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ON SEMIPRIME SUBSEMIMODULES AND RELATED RESULTS

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Abstract. In this paper we introduce the notions of semiprime subsemimodules and semiprime k-subsemimodules and present some characterizations about them. Special attention has been paid, when semimodules are multiplication, to find extra properties of these semimodules. Moreover we prove a result for semiprime subsemimodules of quotient semimodules.

Key words: Semiprime subsemimodule, Strong semiprime k-subsemimodule, S-semiring, S-semimodule.

Abstrak. Pada paper ini kami memperkenalkan ide-ide subsemimodul semiprima, subsemimodul-k semiprima, dan menyajikan beberapa sifat mereka. Perhatian khusus diberikan, saat semimodul adalah perkalian, untuk menemukan sifat-sifat tambahan dari semimodul ini. Lebih jauh, kami membuktikan sebuah hasil untuk subsemimodul semiprima dari semimodul hasil bagi.

Kata kunci: Subsemimodul semiprima, semiprima kuat, semimodul-k, semiring-S, semimodul-S.

1. Introduction

In the recent years a good deal of researches have been done concerning semirings and semimodules (for example see [1]-[6]). Particularly, there are numerous applications of semirings and semimodules in various branches of mathematics and computer sciences (for example see [6]). Also, in the last decade the notion of prime subsemimodules has been studied by many authors (for example see [3], [6] and [8]). Having the vast heritage of ring theory available, a number of authors have tried

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to extend and generalize many classical notions and definitions. For example, from the definition of a prime submodule, they have reached to prime subsemimodule etc. In this paper we define semiprime subsemimodules and strong semiprime k-subsemimodules of semimodules and prove some of their properties. In section 2, we recall some of the known notions and definitions. In sections 3, we give some basic results about semiprime subsemimodules of semimodules and quotient semimodules. Section 4 is devoted to the study of strong semiprime k-subsemimodules, S-semirings and S-semimodules.

2. Preliminaries

We begin this section with some necessary definitions.

Definition 2.1. **(a)** A non-empty set R together with two binary operations (called addition and multiplication and denoted by $+$, $\cdot$ respectively) is called a *semiring* provided that:

(i) $(R,+)$ is a commutative semigroup
(ii) $(R,\cdot)$ is a semigroup
(iii) There exists $0 \in R$ such that $r+0=r$ and $r\cdot 0 = 0\cdot r = 0$ for all $r\in R$
(iv) Multiplication distributes over addition both from the left and right.

If R contains the multiplicative identity 1, then R is called a semiring with identity. A semiring R is commutative if $(R,\cdot)$ is a commutative semigroup. In this paper all semirings are commutative with identity.

(b) A non-empty subset I of a semiring R is called an *ideal* of R if $a,b\in I$ and $r\in R$ implies that $a+b\in I$ and $ra, ar\in I$.

(c) An ideal I of a semiring R is called a *subtractive ideal (or k-ideal)* if $a, a+b\in I$ implies that $b\in I$. For example $\{0\}$ is a k-ideal of R.

(d) An ideal I of a semiring R is called a *partitioning ideal* (or *Q-ideal*) if there exists a subset Q of R such that $R=\bigcup\{q+I|q\in Q\}$ and if $q_1,q_2\in Q$ then $(q_1+I)\bigcap(q_2+I)\neq\emptyset$ if and only if $q_1=q_2$.

Definition 2.2. **(a)** A *semimodule* M over a semiring R (or an *R-semimodule*) is a commutative monoid $(M,+)$ with additive identity 0_M, together with a function $R\times M\longrightarrow M$, defined by $(r,m)\mapsto rm$ (called scalar multiplication) such that:

(i) $r(m+m')=rm+rm'$
(ii) $(r+r')m=rm+r'm$
(iii) $(r\cdot r')m=r(r'm)$
(iv) $1_Rm=m$
(v) $0_Rm=r0_M=0_M$

for all $r,r'\in R$ and $m,m'\in M$. Clearly, every semiring is a semimodule over itself.

(b) A subset N of the R-semimodule M is called a *subsemimodule* of M if $a,b\in N$ and $r\in R$ implies that $a+b\in N$ and $ra\in N$.

(c) Let M be a semimodule over a semiring R. A *subtractive subsemimodule (or k-subsemimodule)* N is a subsemimodule of M such that if $a, a+b \in N$, then $b \in N$. For example $\{0_M\}$ is a k-subsemimodule of M.

(d) A subsemimodule N of a semimodule M over a semiring R is called a *partitioning subsemimodule* (or $Q(M)$-*subsemimodule*) if there exists a non-empty subset $Q(M)$ of M such that:

(i) $RQ(M) \subseteq Q(M)$, where $RQ(M) = \{rq | r \in R, q \in Q(M)\}$
(ii) $M = \bigcup\{q + N | q \in Q(M)\}$
(iii) If $q_1, q_2 \in Q(M)$ then $(q_1 + N)\bigcap(q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$.

Since every semiring is a semimodule over itself, every partitioning ideal of a semiring R is a partitioning subsemimodule of the R-semimodule R.

Example 2.3. Let $R = \{0, 1, 2, \cdots, n\}$ and define $x + y = max\{x, y\}$ and $xy = min\{x, y\}$ for each $x, y \in R$. By [1], Example 5, R together with the two defined operations forms a semiring. Let M denote the set of all non- negative integers. Define $a+b = max\{a, b\}$ for each $a, b \in M$. It is easy to show that $(M, +)$ is a commutative monoid with identity 0. Define a function from $R \times M$ in to M, sending (r, m) to $min\{r, m\}(r \in R, m \in M)$. It is easy to see that M is an R-semimodule. Now we show that $N = R$ is an R-subsemimodule of M. Let $a, b \in N, r \in R$. Since $a, b \leq n$, so $a+b = max\{a, b\} \leq n$. Hence $a+b \in N$. Also we have $ra = min\{r, a\}$. If $a \leq r$ then $ra = a \in N$ and if $a > r, ra = r \in N$. It is clear from the definition of addition in M that $0 + N = N$ and $k + N = \{k\}$ for each $n < k (n \in N)$. Thus N is a $Q(M)$-subsemimodule of M when $Q(M) = \{0\}\bigcup\{k \in M | n < k\}$.

Proposition 2.4. *Let R be a semiring, M an R-semimodule and N a $Q(M)$-subsemimodule of M. Then N is a k-subsemimodule of M.*

Proof. Let $x, x + y \in N$. Since $M = \bigcup\{q + N | q \in Q(M)\}$ we can write $y = q + z$ for some $q \in Q(M)$ and $z \in N$. Therefore $x + y = x + q + z = q + x + z \in q + N$. Also $x + y \in 0_M + N$. Hence $(q + N)\bigcap(0_M + N) \neq \emptyset$ and so $q = 0_M$. Therefore $y \in N$, as required. □

Definition 2.5. Let N be a subsemimodule of an R-semimodule M. Then $(N : M)$ is defined as $(N : M) = \{r \in R \mid rM \subseteq N\}$. Clearly $(N : M)$ is an ideal of R. The *annihilator* of M is defined as $(0 : M)$ and is denoted by $ann(M)$. If $ann(M) = 0$ then M is called *faithful*.

Definition 2.6. Let M and M' be semimodules over the semiring R. A function $f : M \longrightarrow M'$ is said to be a homomorphism of R-semimodules if $f(a+b) = f(a)+f(b)$ and $f(ra) = rf(a)$ for all $r \in R$ and $a, b \in M$. The *kernel* of f, denoted by $kerf$, is the set$\{x \in M | f(x) = 0\}$. Clearly $kerf$ is a k-subsemimodule of M. If f is one-to-one (onto), then f is called a monomorphism (an epimorphism). An isomorphism of R-semimodules is a one-to-one and onto homomorphism of R-semimodules. The R-semimodules M and M' are called *isomorphic* and are denoted by $M \cong M'$ if there exists an isomorphism from M to M'.

Remark 2.7. Let M, M' are semimodules over the semiring R and $f : M \longrightarrow M'$ be a homomorphism of R-semimodules. Then $kerf$ is a k-subsemimodule of M. Because if $x, x+y \in kerf$, we can write $f(x+y) = f(x)+f(y)$. Hence $0 = 0+f(y)$ and so $y \in kerf$, as required.

3. Semiprime Subsemimodules

Let M be a semimodule over a semiring R. A proper subsemimodule N of M is called prime if for each $r \in R$ and $m \in M$, $rm \in N$ implies that $r \in (N : M)$ or $m \in N$.

If N is a prime subsemimodule of M, then $(N : M)$ is a prime ideal of R(see [3], Lemma 4). As it is known a good deal of research has been done for semiprime submodule of a module during the last two decades. In this section we define semiprime subsemimodules and try to find some of their essential properties.

Definition 3.1. A proper subsemimodule N of an R-semimodule M is called *semiprime* if for each $r \in R$, $m \in M$ and positive integer t, $r^t m \in N$ implies that $rm \in N$.

Since the semiring R is an R-semimodule by itself, according to our definition, a proper ideal I of R is a semiprime ideal, if whenever $a^t b \in I$ for every $a, b \in R$ and positive integer t, then $ab \in I$. If the semiprime subsemimodule N of M is a k-subsemimodule ($Q(M)$-subsemimodule), then N is called a semiprime k-subsemimodule (semiprime $Q(M)$-subsemimodule). In a similar way if the semiprime ideal I of R is a k-ideal (Q-ideal), then I is called a semiprime k-ideal (semiprime Q-ideal).

Proposition 3.2. *Let R be a semiring, M an R-semimodule and N a proper subsemimodule of M. Then N is semiprime if and only if $r^2 m \in N (r \in R, m \in M)$ implies that $rm \in N$.*

Proof. Let N be a semiprime subsemimodule of M and $r^2 m \in N$, where $r \in R$ and $m \in M$. Then by definition $rm \in N$. Conversely, let $r^t m \in N$, where $r \in R$, $m \in M$ and $t \in \mathbb{Z}^+$. Then $r^2(r^{t-2}m) \in N$ and by hypothesis $r(r^{t-2}m) = r^{t-1}m \in N$. we write $r^{t-1}m = r^2(r^{t-3}m) \in N$ and so $r(r^{t-3}m) = r^{t-2}m \in N$. In this way we finally obtain $r^2 m \in N$ which implies that $rm \in N$. □

Example 3.3. Let $\mathbb{Z}_+$ denote the semiring of non-negative integers with the usual operations of addition and multiplication and $I \neq \{0\}$ be a proper ideal of $\mathbb{Z}_+$. It is easy to show that I is semiprime if and only if it can be written in the form

$I = (p_1 \cdots p_k) = (p_1) \bigcap \cdots \bigcap (p_k)$, where $p_1, \cdots p_k$ are distinct prime numbers.

Proposition 3.4. *Let R be a semiring and M an R-semimodule. If N is a prime subsemimodule of M, then N is a semiprime subsemimodule of M.*

Proof. Let $r^t m \in N$, where $r \in R$, $m \in M$ and $t \in \mathbb{Z}^+$. Since N is prime so $m \in N$ or $r^t \in (N : M)$. If $m \in N$ then $rm \in N$. Now let $r^t \in (N : M)$. Since $(N : M)$ is a prime ideal of R, so $r \in (N : M)$. Hence $rm \in N$. In any case we have $rm \in N$ and so N is a semiprime subsemimodule of M. □

Proposition 3.5. *Let R be a semiring and M an R-semimodule. If N is a semiprime subsemimodule of M, then $(N : M)$ is a semiprime ideal of R.*

Proof. Since N is a proper subsemimodule of M, so $(N : M) \neq R$. Let $a^t b \in (N : M)$ in which $a, b \in R$ and $t \in \mathbb{Z}^+$. Then $a^t bM \subseteq N$ and so for every $m \in M$, $a^t bm \in N$. Since N is a semiprime subsemimodule of M, so $abm \in N$ and hence $ab \in (N : M)$, as required. □

Definition 3.6. A semiring R is said to be an *integral semidomain* if for every $a, b \in R$, $ab = 0$ implies that $a = 0$ or $b = 0$.

Example 3.7. It is clear that $\mathbb{Z}_+$ is an integral semidomain.

In the next example we show that the converse of Proposition 3.5 is not true in general.

Example 3.8. Let $R = \mathbb{Z}_+$. Then $M = \mathbb{Z}_+ \oplus \mathbb{Z}_+$ is an R-semimodule, $N = \{r(9, 0) | r \in R\}$ is a subsemimodule of M and $(N : M) = 0$. Since R is an integral semidomain, $(N : M) = 0$ is a prime and hence semiprime ideal of R. But N is not a semiprime subsemimodule of M, because $3^2(2, 0) = (18, 0) \in N$ and $3(2, 0) = (6, 0) \notin N$.

The next theorem gives a characterization of semiprime subsemimodules. This is specially useful when we work with multiplication semimodules.

Theorem 3.9. *Let R be a semiring, M an R-semimodule and N a proper subsemimodule of M. Then N is semiprime if and only if for every ideal I of R, subsemimodule K of M and positive integer t, $I^t K \subseteq N$ implies that $IK \subseteq N$.*

Proof. Let I be an ideal of R and K a subsemimodule of M such that $I^t K \subseteq N$, where $t \in \mathbb{Z}^+$. Consider the set $S = \{ax | a \in I, x \in K\}$. Then Ra is an ideal of R, Rx is a subsemimodule of M and we have $(Ra)^t (Rx) \subseteq I^t K \subseteq N$. Hence $a^t x \in N$ which implies that $ax \in N$. So $S \subseteq N$ and since S is a generating set for IK, we must have $IK \subseteq N$. Conversely, let $a \in R$, $m \in M$ and $a^t m \in N$ in which $t \in \mathbb{Z}^+$. Take $I = Ra$ and $K = Rm$. Then we have $I^t K \subseteq N$. Hence $IK \subseteq N$ which implies that $am \in N$. The proof is now completed. □

Remark 3.10. Note that since the semiring R is an R-semimodule by itself, then by Theorem 3.9, I is a semiprime ideal of R, if and only if for every ideals J and K of R and positive integer t, $J^t K \subseteq I$ implies that $JK \subseteq I$.

Multiplication modules play an important rule in module theory. We recall

Definition 3.11. Let R be a semiring. An R-semimodule M is called *multiplication semimodule* provided that for every subsemimodule N of M there exists an ideal I of R such that $N = IM$.

Now we show that if M is a multiplication semimodule then the converse of Proposition 3.5, is true.

Proposition 3.12. *Let M be a multiplication semimodule over a semiring R and N a proper subsemimodule of M. Then N is a semiprime subsemimodule of M if and only if $(N : M)$ is a semiprime ideal of R.*

Proof. Let $(N : M)$ be a semiprime ideal of R. Assume that $I^t K \subseteq N$ in which I is an ideal of R, K a subsemimodule of M and $t \in \mathbb{Z}^+$. Since M is a multiplication R-semimodule, we can write $K = JM$ for some ideal J of R. Therefore $I^t JM \subseteq N$ and so $I^t J \subseteq (N : M)$. Since $(N : M)$ is semiprime, $IJ \subseteq (N : M)$. From this we have $IJM \subseteq N$ and so $IK \subseteq N$. Hence N is a semiprime subsemimodule of M. The converse is true by Proposition 3.5. □

Proposition 3.13. *Let M be a semimodule over a semiring R and N a semiprime k-subsemimodule of M. Then $(N : M)$ is a semiprime k-ideal of R.*

Proof. By Proposition 3.5, $(N : M)$ is a semiprime ideal of R. Now let $x, x + y \in (N : M)$. Then for each $m \in M$ we have $xm, (x + y)m = xm + ym \in N$. Hence, since N is k-subsemimodule, $ym \in N$. Therefore $y \in (N : M)$ and the assertion is proved. □

The next lemma gives conditions for a family of semiprime subsemimodules to be semiprime.

Lemma 3.14. *Let $\{N_\gamma\}_{\gamma\in\Gamma}$ be a non-empty family of semiprime subsemimodules of a semimodule M over a semiring R. Then $N = \bigcap_{\gamma\in\Gamma} N_\gamma$ is a semiprime subsemimodule of M. If, in addition, $\{N_\gamma\}_{\gamma\in\Gamma}$ is totally ordered by inclusion, then $T = \bigcup_{\gamma\in\Gamma} N_\gamma$ is a semiprime subsemimodule of M provided that $T \neq M$.*

Proof. By [5], Lemma 1, N is a subsemimodule of M. Let $r^t m \in N$, where $r \in R$, $m \in M$ and $t \in \mathbb{Z}^+$. Then for every $\gamma \in \Gamma$, $r^t m \in N_\gamma$. SinceN_γ is semiprime, we have $rm \in N_\gamma$ and this is true for every $\gamma \in \Gamma$. Therefore $rm \in \bigcap_{\gamma\in\Gamma} N_\gamma = N$. The fact that $\{N_\gamma\}_{\gamma\in\Gamma}$ is totally ordered by inclusion makes it clear that T is a subsemimodule of M and we can simply show that T is a semiprime subsemimodule of M.□

Definition 3.15. Let R be a semiring and I an ideal of R. The *radical* of I, denoted by $\sqrt{I}$, is the set of all $x \in R$ such that there exists a positive integer n (depending on x) with $x^n \in I$. The ideal I of R is called a *radical ideal* if $I = \sqrt{I}$.

Lemma 3.16. *Let R be a semiring and P a semiprime ideal of R. Then P is a radical ideal .*

Proof. Clearly $P \subseteq \sqrt{P}$. Let $a \in \sqrt{P}$ be an arbitrary element. Then for some $n \in \mathbb{Z}^+$, $a^n = a^n.1 \in P$. Since P is semiprime, so $a = a.1 \in P$, as required. □

The notion of primary submodule is fundamental when we work with modules. Similarly we define

Definition 3.17. Let M be a semimodule over a semiring R. A proper subsemimodule N of M is called *primary* if for each $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r^n \in (N : M)$ for some positive integer n. Clearly every prime subsemimodule is primary.

Since the semiring R is an R-semimodule by itself, according to our definition, a proper ideal I of R is a *primary* ideal, if whenever $ab \in I$ for each $a, b \in R$, then $a \in I$ or $b^n \in I$ for some positive integer n. If the primary subsemimodule N of M is a k-subsemimodule ($Q(M)$-subsemimodule) then N is called primary k-subsemimodule (primary $Q(M)$-subsemimodule). Prime k-subsemimodule (prime $Q(M)$-subsemimodule) is defined in a similar fashion. Also we can define prime and primary k-ideal (Q-ideal) in the same way.

Definition 3.18. Let R be a semiring and M an R-semimodule. M is called a *cancellative semimodule* if whenever $rm = sm$ for elements $m \in M$ and $r, s \in R$, then $r = s$.

A semiring R is called a *cancellative* semiring if it is a cancellative semimodule over itself.

Proposition 3.19. ([3], Proposition 1) *Let R be a semiring, M a cancellative R-semimodule and N a proper $Q(M)$-subsemimodule of M. Then $(N : M)$ is a Q-ideal of R.*

Theorem 3.20. *Let R be a semiring and M a cancellative R-semimodule. If N is a semiprime $Q(M)$-subsemimodule of M, then $(N : M)$ is a semiprime Q-ideal of R.*

Proof. This is clear by using Proposition 3.4 and Proposition 3.17. □

Theorem 3.21. *Let R be a semiring, M a cancellative R-semimodule and N*

a proper $Q(M)$-subsemimodule of M. Then N is a prime $Q(M)$-subsemimodule if and only if N is primary $Q(M)$-subsemimodule and $(N : M)$ is a semiprime Q-ideal of R.

PROOF. Let N be a prime $Q(M)$-subsemimodule of M. Then N is primary $Q(M)$-subsemimodule and by Propositions 3.3, 3.4 and 3.17, $(N : M)$ is a semiprime Q-ideal of R. Conversely, let $rm \in N$ where $r \in R$ and $m \in M$. Assume that $m \notin N$. Since N is primary, so $r^n = r^n.1 \in (N : M)$ for some $n \in \mathbb{Z}^+$. Since $(N : M)$ is semiprime, we have $r = r.1 \in (N : M)$, as required. □

Lemma 3.22. *Let R be a semiring, M be an R-semimodule and N be a $Q(M)$-subsemimodule of M. If $x \in M$, then there exists a unique $q \in Q(M)$ such that $x + N \subseteq q + N$.*

PROOF. Let $x \in M$. Since $\{q + N | q \in Q(M)\}$ is a partition of M, there exists $q \in Q(M)$ such that $x \in q + N$. Let $y \in x + N$ be an arbitrary element. Then there exists $n \in N$ such that $y = x + n$. Since $x \in q + N$, there exists $n' \in N$ such that $x = q + n'$. So $y = x + n = (q + n') + n = q + (n' + n) \in q + N$. Thus $x + N \subseteq q + N$. The uniqueness of q is an immediate consequence of part (iii) of Definition 2.2(d), because if there exists also $q' \in Q(M)$ such $x + N \subseteq q' + N$ then $(q + N) \bigcap (q' + N) \neq \emptyset$ and hence $q = q'$. □

Remark 3.23. Let M be a semimodule over a semiring R and N a $Q(M)$-subsemimodule of M. By Lemma 3.22, we can define a binary operation $\bigoplus$ on $\{q + N | q \in Q(M)\}$ as follows:

$(q_1 + N) \bigoplus (q_2 + N) = q_3 + N$, where q_3 is the unique element in $Q(M)$ such that $q_1 + q_2 + N \subseteq q_3 + N$. Now let $r \in R$ and suppose that $q_1 + N$, $q_2 + N \in \{q + N | q \in Q(M)\}$ are such that $q_1 + N = q_2 + N$. Then $q_1 = q_2$ by Definition 2.2(d) and we must have $rq_1 + N = rq_2 + N$. Hence we can define an operation $\bigodot$ from $R \times \{q + N | q \in Q(M)\}$ into $\{q + N | q \in Q(M)\}$ in the form $r \bigodot (q + N) = rq + N$. It can be shown that $\{q + N | q \in Q(M)\}$ together with $\bigoplus, \bigodot$, denoted by $(\{q + N | q \in Q(M)\}, \bigoplus, \bigodot)$, is an R-semimodule. Also we have the next Theorem.

Theorem 3.24. ([2], Theorem 2.4) *Let R be a semiring, M an R-semimodule, N a subsemimodule of M and $Q_1(M)$ and $Q_2(M)$ non-empty subsets of M such that N is both a $Q_1(M)$-subsemimodule and $Q_2(M)$-subsemimodule. Then $(\{q + N | q \in Q_1(M)\}, \bigoplus, \bigodot) \cong (\{q + N | q \in Q_2(M)\}, \bigoplus, \bigodot)$.*

Remark 3.25. (Quotient semimodule) If R is a semiring, M an R-semimodule and N a subsemimodule of M, then it is possible that N can be considered to be a $Q(M)$-subsemimodule with respect to many different subsets $Q(M)$ of M. However, Theorem 3.24, implies that the structure $(\{q + N | q \in Q(M)\}, \bigoplus, \bigodot)$ is essentially independent of the choice of $Q(M)$. Thus, if N is a $Q(M)$-subsemimodule of M, the semimodule $(\{q + N | q \in Q(M)\}, \bigoplus, \bigodot)$ is called a quotient semimodule of M by N and is denoted by $(M/N, \bigoplus, \bigodot)$ or M/N for short. Also by [2], Lemma 2.3,

there exists a unique element $q_0 \in Q(M)$ such that $q_0 + N = N$. Thus $q_0 + N$ is the zero element of M/N.
To prove our next result we need the following three theorems.

Theorem 3.26. ([3], Theorem 1) *Let R be a semiring, M an R-semimodule, N a $Q(M)$-subsemimodule of M and L a k-subsemimodule of M with $N \subseteq L$. Then $L/N = \{q + N | q \in L \bigcap Q(M)\}$ is a k-subsemimodule of M/N.*

Theorem 3.27. ([3], Theorem 2) *Let R be a semiring, M an R-semimodule, N a $Q(M)$-subsemimodule of M and L a k-subsemimodule of M/N. Then L is in the form T/N, where T is a k-subsemimodule of M which contains N.*

Theorem 3.28. ([3], Theorem 4) *Let R be a semiring, M an R-semimodule, N a $Q(M)$-subsemimodule of M and let T, L be k-subsemimodules of M containing N. Then $T/N = L/N$ if and only if $T = L$.*

Theorem 3.29. *Let N be a $Q(M)$-subsemimodule of a semimodule M over a semiring R and T be a k-subsemimodule of M with $N \subseteq T$. Then T is a semiprime R-subsemimodule of M if and only if T/N is a semiprime R-subsemimodule of M/N.*

Proof. Let T be a semiprime R-subsemimodule of M. By Definition 3.1, T is a proper subsemimodule of M and so, $T/N \neq M/N$, by Theorem 3.28. To show that T/N is a semiprime R-subsemimodule of M/N, let $r^t(q_1 + N) = r^t q_1 + N \in T/N$ where $q_1 \in Q(M)$, $r \in R$ and $t \in \mathbb{Z}^+$. It follows from Theorem 3.26, that $r^t q_1 \in T$ and since T is semiprime, we have $rq_1 \in T$. Hence $r(q_1 + N) \in T/N$, as required. Conversely, let T/N be a semiprime R-subsemimodule of M/N. Hence T/N is a proper R-subsemimodule of M/N and so $T \neq M$, by Theorem 3.28. Let $r^l a \in T$ where $r \in R$, $a \in M$ and $l \in \mathbb{Z}^+$. Since $a \in M$ and N is a $Q(M)$-subsemimodule of M, there are elements $q \in Q(M)$ and $n \in N$ such that $a = q + n$, by Lemma 3.22. So $r^l a = r^l q + r^l n \in T$. Since T is a k-subsemimodule of M and $r^l n \in N \subseteq T$, so $r^l q \in T$. Therefore, $r^l(q + N) = r^l q + N \in T/N$. Then T/N semiprime gives $r(q + N) \in T/N$ and so $rq + N \in T/N$. Hence $rq \in T$ and so $ra \in T$. The proof is now completed. □

4. Strong Semiprime k-Subsemimodules

In this section we investigate strong semiprime k-subsemimodules will lead us to some interesting results. First we recall some definitions.

Definition 4.1. **(a)** A proper ideal I of a semiring R is said to be a *strong ideal* if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$. If the strong ideal I of R is a k-ideal, then we call I a *strong k-ideal*. If the strong k-ideal I of R is semiprime, then I is called a *strong semiprime k-ideal*. If the strong k-ideal I of R is a radical ideal, then we call I a *strong k-radical ideal*.

(b) A subsemimodule N of an R-semimodule M is said to be a *strong subsemimodule* if for each $a \in N$ there exists $b \in N$ such that $a + b = 0$. Clearly, every submodule of a module over a ring R is a strong subsemimodule. If a strong subsemimodule N of M is a k-subsemimodule, then we call N a *strong k-subsemimodule*. If a strong k-subsemimodule N of M is semiprime, then N is called a *strong semiprime k-subsemimodule*.

Proposition 4.2. ([5], Proposition 1) *Let M be a semimodule over a semiring R. Then the following statements hold*:

(i) *If N is a strong subsemimodule of M, then N is a k-subsemimodule.*
(ii) *If I is a strong ideal of R, then IM is a strong k-subsemimodule.*

Example 4.3. The monoid $M = (\mathbb{Z}_6, +_6)$ is a semimodule over $(\mathbb{Z}_+, +, \cdot)$. We can show that $N = \{0, 2, 4\}$ is a strong $Q(M)$-subsemimodule of M (and so k-strong subsemimodule of M, by Proposition 4.2(i)), where $Q(M) = \{0, 1\}$.

Lemma 4.4. ([4], Proposition 2) *Let M be a finitely generated semimodule over a semiring R and I be a strong k-radical ideal of R. Then $(IM : M) = I$ if and only if $ann(M) \subseteq I$.*

Proposition 4.5. *Let M be a finitely generated semimodule over a semiring R, P a strong k-radical ideal of R containing $ann(M)$ and I an ideal of R such that $IM \subseteq PM$. Then $I \subseteq P$.*

PROOF. By Lemma 4.4, we have $(PM : M) = P$. Now let $r \in I$ be an arbitrary element. Then $rM \subseteq IM \subseteq PM$ which implies $r \in (PM : M) = P$, as required. □

Theorem 4.6. *Let M be a finitely generated multiplication semimodule over a semiring R and P a strong semiprime k-ideal of R containing $ann(M)$. Then PM is a strong semiprime k-subsemimodule of M.*

PROOF. By Lemma 3.16, P is a radical ideal of R and so $(PM : M) = P$, by Lemma 4.4. On the other hand, by Proposition 4.2(ii), PM is a strong k-subsemimodule of M. Now it is enough to show that PM is a semiprime subsemimodule of M. Let I be an ideal of R, K a subsemimodule of M and $t \in \mathbb{Z}^+$ such that $I^t K \subseteq PM$. Since M is a multiplication R-semimodule, there exists an ideal J of R such that $K = JM$. Hence $I^t JM \subseteq PM$ and by Proposition 4.5, $I^t J \subseteq P$. But P is a semiprime ideal of R and so $IJ \subseteq P$. Hence $IJM \subseteq PM$, that is, $IK \subseteq PM$. The theorem is now proved. □

Definition 4.7. **(a)** A semiring R is called an *S-semiring* if every proper ideal in R is a product of strong semiprime k-ideals.

(b) A semimodule M over a semiring R is called an *S-semimodule* if every proper subsemimodule N of M either is strong semiprime k-subsemimodule or has a factorization in the form $N = P_1 \cdots P_n N^\star$ in which $P_1, \cdots, P_n$ are strong semiprime k-ideals of R and $N^\star$ is a strong semiprime k-subsemimodule of M.

Example 4.8. A Dedekind domain is an integral domain R in which every proper ideal is the product of a finite number of prime ideals. Let M be a multiplication module over a Dedekind domain R. Then M is clearly an S-semimodule.

Theorem 4.9. *Let M be a faithful finitely generated multiplication semimodule over an S-semiring R. Then M is an S-semimodule.*

Proof. Let N be a proper subsemimodule of M. Since M is a multiplication semimodule, we can write $N = IM$ for some ideal I of R. Since R is an S-semiring, so $I = P_1 \cdots P_n$, where $P_i (1 \leq i \leq n)$ is a strong semiprime k-ideal of R and so $N = P_1 \cdots P_n M$. By Theorem 4.6, $P_i M (1 \leq i \leq n)$ is a strong semiprime k-subsemimodule of M. Therefore N either is a strong semiprime k-subsemimodule of M or has a factorization in the form $N = P_1 \cdots P_{n-1} N^\star$, where $N^\star = P_n M$. □

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