then we have

$$rm = p_l p_j^{f_{j,i_j}} m_{j,i_j} + p_l^{f_{l,i_l}} m_{l,i_l} \in N - (p_1^{f_1} ... p_k^{f_k})N = N - (N : M)N.$$

But

$$r \notin (N : M) \text{ and } m \notin N,$$

it contradicts that N is almost prime.

By Theorem 3.5, if the module $M = \mathbb{Z}_8 \oplus \mathbb{Z}_9$ is given, then its nonzero almost prime submodules are $\langle 2 \rangle \oplus \mathbb{Z}_9$, $\mathbb{Z}_8 \oplus \langle 3 \rangle$, $\langle 0 \rangle \oplus \mathbb{Z}_9$ and $\mathbb{Z}_8 \oplus \langle 0 \rangle$. The last two are almost prime submodules which is not prime.

If $M = M_{p_1} \oplus M_{p_2} \oplus M_{p_3}$ and P_1, P_2 are almost prime submodules of M_{p_1}, M_{p_2} respectively. Then by the Theorem 3.5 $P_1 \oplus P_2 \oplus M_{p_3}$ is not an almost prime submodule of $M_{p_1} \oplus M_{p_2} \oplus M_{p_3}$, because we can choose $p_1 \notin (N : M)$ and $1 + p_2 + 1 \notin N$ but $p_1(1 + p_2 + 1) \in N$.

The characterization of an almost prime submodule of a finitely generated module is given in the next theorem.

Theorem 3.6. *Let $M = M_F \oplus M_T$ be a finitely generated module over a principal ideal domain R, let M_F be its free part and M_T be its torsion part. Let $N = N_F \oplus N_T$ be submodule of M with $N_F \subset M_F$, with $(N_F : M_F) = p$ and $(N_T : M_T) = q$. Submodule N is almost prime submodule of M if and only if N is of the form:*

(1) $N = N_F \oplus M_T$, N_F is an almost prime submodule of M_F, or
(2) $N = M_F \oplus N_T$, N_T is an almost prime submodule of M_T, or
(3) $N = N_F \oplus N_T$, N_F and N_T are an almost prime submodule of respectively M_F and M_T with $(N_F : M_F) = (N_T : M_T)$

Proof. First, let $N = N_F \oplus M_T$, N_F be an almost prime submodule of M_F, then $(N : M) = (N_F : M_F)$. Let $rm = r(m_f + m_r) \in N - (N : M)N$, then we have $rm_f \in N_F - (N_F : M_F)N_F$. Since N_F is an almost prime submodule, we have $r \in (N_F : M_F)$ or $m_f \in N_F$. Hence $r \in (N : M)$ or $m \in N$, then N almost prime.

Second, let $N = M_F \oplus N_T$, N_T be an almost prime submodule of M_T, then $(N : M) = (N_T : M_T)$. Let $rm = r(m_f + m_r) \in N - (N : M)N$, then we have $rm_t \in N_T - (N_T : M_T)N_T$. Since N_T is an almost prime submodule, we have $r \in (N_T : M_T)$ or $m_t \in N_T$. Hence $r \in (N : M)$ or $m \in N$, then N is almost prime.

Third, let $N = N_F \oplus N_T$, N_F and N_T be almost prime submodule of M_F and M_T with $(N_F : M_F) = (N_T : M_T)$, then we have $(N : M) = (N_T : M_T) = (N_F : M_F)$. Let $rm = r(m_f + m_t) \in N - (N : M)N$, then $rm_f \in N_F - (N_F : M_F)N_F$ and $rm_t \in N_T - (N_T : M_T)N_T$. Since N_F and N_T is almost prime, we have $r \in (N_F : M_F)$ or $m_f \in N_F$ and $r \in (N_T : M_T)$ or $m_t \in N_T$. Hence $r \in (N : M)$ or $m \in N$, then N almost prime.

Conversely, let N be almost prime submodule. Supposed N_F not an almost prime submodule and $N_F \neq M_F$, then we have $r \notin (N_F : M_F)$ and $m_f \notin N_F$ such that $rm_f \in N_F - (N_F : M_F)N_F$. According to Lemma 2.8 we have $r \notin (N : M)$ and $m_f \notin N$ such that $rm_f \in N - (N_F : M_F)N_F$. This contradict to N an almost prime submodule, hence N_F almost prime or $N_F = M_F$. With same technique we will have N_T is an almost prime submodule or $N_T = M_T$. Base on this fact we will show you that if $N_F \neq M_F$ and $N_T \neq M_F$ then $(N_F : M_F) = (N_T : M_T)$.

Write $(N_F : M_F) = \langle p \rangle$ and $(N_T : M_T) = \langle q \rangle$. Since N_F is an almost prime submodule, we have p is prime or $p = 0$ according to Theorem 2.8. Hence $lcd(p, q) = q$ if $p|q$ or $lcd(p, q) = pq$ if $p \not| q$.

Suppose $p \not| q$, then we have $(N : M) = \langle pq \rangle$, note that both p and q cannot be zero. If p nonzero, we can choose $r = p$ and $m = m_f$ where m_f is one of basis of M_F such that pm_f is one of basis of N_F. Hence $rm \in N - (N : M)N$ where $r \notin (N : M)$ and $m \notin N$, this contradicts to N an almost prime submodule. If q nonzero we can choose $r = q$ and $m = m_t$ where m_t one of cyclic generator of M_T that not in N_T. Hence $rm \in N - (N : M)N$ where $r \notin (N : M)$ and $m \notin N$, this contradicts to N an almost prime submodule. Then we can conclude that $p|q$.

Suppose that $q \not| p$, then $q = px$ for x not an unit element, then we can choose $r = x$ and $m = pm_t$ where $pm_t \in M_T - N_T$ since $(N_T : M_T) = \langle px \rangle$. Hence $rm = pxm_t = qm_t \in N - (N : M)N$ where $r \notin (N : M)$ and $m \notin N$, this contradicts to N an almost prime submodule. Then we can conclude that x is unit element, hence $(N_F : M_F) = (N_T : M_T)$.

REFERENCES

[1] Kashan, H.A., On Almost Prime Submodules, *Acta Mathematica Scientia* **32B(2)** (2012), 645-651.

[2] Wardhana, I.G.A.W., Nghiem, N.D.H., Switrayni, N.W., Aini, Q., A note on almost prime submodule of CSM module over principal ideal domain, *Journal of Physics: Conference Series* **2106(1)** (2021), 012011.

[3] Wardhana, I.G.A.W., Astuti, P., Muchtadi-Alamsyah, I., On almost prime submodules of a module over a principal ideal domain, *JP Journal of Algebra, Number Theory and Applications*, **38(1)** (2016), 121-130.

[4] Wardhana, I.G.A.W., The Decomposition of a Finitely Generated Module over Some Special Ring, *Jurnal Teori dan Aplikasi Matematika*, **6(1)** (2022), 261-267.

[5] Kaplansky I., Infinite Abelian groups, Bull Oz. AW, 2022.

[6] Hedayat, S., Nekooei, R., Characterization of Prime Submodules of a Finitely Generated Free Module over a PID, *Houston Journal of Mathematics*, **31**(1) (2005), 75-85.