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THE PARTIAL SUMS OF THE LEAST SQUARES RESIDUALS OF SPATIAL OBSERVATIONS SAMPLED ACCORDING TO A PROBABILITY MEASURE

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Abstract. A functional central limit theorem for a sequence of partial sums processes of the least squares residuals of a spatial linear regression model in which the observations are sampled according to a probability measure is established. Under mild assumptions to the model, the limit of the sequence of the least squares residual partial sums processes is explicitly derived. It is shown that the limit process which is a function of the Brownian sheet depends on the regression functions and the probability measure under which the design is constructed. Several examples of the limit processes when the model is true are presented. Lower and upper bounds for boundary crossing probabilities of signal plus noise models when the noises come from the residual partial sums processes are also investigated.

Key words and Phrases: Least squares residuals, partial sums process, spatial linear regression model, standard Brownian sheet, Riemann-Stieltjes integral.

Abstrak. Sebuah teorema limit pusat fungsional untuk barisan proses jumlah parsial dari sisaan kuadrat terkecil suatu model regresi linear spasial yang pengamatannya dilakukan berdasarkan suatu fungsi peluang telah ditemukan. Berdasarkan asumsi-asumsi yang tidak kuat terhadap model, limit barisan proses jumlah parsial dari sisaan kuadrat terkecil dimunculkan secara jelas. Proses limit yang diperoleh yang merupakan fungsi dari lembaran Brown bergantung pada fungsi-fungsi regresi. dan fungsi peluang yang digunakan dalam mengonstruksikan rancangan percobaannya. Beberapa contoh proses limit untuk model yang diasumsikan benar disajikan. Batas bawah dan atas dari peluang-peluang melewati perbatasan dari model sinyal ditambah pengganggu juga diselidiki untuk kasus dimana pengganggu berupa proses jumlah parsial dari sisaan kuadrat terkecil

Kata kunci: Sisaan kuadrat terkecil, proses jumlah parsial, model regresi linear spasial, lembaran Brown standar, integral Riemann-Stieltjes.

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1. INTRODUCTION

In the literatures of model-check and boundary detection problems for spatial linear regression models, the partial sums of the least squares residuals are commonly investigated. To test whether the assumed model holds true, Kolmogorov (-Smirnov) and Cramér-von Mises functionals of the partial sums process of the residuals are defined and their limiting distribution are studied. MacNeill [14, 15] and Xie [22] derived the limit of such a process to detect the existence of a boundary on the experimental region. In Bischoff and Somayasa [6] the limit process was established by applying the geometrical approach proposed by Bischoff [5] [see also Somayasa [19]].

It is worth noting that in the literatures just mentioned the limit of the sequence of the partial sums processes of the residuals were obtained under an equidistance experimental design or a so-called regular lattice only. However, in practice for economic, technical or ecological reasons it is possible that the statistician cannot or will not sample equidistantly. For change-point problems it is sometimes not optimal to sample equidistantly, see e.g. Bischoff, and Miller [4]. By those practical reasons it is urgent to extend the results given in [14], [15], and [6] to those under a more general experimental design rather than a regular lattice.

To explain the problem in more detail let us consider an experiment conducted on an experimental region given by a closed rectangle $E := [a,b] \times [c,d] \subset \mathbb{R}^2$, $a < b$, and $c < d$. Let $\Xi_n := \{(t_{n\ell}, s_{\ell k}) \in E : 1 \leq k, \ell \leq n\}$, $n \geq 1$ be the $n \times n$ experimental conditions. Throughout this paper for any function $h : E \mapsto \mathbb{R}$ let $h(\Xi_n) := (h(t_{n\ell}, s_{\ell k}))_{k=1,\ell=1}^{n,n}$ be an $n \times n$-dimensional matrix whose entry in the k-th row and ℓ-th column is given by $h(t_{n\ell}, s_{\ell k})$. Correspondingly suppose we have a linear model

$$\mathbf{Y}_{n\times n} = \sum_{i=1}^{p} \beta_i f_i(\Xi_n) + \mathbf{E}_{n\times n}, \tag{1}$$

where $\mathbf{Y}_{n\times n} := (Y_{\ell k})_{k=1,\ell=1}^{n,n}$ is the $n \times n$-dimensional matrix of observations, $Y_{\ell k}$ is the observation in $(t_{n\ell}, s_{\ell k})$, $\mathbf{E}_{n\times n} := (\varepsilon_{\ell k})_{k=1,\ell=1}^{n,n}$ is the $n\times n$-dimensional matrix of random errors having independent and identically distributed entries with $\mathbb{E}(\varepsilon_{\ell k}) = 0$ and $Var(\varepsilon_{\ell k}) = \sigma^2 < \infty$, $f_i : E \mapsto \mathbb{R}$ is a known regression function, and β_i is an unknown constant, $1 \leq i \leq p$.

Let $\mathbf{W}_n := [f_1(\Xi_n), \ldots, f_p(\Xi_n)]$ be a linear subspace of the space of $n \times n$-dimensional real matrices $\mathbb{R}^{n\times n}$ generated by $f_1(\Xi_n), \ldots, f_p(\Xi_n)$. The matrix of least squares residuals of (1) is given by

$$\widehat{\mathbf{R}}_{n\times n} := (r_{\ell k})_{k=1,\ell=1}^{n,n} = pr_{\mathbf{W}_n^\perp} \mathbf{Y}_{n\times n} = pr_{\mathbf{W}_n^\perp} \mathbf{E}_{n\times n} \tag{2}$$

(cf. Seber and Lee [18], p. 38, and Arnold [1], p. 62), where $pr_{\mathbf{W}_n}$ and $pr_{\mathbf{W}_n^\perp} := id - pr_{\mathbf{W}_n}$ stand for the orthogonal projectors onto $\mathbf{W}_n$ and onto the orthogonal complement of $\mathbf{W}_n$, respectively.

MacNeill and Jandhyala [14] and Bischoff and Somayasa [6] embedded the sequence of the matrix of residuals $\widehat{\mathbf{R}}_{n\times n}$ into a sequence of stochastic processes

$\Big\{\mathbf{T}_n(\widehat{\mathbf{R}}_{n\times n})(t,s) : (t,s) \in \mathbf{I} := [0,1]\times[0,1], n \geq 1\Big\}$ which is further called the sequence of the least squares residual partial sums processes of Model (1), where for any $\mathbf{A}_{n\times n} := (a_{\ell k})_{k=1,\ell=1}^{n,n} \in \mathbb{R}^{n\times n}$ and $(t,s) \in \mathbf{I}$,

$$\mathbf{T}_n(\mathbf{A}_{n\times n})(t,s) := \sum_{k=1}^{[ns]}\sum_{\ell=1}^{[nt]} a_{\ell k} + (nt-[nt])\sum_{k=1}^{[ns]} a_{[nt]+1,k} + (ns-[ns])\sum_{\ell=1}^{[nt]} a_{\ell,[ns]+1} + (nt-[nt])(ns-[ns])a_{[nt]+1,[ns]+1},$$

thereby for $x \in \mathbb{R}$, $[x] := \max\{n \in \mathbb{N} : \ n \leq x\}$, and $\mathbf{T}_n(\mathbf{A}_{n\times n})(t,s) = 0$, for $t = 0$ or $s = 0$. By the definition these processes have sample paths in $\mathcal{C}(\mathbf{I})$, where $\mathcal{C}(\mathbf{I})$ is the space of continuous functions on $\mathbf{I}$. As usual $\mathcal{C}(\mathbf{I})$ is endowed with the uniform topology. Throughout we will use the acronym, LSRPS, as shorthand for least squares residual partial sums.

Under the condition that $E = \mathbf{I}$, $t_{n\ell} = \frac{\ell}{n}$, $s_{\ell k} = \frac{k}{n}$, $1 \leq \ell, k \leq n$, and $f_1, \ldots, f_p$ are linearly independent and continuously differentiable on $\mathbf{I}$, it was shown in [14], that $\frac{1}{n\sigma}\mathbf{T}_n(\widehat{\mathbf{R}}_{n\times n})$ converges weakly in $\mathcal{C}(\mathbf{I})$ to a centered Gaussian process defined by

$$B(t,s) - \int_0^t\int_0^s\int_0^1\int_0^1 \widetilde{f}^\top(u,v)\mathbf{G}^{-1}\widetilde{f}(u',v')dB(u',v')dudv, (t,s) \in \mathbf{I},$$

where B is the standard Brownian sheet on $\mathcal{C}(\mathbf{I})$, and $\widetilde{f} := (f_1, \ldots, f_p)^\top$. Note that [6] proposed a geometrical approach due to Bischoff [5] which is different with that proposed in [14] in obtaining the limit process of $\frac{1}{n\sigma}\mathbf{T}_n(\widehat{\mathbf{R}}_{n\times n})$.

As a matter of fact, in this paper we aim to give a generalization of the preceding result by sampling the observations according to a probability measure instead of sampling equidistantly. We also derive the limit of the sequence of the LSRPS processes under different assumptions given to the regression functions.

It is obvious that the sequence of the experimental conditions $(\Xi_n)_{n\geq 1}$ corresponds uniquely to a sequence of discrete probability measures $(P_n)_{n\geq 1}$ defined on the measure space $(E, \mathcal{B}(E))$, given by

$$P_n(A) := \frac{1}{n^2}\sum_{\ell=1}^n\sum_{k=1}^n \delta_{\{(t_{n\ell},s_{\ell k})\}}(A), A \in \mathcal{B}(E), n \geq 1,$$

where $\delta_{\{(t,s)\}}$ is the Dirac measure in $(t,s) \in E$, defined by $\delta_{\{(t,s)\}}(A) = 1$, for $(t,s) \in A$, and $\delta_{\{(t,s)\}}(A) = 0$, for $(t,s) \notin A$. Let $(F_n)_{n\geq 1}$ be the corresponding sequence of the distribution functions of $(P_n)_{n\geq 1}$, and P_0 be a probability measure on $(E, \mathcal{B}(E))$ with the distribution function F_0 on E, such that $F_0 = F_{01} \times F_{02}$, for some distribution functions F_{01} and F_{02} on $[a,b]$ and $[c,d]$, respectively. We need for our result

$$\sup_{(t,s)\in E} |F_n(t,s) - F_0(t,s)| \to 0, \text{ as } n \to \infty. \quad (3)$$

The sequence $(\Xi_n)_{n\geq 1}$ can be constructed in a natural way so that (3) is fulfilled. For instance, for a fixed $n \geq 1$ and $1 \leq \ell \leq n$, let us determine first a partition $\{t_{n1}, t_{n2}, \ldots, t_{nn}\}$ on the interval $(a, b]$ based on the equation $F_0(t_{n\ell}, d) = \frac{\ell}{n}$. Accordingly, for a fixed $\ell \in \{1, \ldots, n\}$ the design point $(t_{n\ell}, s_{\ell k}) \in E$ is then developed as the solution of the equation $F_0(t_{n\ell}, s_{\ell k}) = \frac{\ell k}{n^2}$, $1 \leq k \leq n$, which is unique as long as F_0 is continuous and strictly increasing on E. Under this sampling scheme the obtained experimental condition is not a regular lattice, unless P_0 is the uniform probability measure on $(E, \mathcal{B}(E))$. For $1 \leq \ell \leq n$, the second component of the design pints $(t_{n\ell}, s_{\ell k})$ does not depend anymore on ℓ. Therefore for a fixed k and $1 \leq \ell \leq n$, $s_{\ell k}$ will be denoted by s_{nk}, $1 \leq k \leq n$. It can be shown that under this sampling procedure the sequence $(F_n)_{n\geq 1}$ satisfies (3). To see this, let us consider the family of closed rectangles $\{[t_{n\ell-1}, t_{n\ell}] \times [s_{nk-1}, s_{nk}] : 1 \leq \ell, k \leq n\}$, where $t_{n0} := a$, and $s_{n0} := c$, and let $(t, s) \in E$ be arbitrary. Then by construction there exist ℓ and k, $1 \leq \ell, k \leq n$, such that $(t, s) \in [t_{n\ell-1}, t_{n\ell}] \times [s_{nk-1}, s_{nk}]$, and it holds $|F_n(t, s) - F_0(t, s)| \leq |F_n(t_{n\ell-1}, s_{nk-1}) - F_0(t_{n\ell}, s_{nk})| = \left|\frac{(\ell-1)(k-1)}{n^2} - \frac{\ell k}{n^2}\right| \leq 2/n$. In this paper we assume that F_0 is continuous and strictly increasing on E and satisfies $F_0 = F_{01} \times F_{02}$, for some marginal distribution functions F_{01} and F_{02} defined on $[a, b]$ and $[c, d]$, respectively.

It is worth mentioning that for the case of linear regression models with an experimental region given by a closed interval $[a, b]$, $a < b$, Bischoff [3, 4] generalized the results of MacNeill [12, 13] under Assumption (3) by proposing a sampling scheme according to the quantile function of a given probability measure on $([a, b], \mathcal{B}([a, b]))$. Their results can not be straightforwardly extended however to the spatial context, since the quantile function of a probability measure on $(E, \mathcal{B}(E))$ is not uniquely determined. For this reason we need more effort in deriving the limit of the sequence of the LSRPS processes when the consideration is extended to the spatial observations.

The rest of this paper discusses the limit process of the sequence of the spatial LSRPS processes under Assumption (3) when the model is true, see Section 2. There we also present examples of the limit process associated with polynomial models. Lower and upper bounds of boundary crossing probabilities involving signal plus noise models when the involved noise is the limit of the sequence of spatial LSRPS processes are presented at the end of Section 2.

2. LIMIT PROCESS

Let us consider Model (1) with the experimental region E and experimental design Ξ_n. We suppose that (3) is fulfilled. Let $\mathbf{X}_{n^2}$ be the $n^2 \times p$-dimensional design matrix whose j-th column is given by $vec(f_j(\Xi_n))$, $1 \leq j \leq p$, where "vec" is the well known vec operator defined e.g. in Harville [9], p.340-344. The entry in the i-th row and j-th column of $\mathbf{X}_{n^2}$ is nothing but $f_j(t_{n\ell}, s_{\ell k})$, for $1 \leq \ell, k \leq n$ that satisfies the relation $(\ell - 1)n + k = i$. Since P_n converges weakly to P_0, it holds by the convergence componentwise that

$$\frac{1}{n^2}\mathbf{X}_{n^2}^\top\mathbf{X}_{n^2} = \left(\int_E f_i(x,y)f_j(x,y)P_n(dx,dy)\right)_{i=1,j=1}^{p,p} \to \mathbf{G}, \text{ as } n\to\infty,$$

provided $f_1,\ldots,f_p$ are continuous on E, see e.g. Theorem 2.1 (Portmanteau Theorem) in [2], where $\mathbf{G}$ is a $p\times p$-dimensional matrix whose entry in the i-th row and j-th column is given by $\int_E f_i(x,y)f_j(x,y)P_0(dx,dy)$, $1\le i,j\le p$. Note that if $f_1,\ldots,f_p$ are linearly independent as functions in $L_2(P_0,E)$, then $\mathbf{G}$ is invertible, where $L_2(P_0,E)$ is the space of squared integrable function on E with respect to P_0. The well known continuous mapping theorem further implies

$$\left(\frac{1}{n^2}\mathbf{X}_{n^2}^\top\mathbf{X}_{n^2}\right)^{-1} \to \mathbf{G}^{-1}, \text{ as } n\to\infty. \tag{4}$$

Hence for large enough $n\in\mathbb{N}$, the projection matrix $\mathbf{X}_{n^2}(\mathbf{X}_{n^2}^\top\mathbf{X}_{n^2})^{-1}\mathbf{X}_{n^2}^\top$ exists by the reason $vec(f_1(\Xi_n)),\ldots,vec(f_p(\Xi_n))$ are linearly independent in $\mathbb{R}^{n^2}$ for large enough $n\in\mathbb{N}$.

Theorem 2.1. *(Invariance Principle). Let $(\mathbf{E}_{n\times n})_{n\ge1}$, $\mathbf{E}_{n\times n} = (\varepsilon_{\ell k})_{k=1,\ell=1}^{n,n}$ be a sequence of $n\times n$ dimensional random matrices such that $\varepsilon_{\ell k}$ are independent and identically distributed random variables with $\mathbb{E}(\varepsilon_{\ell k}) = 0$ and $Var(\varepsilon_{\ell k}) = \sigma^2 < \infty$. Then*

$$\frac{1}{n\sigma}S_n(\mathbf{E}_{n\times n}) \xrightarrow{\mathcal{D}} B_{F_0} \text{ in } \mathcal{C}(E), \text{ as } n\to\infty,$$

where $S_n : \mathbb{R}^{n\times n}\mapsto\mathcal{C}(E)$ is a linear operator on $\mathbb{R}^{n\times n}$ defined by

$$S_n(\mathbf{A}_{n\times n})(x,y) := \mathbf{T}_n(\mathbf{A}_{n\times n})(F_{01}(x),F_{02}(y)),\ \mathbf{A}_{n\times n}\in\mathbb{R}^{n\times n},$$

and B_{F_0} is the Brownian sheet on $\mathcal{C}(E)$, such that

$$B_{F_0}(x,y) := B(F_{01}(x),F_{02}(y)), \text{ for } (x,y)\in E,$$

i.e., B_{F_0} is a centered Gaussian process whose covariance function is given by

$$Cov(B_{F_0}(x,y),B_{F_0}(x',y')) = F_0(x\wedge x', y\wedge y'),$$

for $(x,y),(x',y')\in E$. Here and throughout this paper $a\wedge b$ means the minimum between a and b, for any real numbers a and b.

Proof. The first assertion is a direct consequence of the well known continuous mapping theorem, (see e.g. [2], p. 20-22) and the result of Park [17]. The second assertion follows from the definition of the covariance function of the standard Brownian sheet (cf. [10], [23], and [17]) and by the monotonicity of F_{01} and F_{02}. □

Theorem 2.2. *Let $f_1,\ldots,f_p$ be linearly independent as functions in $\mathcal{C}(E)\cap BV_H(E)$, where $BV_H(E)$ is the space of functions that have bounded variations in the sense of Hardy on E. If Assumption (3) is fulfilled, then for $n\to\infty$,*

$$\frac{1}{n\sigma}S_n(\widehat{\mathbf{R}}_{n\times n}) \xrightarrow{\mathcal{D}} B_{\widetilde{f},F_0} \text{ in } \mathcal{C}(E),$$

where for $(x,y)\in E$,

$$B_{\widetilde{f},F_0}(x,y) := B_{F_0}(x,y) - \int_{[a,x]\times[c,y]} \widetilde{f}^\top(u,v)P_0(du,dv)\mathbf{G}^{-1}B^*_{F_0,\widetilde{f}},$$

with $\widetilde{f} := (f_1,\ldots,f_p)^\top$, *and*

$$\begin{aligned} B^*_{F_0,\widetilde{f}} := \Delta_E(B_{F_0}\widetilde{f}) &- \int_{[a,b]}^{(R)} B_{F_0}(t,d)d\widetilde{f}(t,d) - \int_{[c,d]}^{(R)} B_{F_0}(b,s)d\widetilde{f}(b,s) \\ &+ \int_{[a,b]}^{(R)} B_{F_0}(t,c)d\widetilde{f}(t,c) + \int_{[c,d]}^{(R)} B_{F_0}(a,s)d\widetilde{f}(a,s) \\ &+ \int_{E}^{(R)} B_{F_0}(t,s)d\widetilde{f}(t,s). \end{aligned}$$

Thereby $\Delta_{[a_1,b_1]\times[c_1,d_1]}\psi := \psi(b_1,d_1)-\psi(b_1,c_1)-\psi(a_1,d_1)+\psi(a_1,c_1)$, *and* $\Delta_\emptyset\psi := 0$, *for any rectangle* $[a_1,b_1]\times[c_1,d_1]\subseteq E$, *and any real-valued function* ψ *on* E. *We refer the reader to Clarkson and Adams* [8] *for the definition of* $BV_H(E)$.

PROOF. Since S_n is linear on $\mathbb{R}^{n\times n}$, by Equation (2) we have for any $(x,y)\in E$,

$$\frac{1}{n\sigma}S_n(\widehat{\mathbf{R}}_{n\times n})(x,y) = \frac{1}{n\sigma}S_n(\mathbf{E}_{n\times n})(x,y) - \frac{1}{n\sigma}S_n(pr_{\mathbf{W}_n}\mathbf{E}_{n\times n})(x,y).$$

Let $\mathbf{1}_{[nF_{01}(x)][nF_{02}(y)]}$ be an n^2-dimensional vector that has 1's for elements where Ξ_n has its component $(t_{n\ell},s_{\ell k})$ with $\ell\le[nF_{01}(x)]$ and $k\le[nF_{02}(y)]$, whereas the remainder is zero. Then by the definition of $\mathbf{T}_n$, we have

$$\begin{aligned} &\frac{1}{n\sigma}S_n(pr_{\mathbf{W}_n}\mathbf{E}_{n\times n})(x,y) \\ &\quad= \left(\frac{1}{n^2}\mathbf{1}^\top_{[nF_{01}(x)][nF_{02}(y)]}\mathbf{X}_{n^2}+o(1)\right)\left(\frac{1}{n^2}\mathbf{X}^\top_{n^2}\mathbf{X}_{n^2}\right)^{-1}\frac{1}{n\sigma}\mathbf{X}^\top_{n^2}vec(\mathbf{E}_{n\times n}) \\ &\quad= \left(\frac{1}{n^2}\sum_{\ell=1}^{[nF_{01}(x)]}\sum_{k=1}^{[nF_{02}(y)]}\widetilde{f}^\top(t_{n\ell},s_{\ell k})+o(1)\right)\left(\frac{1}{n^2}\mathbf{X}^\top_{n^2}\mathbf{X}_{n^2}\right)^{-1}\frac{1}{n\sigma}\mathbf{X}^\top_{n^2}vec(\mathbf{E}_{n\times n}) \\ &\quad= \left(\int_{[a,x]\times[c,y]}\widetilde{f}^\top(u,v)P_n(du,dv)+o(1)\right)\left(\frac{1}{n^2}\mathbf{X}^\top_{n^2}\mathbf{X}_{n^2}\right)^{-1}\frac{1}{n\sigma}\mathbf{X}^\top_{n^2}vec(\mathbf{E}_{n\times n}), \end{aligned}$$

where $o(1)$ is the collection of terms that goes to zero as $n\to\infty$. By Assumption (3), we further get the following componentwise convergence

$$\int_{[a,x]\times[c,y]}\widetilde{f}^\top(u,v)P_n(du,dv) \to \int_{[a,x]\times[c,y]}\widetilde{f}(u,v)P_0(du,dv), \text{ as } n\to\infty.$$

The term $\frac{1}{n\sigma}\mathbf{X}^\top_{n^2}vec(\mathbf{E}_{n\times n})$ is equal to $\frac{1}{n\sigma}\sum_{\ell=1}^n\sum_{k=1}^n\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k}$ which is the sequence of the partial sums of the sequence of independent random vectors

$$\left\{\frac{1}{n\sigma}\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k},\ 1\le\ell,k\le n\right\}, n\ge 1, \tag{5}$$

with $\mathbb{E}(\frac{1}{n\sigma}\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k})=\mathbf{0}\in\mathbb{R}^p$, and

$$Cov\left(\frac{1}{n\sigma}\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k}\right)=\left(\frac{1}{n^2}\sum_{\ell=1}^{n}\sum_{k=1}^{n}f_i(t_{n\ell},s_{\ell k})f_j(t_{n\ell},s_{\ell k})\right)_{i=1,j=1}^{p,p},$$

which converges componentwise to $\mathbf{G}$, as $n\to\infty$. Furthermore, for every $\varepsilon>0$, it holds

$$\begin{aligned}&\sum_{\ell=1}^{n}\sum_{k=1}^{n}\mathbb{E}\left\{\left\|\frac{1}{n\sigma}\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k}\right\|^2\mathbf{1}_{\left\{\left\|\frac{1}{n\sigma}\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k}\right\|>\varepsilon\right\}}\right\}\\&\le\frac{C}{\sigma^2}\mathbb{E}\left\{\varepsilon_{11}^2\mathbf{1}_{\left\{\varepsilon_{11}^2>\frac{(\varepsilon n\sigma)^2}{C}\right\}}\right\}\to0,\ \text{as } n\to\infty,\end{aligned}$$

where $C:=\sum_{i=1}^{p}\|f_i\|_\infty^2<\infty$ and $\mathbf{1}_A$ is the indicator of the set A. This shows that the Lindeberg condition is satisfied by Sequence (5). Hence, by the multivariate Lindeberg-Feller central limit theorem (cf. Van der Vaart [21], p. 20), we have

$$\frac{1}{n\sigma}\sum_{\ell=1}^{n}\sum_{k=1}^{n}\widetilde{f}(t_{n\ell},s_{\ell k})\varepsilon_{\ell k}\xrightarrow{\mathcal{D}}N_p(\mathbf{0},\mathbf{G}),\ \text{as } n\to\infty,$$

where N_p stands for the p-variate normal distribution. Let us consider the random vector $B^*_{F_0,\widetilde{f}}$ defined above. It is obvious that the Riemann-Stieltjes integrals involved therein are well defined by the fact B_{F_0} has the sample paths in $\mathcal{C}(E)$, whereas $f_1,\ldots,f_p$ are assumed to be in $BV_H(E)$ [see also Lemma 2 in Móricz [16] or Theorem 2 in Yeh [23]]. Furthermore, by the integration by parts for the Riemann-Stieltjes integral on E (cf. Young [24]), it can be shown that $B^*_{F_0,\widetilde{f}}$ coincides with $\int_E^{(R)}\widetilde{f}(u,v)dB_{F_0}(u,v)$ which is clearly p-variate normally distributed with mean $\mathbf{0}$ and covariance matrix given by the Lebesgue-Stieltjes integral $\int_E\widetilde{f}(u,v)\widetilde{f}^\top F_0(du,dv)=\mathbf{G}$. Thus, it can be concluded that

$$\frac{1}{n\sigma}\mathbf{X}_{n^2}^\top vec(\mathbf{E}_{n\times n})\xrightarrow{\mathcal{D}}B^*_{F_0,\widetilde{f}}=\int_E^{(R)}\widetilde{f}(u,v)dB_{F_0}(u,v),\ \text{as } n\to\infty,$$

by the fact a p-variate normal distribution is uniquely determined by its mean vector and its covariance matrix. Finally by applying (4) and Theorem 2.1 the proof of the theorem is complete. □

Lemma 2.3. *The covariance function of $B_{\widetilde{f},F_0}$ is given by*

$$\begin{aligned}K_{B_{\widetilde{f},F_0}}((x,y),(x',y')):=&Cov\left(B_{\widetilde{f},F_0}(x,y),B_{\widetilde{f},F_0}(x',y')\right)\\=&F_0(x\wedge x',y\wedge y')-\mathbf{a}_{\widetilde{f}}^\top(x,y)\mathbf{G}^{-1}\mathbf{a}_{\widetilde{f}}(x',y'),\end{aligned}$$

for $(x,y),(x',y')\in E$, where $\mathbf{a}_{\widetilde{f}}:E\mapsto\mathbb{R}^p,\ (t,s)\mapsto\int_{[a,t]\times[c,s]}\widetilde{f}(u,v)P_0(du,dv)$.

PROOF.

$$\begin{aligned}
&Cov\left(B_{\widetilde{f},F_0}(x,y), B_{\widetilde{f},F_0}(x',y')\right) = Cov\left(B_{F_0}(x,y), B_{F_0}(x',y')\right)\\
&\quad - Cov\left(B_{F_0}(x,y), \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}\right)\\
&\quad - Cov\left(\int_{[a,x]\times[c,y]} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}, B_{F_0}(x',y')\right)\\
&\quad + Cov\left(\int_{[a,x]\times[c,y]} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}, \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}\right).
\end{aligned}$$

Let $\Gamma := \{\mathbf{I}_{\ell k} := [x_{\ell-1}, x_\ell] \times [y_{k-1}, y_k] : \ 1 \leq \ell, k \leq n\}$ be a non-overlapping, finite exact cover of E. This is a simple generalization of the notion of finite exact cover discussed in Stroock [20], p.5. Then by the definition of the Riemann-Stieltjes integral on E, we have

$$\begin{aligned}
&Cov\left(B_{F_0}(x,y), \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}\right)\\
&\quad = \lim_{n\to\infty} \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} Cov\left(B_{F_0}(x,y), \sum_{\ell=1}^{n}\sum_{k=1}^{n} \widetilde{f}(x_\ell, y_k) \Delta_{\mathbf{I}_{\ell k}} B_{F_0}\right)\\
&\quad = \lim_{n\to\infty} \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} \sum_{\ell=1}^{n}\sum_{k=1}^{n} \widetilde{f}(x_\ell, y_k) \Delta_{\mathbf{I}_{\ell k}\cap[a,x]\times[c,y]} F_0\\
&\quad = \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} \int_{[a,x]\times[c,y]} \widetilde{f}\, dP_0.
\end{aligned}$$

Analogously, it holds

$$\begin{aligned}
&Cov\left(\int_{[a,x]\times[c,y]} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}, B_{F_0}(x',y')\right)\\
&\quad = \int_{[a,x]\times[c,y]} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} \int_{[a,x']\times[c,y']} \widetilde{f}\, dP_0.
\end{aligned}$$

Now by applying the multivariate technique (Theorem 1.3 in [18]), we get

$$\begin{aligned}
&Cov\left(\int_{[a,x]\times[c,y]} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}, \int_{[a,x']\times[c,y']} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} B^*_{F_0,\widetilde{f}}\right)\\
&\quad = \int_{[a,x]\times[c,y]} \widetilde{f}^{\top}\, dP_0 \mathbf{G}^{-1} Cov\left(B^*_{F_0,\widetilde{f}}, B^*_{F_0,\widetilde{f}}\right) \mathbf{G}^{-1} \int_{[a,x']\times[c,y']} \widetilde{f}\, dP_0.
\end{aligned}$$

Since for such a non-overlapping finite exact cover Γ, the definition of the Riemann-Stiletjes integral and independent increments of B_{F_0} result in

$$\begin{aligned}
&Cov\left(B^*_{F_0,\widetilde{f}}, B^*_{F_0,\widetilde{f}}\right)\\
&= \lim_{n\to\infty} Cov\left(\sum_{\ell=1}^{n}\sum_{k=1}^{n}\widetilde{f}(x_\ell,y_k)\Delta_{\mathbf{I}_{\ell k}}B_{F_0}, \sum_{\ell=1}^{n}\sum_{k=1}^{n}\widetilde{f}(x_\ell,y_k)\Delta_{\mathbf{I}_{\ell k}}B_{F_0}\right)\\
&= \lim_{n\to\infty}\sum_{\ell=1}^{n}\sum_{k=1}^{n}\widetilde{f}(x_\ell,y_k)\widetilde{f}^\top(x_\ell,y_k)\Delta_{\mathbf{I}_{\ell k}}F_0 = \mathbf{G},
\end{aligned}$$

the proof of the lemma is complete. $\square$

Remark 2.4. *Without altering the convergence result we can replace σ in Theorem 2.2 by any consistent estimator of σ. One of such an estimator is provided by $\widehat{\sigma}_n := \sqrt{\frac{1}{n^2-p}\sum_{\ell=1}^n\sum_{k=1}^n r_{\ell k}^2}$ [see Theorem 10.5 of Arnold* [1]*].*

Let have a look at the following hypotheses:

$$H_0: \mathbf{Y}_{n\times n} = \sum_{i=1}^{p}\beta_i f_i(\Xi_n) + \mathbf{E}_{n\times n} \text{ versus } H_1: \mathbf{Y}_{n\times n} = g(\Xi_n) + \mathbf{E}_{n\times n}, \quad (6)$$

for an unknown-true regression function $g: E \mapsto \mathbb{R}$. It is usual in practice to test (6) by a type of Kolmogorov statistic, defined by

$$KS_n := \sup_{(x,y)\in E}\frac{1}{n\sigma}S_n(\widehat{\mathbf{R}}_{n\times n})(x,y),$$

where H_0 will be rejected at level $\alpha \in (0,1)$ if and only if $KS_n \geq t_{n;1-\alpha}$. Thereby $t_{n;1-\alpha}$ is a constant that satisfies the equation

$$\mathbb{P}\left\{\sup_{(x,y)\in E}\frac{1}{n\sigma}S_n(\widehat{\mathbf{R}}_{n\times n})(x,y) \geq t_{n;1-\alpha} \mid H_0\right\} = \alpha.$$

If H_0 is true, then under Assumption (3) it holds

$$KS_n \xrightarrow{\mathcal{D}} \sup_{(x,y)\in E} B_{\widetilde{f},F_0}(x,y), \text{ for } n\to\infty.$$

Based on this result, $t_{n;1-\alpha}$ can be approximated by a number $t_{1-\alpha}$, where $t_{1-\alpha}$ is a constant that satisfies the equation $\mathbb{P}\left\{\sup_{(x,y)\in E} B_{\widetilde{f},F_0}(x,y) \geq t_{1-\alpha}\right\} = \alpha$.

To get the power of the test, we consider under H_1 a localized non parametric model $\mathbf{Y}^{loc}_{n\times n} := \frac{1}{n}g(\Xi_n) + \mathbf{E}_{n\times n}$, $n \geq 1$.

Corollary 2.5. *Suppose $f_1,\ldots,f_p$ and Ξ_n satisfy the situation in Theorem 2.2, and g is continuous on E. Then under the localized alternative we have*

$$KS_n \xrightarrow{\mathcal{D}} \sup_{(x,y)\in E}(B_{\widetilde{f},F_0}(x,y) + h_g(x,y)), \textit{ for } n\to\infty,$$

where for $(x, y) \in E$,

$$h_g(x, y) := \frac{1}{\sigma} \int_{[a,x]\times[c,y]} g \; dP_0 - \frac{1}{\sigma} \int_{[a,x]\times[c,y]} \widetilde{f}^\top \; dP_0 \mathbf{G}^{-1} \int_E \widetilde{f} g \; dP_0.$$

PROOF. By the linearity of S_n, under the localized alternative we have

$$\frac{1}{n\sigma} S_n(\widehat{\mathbf{R}}_{n\times n})(x, y) = \frac{1}{n\sigma} S_n(pr_{\mathbf{W}_n^\perp} \mathbf{E}_{n\times n})(x, y) + \frac{1}{n\sigma} S_n(pr_{\mathbf{W}_n^\perp} \frac{1}{n} g(\Xi_n))(x, y)$$

The second term in the right-hand side of the last equation is equivalent to

$$\frac{1}{\sigma} \int_{[a,x]\times[c,y]} g \; dP_n - \frac{1}{\sigma} \int_{[a,x]\times[c,y]} \widetilde{f}^\top \; dP_n \left(\frac{1}{n^2} \mathbf{X}_{n^2}^\top \mathbf{X}_{n^2} \right)^{-1} \int_E \widetilde{f} g \; dP_n,$$

which converges to $h_g(x, y)$, as $n \to \infty$. The proof is complete by Theorem 2.2 and the continuous mapping theorem in [2], p.20-22. $\square$

2.1. **Examples.** In the following we present examples of the limit process under H_0 for various polynomial models. For computational reason we consider the experimental region $\mathbf{I}$ and a distribution function F_0 on $\mathbf{I}$ defined by $F_0(x, y) = x^2 y^2$, $(x, y) \in \mathbf{I}$, having an $L_2(\lambda, E)$ density $f_0(x, y) = 4xy$, on $\mathbf{I}$, where λ in Lebesgue measure on $(\mathbf{I}, \mathcal{B}(\mathbf{I}))$.

2.1.1. *Constant Model.* Suppose under H_0 we assume a constant model

$$\mathbf{Y}_{n\times n} = \beta_1 f_1(\Xi_n) + \mathbf{E}_{n\times n},$$

where $f_1 : \mathbf{I} \mapsto \mathbb{R}$, $f_1(x, y) = 1$, for $(x, y) \in \mathbf{I}$, and β is an unknown constant. Then we get $\int_{[0,x]\times[0,y]} f_1 \; dF_0 = x^2 y^2$, $\mathbf{G}^{-1} = (1)$, and $B^*_{F_0,f_1} = B_{F_0}(1, 1)$. The last follows from the property that $B_{F_0}(x, y) = 0$ almost surely, if $x = 0$ or $y = 0$. The limit process under H_0 is then given by

$$B_{f_1,F_0}(x, y) = B_{F_0}(x, y) - x^2 y^2 B_{F_0}(1, 1), \; (x, y) \in \mathbf{I},$$

with the covariance function

$$K_{B_{f_1,F_0}}((x, y), (x', y')) = (x \wedge x')^2 (y \wedge y')^2 - x^2 x'^2 y^2 y'^2,$$

for $(x, y), (x', y') \in \mathbf{I}$.

2.1.2. *First-Order Model.* For the next example let us consider under H_0 a first-order polynomial model

$$Y(x, y) = \widetilde{f}^\top(x, y)\beta + \varepsilon(x, y), \; (x, y) \in \mathbf{I},$$

where $\beta = (\beta_1, \beta_2, \beta_3)^\top \in \mathbb{R}^3$ is a vector of unknown constants, and $\widetilde{f}_3 := (f_1, f_2, f_3)^\top :$ $\mathbf{I} \mapsto \mathbb{R}^3$ is a vector of known regression functions, given by $\widetilde{f}_3(x, y) = (1, x, y)^\top$, for

$(x,y) \in \mathbf{I}$. The calculation of the integrals result in

$$\int_{[0,x]\times[0,y]} \widetilde{f}_3 \, dP_0 = \left(x^2y^2, \frac{2}{3}x^3y^2, \frac{2}{3}x^2y^3 \right)^\top$$
$$\mathbf{G} = \begin{pmatrix} 1 & 2/3 & 2/3 \\ 2/3 & 1/2 & 4/9 \\ 2/3 & 4/9 & 1/2 \end{pmatrix}, \; \mathbf{G}^{-1} = \begin{pmatrix} 17 & -12 & -12 \\ -12 & 18 & 0 \\ -12 & 0 & 18 \end{pmatrix}$$
$$B^*_{F_0,\widetilde{f}_3} = \left(B_{F_0}(1,1), B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(t,1)dt, B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(1,s)ds \right)^\top.$$

Hence, we obtain the limit process under H_0:

$$\begin{aligned} B_{\widetilde{f}_3,F_0}(x,y) = B_{F_0}(x,y) &- (17x^2y^2 - 8x^3y^2 - 8x^2y^3)B_{F_0}(1,1) \\ &- 12(x^3y^2 - x^2y^2)\left(B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(t,1)dt \right) \\ &- 12(x^2y^3 - x^2y^2)\left(B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(1,s)ds \right) \\ = B_{F_0}(x,y) &+ (7x^2y^2 - 4x^3y^2 - 4x^2y^3)B_{F_0}(1,1) \\ &+12(x^3y^2 - x^2y^2)\int_{[0,1]} B_{F_0}(t,1)dt + 12(x^2y^3 - x^2y^2)\int_{[0,1]} B_{F_0}(1,s)ds. \end{aligned}$$

The covariance function of this process is given by

$$\begin{aligned} K_{B_{\widetilde{f}_3,F_0}}((x,y),(x',y')) &= (x \wedge x')^2(y \wedge y')^2 - (17x^2y^2 - 8x^3y^2 - 8x^2y^3)x'^2y'^2 \\ &- 8(x^3y^2 - x^2y^2)x'^3y'^2 - 8(x^2y^3 - x^2y^2)x'^2y'^3, \text{ for } (x,y),(x',y') \in \mathbf{I}. \end{aligned}$$

2.1.3. *Second-Order Model.* For the last example we consider a second-order polynomial model

$$Y(t,s) = \widetilde{f}_6^\top(t,s)\beta + \varepsilon(t,s), \; (t,s) \in \mathbf{I},$$

where $\widetilde{f}_6 := (f_1, f_2, f_3, f_4, f_5, f_6)^\top : \mathbf{I} \mapsto \mathbb{R}^6$ is the vector of known regression functions, defined by $\widetilde{f}_6(t,s) = (1, t, s, t^2, ts, s^2)^\top$, for $(t,s) \in \mathbf{I}$, and $\beta := (\beta_1, \ldots, \beta_6)^\top \in$

$\mathbb{R}^p$ is a vector of unknown constants. Associated to this model we consequently get

$$\int_{[0,x]\times[0,y]} \widetilde{f}_6 \, dP_0 = \left(x^2y^2, \frac{2}{3}x^3y^2, \frac{2}{3}x^2y^3, \frac{1}{2}x^4y^2, \frac{4}{9}x^3y^3, \frac{1}{2}x^2y^4\right)^\top,$$

$$\mathbf{G} = \begin{pmatrix} 1 & 2/3 & 2/3 & 1/2 & 4/9 & 1/2 \\ 2/3 & 1/2 & 4/9 & 2/5 & 1/3 & 1/3 \\ 2/3 & 4/9 & 1/2 & 1/3 & 1/3 & 2/5 \\ 1/2 & 2/5 & 1/3 & 1/3 & 4/15 & 1/4 \\ 4/9 & 1/3 & 1/3 & 4/15 & 1/4 & 4/15 \\ 1/2 & 1/3 & 2/5 & 1/4 & 4/15 & 1/3 \end{pmatrix},$$

$$\mathbf{G}^{-1} = \begin{pmatrix} 135 & -216 & -216 & 90 & 144 & 90 \\ -216 & 594 & 144 & -360 & -216 & 0 \\ -216 & 144 & 594 & 0 & -216 & -360 \\ 90 & -360 & 0 & 300 & 0 & 0 \\ 144 & -216 & -216 & 0 & 324 & 0 \\ 90 & 0 & -360 & 0 & 0 & 300 \end{pmatrix}.$$

By the definition of $B^*_{F_0,\widetilde{f}}$, we also have

$$\begin{aligned} B^*_{F_0,\widetilde{f}_6} = \Bigg(& B_{F_0}(1,1), B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(t,1)dt, B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(1,s)ds, \\ & B_{F_0}(1,1) - 2\int_{[0,1]} B_{F_0}(t,1)tdt, \\ & B_{F_0}(1,1) - \int_{[0,1]} B_{F_0}(t,1)dt - \int_{[0,1]} B_{F_0}(1,s)ds + \int_{\mathbf{I}} B_{F_0}(t,s)dtds, \\ & B_{F_0}(1,1) - 2\int_{[0,1]} B_{F_0}(1,s)sds \Bigg)^\top. \end{aligned}$$

After some simplification in the computation, the limit process corresponding to this model is given by

$$\begin{aligned} B_{\widetilde{f}_6,F_0}(x,y) = {} & B_{F_0}(x,y) \\ & - (27x^2y^2 - 36x^3y^2 - 36x^2y^3 + 15x^4y^2 + 16x^3y^3 - 120x^2y^4)B_{F_0}(1,1) \\ & + (-72x^2y^2 + 252x^3y^2 - 48x^2y^3 - 180x^4y^2 + 48x^3y^3)\int_{[0,1]} B_{F_0}(t,1)dt \\ & + (-72x^2y^2 - 48x^3y^2 + 252x^2y^3 + 48x^3y^3 - 180x^2y^4)\int_{[0,1]} B_{F_0}(1,s)ds \end{aligned}$$

$$+\,(180x^2y^2 - 480x^3y^2 + 300x^4y^2)\int_{[0,1]} B_{F_0}(t,1)tdt$$
$$+\,(180x^2y^2 - 4800x^2y^3 + 300x^2y^4)\int_{[0,1]} B_{F_0}(1,s)sds$$
$$-\,(144x^2y^2 - 144x^3y^2 - 144x^2y^3 + 144x^3y^3)\int_{\mathbf{I}} B_{F_0}(t,s)dtds,$$

with the covariance function

$$\begin{aligned}
K_{B_{\tilde{f}_6,F_0}}((x,y),(x',y')) &= (x\wedge x')^2(y\wedge y')^2 \\
&- (135x^2y^2 - 144x^3y^2 - 144x^2y^3 + 45x^4y^2 + 64x^3y^3 + 45x^2y^4)x'^2y'^2 \\
&- \frac{2}{3}(-216x^2y^2 + 396x^3y^2 + 96x^2y^3 - 180x^4y^2 - 96x^3y^3)x'^3y'^2 \\
&- \frac{2}{3}(-216x^2y^2 + 96x^3y^2 + 396x^2y^3 - 96x^3y^3 - 180x^2y^4)x'^2y'^3 \\
&- \frac{1}{2}(90x^2y^2 - 240x^3y^2 + 150x^4y^2)x'^4y'^2 \\
&- \frac{4}{9}(144x^2y^2 - 144x^3y^2 - 144x^2y^3 + 144x^3y^3)x'^3y'^3 \\
&- \frac{1}{2}(90x^2y^2 - 240x^2y^3 + 150x^2y^4)x'^2y'^4,\ \text{for } (x,y),(x',y') \in \mathbf{I}.
\end{aligned}$$

2.2. **Upper and Lower Bounds for the Localized Power.** Let us consider the boundary crossing probability

$$\mathbb{P}\left\{\exists (t,s)\in E:\ \rho\varphi(t,s) + B_{\tilde{f},F_0}(t,s) \geq u(t,s)\right\}, \tag{7}$$

having a known trend $\varphi : E \to \mathbb{R}$, and a general known *boundary* $u : E \to \mathbb{R}$, for any real numbers $\rho > 0$. Note that in case $\varphi = h_g$ for a function g, such that $g(\Xi_n) \notin \mathbf{W}_n$, $n \geq 1$, and $u(t,s) = t_{1-\alpha}$, for $(t,s) \in E$, we get the power of the size α test derived in Corollary 2.5 evaluated at g. We aim to derive the lower and upper bounds for such a probability when the trend is restricted in the certain subset of the reproducing kernel Hilbert space of $B_{\tilde{f},F_0}$, the one defined by

$$\mathcal{H}_{B_{\tilde{f},F_0}} := \left\{h : E \to \mathbb{R} : \exists f \in L_2(P_0,E),\ h(t,s) = \langle f, m_{(t,s)}\rangle_{L_2(P_0,E)}\right\},$$

where the family $\{m_{(t,s)} : E \to \mathbb{R}, (t,s) \in E\}$ constitutes a model for $B_{\tilde{f},F_0}$ (cf. Lifshits [11], p.93), and $\langle\cdot,\cdot\rangle_{L_2(P_0,E)}$ is the inner product on $L_2(P_0,E)$. The function f having the property $h(t,s) = \langle f, m_{(t,s)}\rangle_{L_2(P_0,E)}$ is called the reproducing function of $h \in \mathcal{H}_{B_{\tilde{f},F_0}}$. The space $\mathcal{H}_{B_{\tilde{f},F_0}}$ is furnished with the inner product and the corresponding norm defined by

$$\langle h_1,h_2\rangle_{\mathcal{H}_{B_{\tilde{f},F_0}}} := \langle f_1,f_2\rangle_{L_2(P_0,E)}, \|h_1\|_{\mathcal{H}_{B_{\tilde{f},F_0}}} := \|f_1\|_{L_2(P_0,E)},$$

for any $h_1, h_2 \in \mathcal{H}_{B_{\tilde{f},F_0}}$ having the reproducing functions $f_1, f_2 \in L_2(P_0,E)$, respectively.

Proposition 2.6. *(Cameron-Martin-Girsanov formula) Let $\mathcal{P}_{B_{\tilde{f},F_0}}$ be the distribution of $B_{\tilde{f},F_0}$ on the space $(\mathcal{C}(E),\mathcal{B}_{\mathcal{C}})$. For any $h \in \mathcal{H}_{B_{\tilde{f},F_0}}$, let $\mathcal{P}^h_{B_{\tilde{f},F_0}}$ be a probability measure defined on $(\mathcal{C}(E),\mathcal{B}_{\mathcal{C}})$, given by $\mathcal{P}^h_{B_{\tilde{f},F_0}}(A) := \mathcal{P}_{B_{\tilde{f},F_0}}(A-h)$, for $A \in \mathcal{B}_{\mathcal{C}}$, where $A - h := \{x-h : x \in A\}$. If $f \in BV_H(E)$ is the reproducing function of h, then the density of $\mathcal{P}^h_{B_{\tilde{f},F_0}}$ with respect to $\mathcal{P}_{B_{\tilde{f},F_0}}$ is given by*

$$\frac{d\mathcal{P}^h_{B_{\tilde{f},F_0}}}{d\mathcal{P}_{B_{\tilde{f},F_0}}}(x) = \exp\left\{\int_E^{(R)} f(t,s)\ dx(t,s) - \frac{1}{2}\left\|h\right\|^2_{\mathcal{H}_{B_{\tilde{f},F_0}}}\right\}.$$

PROOF. See Theorem 3 in Lifshits [11], p.88. □

Now let us consider the case where $B_{\tilde{f},F_0} = B_{F_0}$. By the definition of the covariance function of B_{F_0} and by the result in Lifshits [11], p. 93, the reproducing kernel Hilbert space of B_{F_0} is given by

$$\mathcal{H}_{B_{F_0}} = \left\{h : E \to \mathbb{R} : \exists w \in L_2(P_0,E), h(t,s) = \int_{[a,t]\times[c,s]} w dP_0, (t,s) \in E\right\}.$$

It is clear that for every $h \in \mathcal{H}_{B_{F_0}}$, there exists uniquely an absolutely continuous signed measure ν_h, say, defined on $(E,\mathcal{B}(E))$ with an $L_2(P_0,E)$ density with respect to P_0. Hence by Proposition 2.6, for any $h \in \mathcal{H}_{B_{F_0}}$ it holds

$$\frac{d\mathcal{P}^h_{B_{F_0}}}{d\mathcal{P}_{B_{F_0}}}(x) = \exp\left\{\int_E^{(R)} \frac{d\nu_h}{dP_0}(t,s)\ dx(t,s) - \frac{1}{2}\left\|h\right\|^2_{\mathcal{H}_{B_{F_0}}}\right\}, \tag{8}$$

provided $\frac{d\nu_h}{dP_0} \in BV_H(E)$.

Theorem 2.7. *Let u be continuous on E and φ be in $\mathcal{H}_{B_{F_0}}$. If $w := \frac{d\nu_\varphi}{dP_0}$ is non decreasing on E and the corresponding marginal functions $w(b,\cdot) : [c,d] \to \mathbb{R}, s \mapsto w(b,s)$, $s \in [c,d]$, and $w(\cdot,d) : [a,b] \to \mathbb{R}$, $t \mapsto w(t,d)$, $t \in [a,b]$ are non increasing on $[c,d]$ and $[a,b]$,respectively, then*

$$\begin{aligned}&\mathbb{P}\left\{\forall(t,s) \in E :\ \rho\varphi(t,s) + B_{F_0}(t,s) < u(t,s)\right\}\\ &\quad \le k^*\mathbb{P}\left\{\forall(t,s) \in E :\ B_{F_0}(t,s) < u(t,s)\right\},\end{aligned}$$

where

$$\begin{aligned}k^* := \exp\Bigg\{&\rho w(b,d)u(b,d) + \rho\int_{[a,b]}^{(R)} u(t,d)d(-w(t,d)) + \rho\int_{[c,d]}^{(R)} u(b,s)d(-w(b,s))\\ &+\rho\int_E^{(R)} u(t,s)dw(t,s) - \frac{1}{2}\rho^2\left\|\varphi\right\|^2_{\mathcal{H}_{\mathbf{B}_{F_0}}}\Bigg\},\ \rho > 0.\end{aligned}$$

Proof. By transformation of variable and Equation (8), we obtain

$$
\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : \forall\ (t,s) \in E, \rho\varphi(t,s) + B_{F_0}(\omega)(t,s) < u(t,s)\right\} \\
&= \int_{\Omega} \mathbf{1}_{\{\omega\in\Omega:\forall(t,s)\in E,\rho\varphi(t,s)+B_{F_0}(\omega)(t,s)<u(t,s)\}}\mathbb{P}(d\omega) \\
&= \int_{\mathcal{C}(E)} \mathbf{1}_{\{y\in\mathcal{C}(E):\ \forall\ (t,s)\in E, y(t,s)<u(t,s)\}}\mathcal{P}^{\rho\varphi}_{B_{F_0}}(dy) \\
&= \int_{\mathcal{C}(E)} \mathbf{1}_{\{y\in\mathcal{C}(E):\forall(t,s)\in E, y(t,s)<u(t,s)\}} \\
&\qquad \times \exp\left\{\int_E^{(R)} \rho w(t,s)\ dy(t,s) - \frac{1}{2}\rho^2 \|\varphi\|^2_{\mathcal{H}_{B_{F_0}}}\right\}\mathcal{P}_{B_{F_0}}(dy) \\
&= \int_{\Omega} \mathbf{1}_{\{\omega\in\Omega:\forall\ (t,s)\in E,\ B_{F_0}(\omega)(t,s)<u(t,s)\}} \\
&\qquad \times \exp\left\{\int_E^{(R)} \rho w(t,s)\ dB_{F_0}(\omega)(t,s) - \frac{1}{2}\rho^2 \|\varphi\|^2_{\mathcal{H}_{B_{F_0}}}\right\}\mathbb{P}(d\omega).
\end{aligned}
$$

Since $B_{F_0}(t,c) = 0$ a.s. for $t \in [a,b]$ and $B_{F_0}(a,s) = 0$ a.s. for $s \in [c,d]$, then $\Delta_E w B_{F_0} = w(b,d)B_{F_0}(b,d)$ a.s. The result follows immediately from integration by parts and the assumption that w is non decreasing on E, with $-w(\cdot,d)$ and $-w(b,\cdot)$ are non decreasing on $[a,b]$ and $[c,d]$, respectively. □

Corollary 2.8. *Under the conditions of Theorem 2.7 it holds*

$$
\begin{aligned}
&\mathbb{P}\{\exists(t,s) \in E : \rho\varphi(t,s) + B_{F_0}(t,s) \geq u(t,s)\} \\
&\geq 1 - k^*\mathbb{P}\{\forall(t,s) \in E :\ B_{F_0}(t,s) < u(t,s)\} \\
&= 1 - k^* + k^*\mathbb{P}\{\exists(t,s) \in E :\ B_{F_0}(t,s) \geq u(t,s)\}.
\end{aligned}
$$

In particular, for the case $u(t,s) = t_{1-\alpha}$, $(t,s) \in E$, *where* $\mathbb{P}\{\sup_{(t,s)\in E} B_{F_0}(t,s) \geq t_{1-\alpha}\} = \alpha$, *we get*

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{(t,s)\in E}(\rho\varphi(t,s) + B_{F_0}(t,s)) \geq t_{1-\alpha}\right\} \\
&\geq 1 - k_1^*\mathbb{P}\{\sup_{(t,s)\in E} B_{F_0}(t,s) < t_{1-\alpha}\} = 1 - k_1^*(1-\alpha),\ \ \rho > 0,
\end{aligned}
$$

where

$$
\begin{aligned}
k_1^* :=& \exp\{\rho t_{1-\alpha} w(b,d) - \rho t_{1-\alpha}\Delta_{[a,b]} w(\cdot,d) - \rho t_{1-\alpha}\Delta_{[a,b]} w(b,\cdot) \\
&+ \rho t_{1-\alpha}\Delta_E w(\cdot) - \frac{1}{2}\rho^2 \|\varphi\|^2_{\mathcal{H}_{B_{F_0}}}\}.
\end{aligned}
$$

Corollary 2.9. *Suppose φ and u satisfy the conditions of Theorem 2.7, then*

$$\begin{aligned}&\mathbb{P}\left\{\exists(t,s)\in E:\rho\varphi(t,s)+B_{F_0}(t,s)\geq u(t,s)\right\}\\&\leq 1-\exp\left\{\mathbb{E}_{\mathbb{P}}\left(k^*_{B_{F_0}}\mathbf{1}_{\{\forall(t,s)\in E:B_{F_0}(t,s)<u(t,s)\}}\right)-\frac{1}{2}\rho^2\left\|\varphi\right\|^2_{\mathcal{H}_{B_{F_0}}}\right\},\end{aligned}$$

where

$$\begin{aligned}k^*_{B_{F_0}}:=\ &\rho w(b,d)B_{F_0}(b,d)+\rho\int^{(R)}_{[a,b]}B_{F_0}(t,d)d(-w(t,d))\\&+\rho\int^{(R)}_{[c,d]}B_{F_0}(b,s)d(-w(b,s))+\rho\int^{R}_{E}B_{F_0}(t,s)dw(t,s).\end{aligned}$$

PROOF. By Theorem 2.7, integration by parts and Jensen's inequality (cf. Chow [7]), we get

$$\begin{aligned}&\mathbb{P}\left\{\forall(t,s)\in E:\rho\varphi(t,s)+B_{F_0}(t,s)<u(t,s)\right\}\\&=\exp\left\{-\frac{1}{2}\rho^2\left\|\varphi\right\|^2_{\mathcal{H}_{B_{F_0}}}\right\}\int_{\Omega}\mathbf{1}_{\{\forall(t,s)\in E:B_{F_0}(t,s)<u(t,s)\}}\exp\{k^*_{B_{F_0}}\}d\mathbb{P}\\&\geq\exp\left\{-\frac{1}{2}\rho^2\left\|\varphi\right\|^2_{\mathcal{H}_{B_{F_0}}}\right\}\exp\left\{\mathbb{E}\left(k^*_{B_{F_0}}\mathbf{1}_{\{\forall(t,s)\in E:B_{F_0}(t,s)<u(t,s)\}}\right)\right\}\\&=\exp\left\{\mathbb{E}\left(k^*_{B_{F_0}}\mathbf{1}_{\{\forall(t,s)\in E:B_{F_0}(t,s)<u(t,s)\}}\right)-\frac{1}{2}\rho^2\left\|\varphi\right\|^2_{\mathcal{H}_{B_{F_0}}}\right\}.\end{aligned}$$

The proof is complete by the probability formula for the complement of an event. □

For the second example we observe the case $B_{\widetilde{f},F_0}=B_{f_1,F_0}$, where B_{f_1,F_0} is the limit process associated with the constant model presented in Example 2.1.1, with $F_0(t,s)=t^2s^2$, $(t,s)\in\mathbf{I}$. Being a process with the covariance function $K_{B_{f_1,F_0}}((t,s),(t',s'))=(t\wedge t')^2(s\wedge s')^2-t^2t'^2s^2s'^2$, for $(t,s),(t',s')\in\mathbf{I}$, which can be represented as

$$K_{B_{f_1,F_0}}((t,s),(t',s'))=\langle\mathbf{1}_{[0,t]\times[0,s]}-t^2s^2\mathbf{1}_{\mathbf{I}},\mathbf{1}_{[0,t']\times[0,s']}-t'^2s'^2\mathbf{1}_{\mathbf{I}}\rangle_{L_2(P_0,E)},$$

B_{f_1,F_0} has the reproducing kernel Hilbert space given by

$$\mathcal{H}_{B_{f_1,F_0}}:=\left\{h:\mathbf{I}\to\mathbb{R}:\exists u\in L_2(P_0,E),h(t,s)=\int_{[0,t]\times[0,s]}udP_0-t^2s^2\int_{\mathbf{I}}udP_0\right\},$$

(cf. Lifshits [11], p.93). Thus for every $h\in\mathcal{H}_{B_{f_1,F_0}}$, $h(1,1)=0$ and it determines uniquely an absolutely continuous signed measure μ_h, say, defined on the measurable space $(\mathbf{I},\mathcal{B}(\mathbf{I}))$, having an $L_2(P_0,E)$ density with respect to P_0. Hence, as a direct consequence of Proposition 2.6, we have for every $h\in\mathcal{H}_{B_{f_1,F_0}}$,

$$\frac{d\mathcal{P}^h_{B_{f_1,F_0}}}{d\mathcal{P}_{B_{f_1,F_0}}}(x)=\exp\left\{\int^{(R)}_{\mathbf{I}}\frac{d\mu_h}{dP_0}(t,s)\ dx(t,s)-\frac{1}{2}\left\|h\right\|^2_{\mathcal{H}_{B_{f_1,F_0}}}\right\},$$

provided $\frac{d\mu_h}{dP_0}\in BV_H(\mathbf{I})$.

Theorem 2.10. *Suppose the boundary u is continuous on $\mathbf{I}$, and the trend $\varphi \in \mathcal{H}_{B_{f_1,F_0}}$, such that $g := \frac{d\mu_\varphi}{dP_0}$ is non decreasing on $\mathbf{I}$. If the marginal functions $g(\cdot,1) : [0,1] \to \mathbb{R}, t \mapsto g(t,1)$ and $g(1,\cdot) : [0,1] \to \mathbb{R}$, $s \mapsto g(1,s)$ are non increasing, then*

$$\begin{aligned}&\mathbb{P}\left\{\forall(t,s) \in \mathbf{I} :\ \rho\varphi(t,s) + B_{f_1,F_0}(t,s) < u(t,s)\right\}\\ &\quad \le m^*\mathbb{P}\left\{\forall(t,s) \in \mathbf{I} :\ B_{f_1,F_0}(t,s) < u(t,s)\right\},\ \rho > 0,\end{aligned}$$

where

$$\begin{aligned}m^* := \exp\Bigg\{&\rho g(1,1)u(1,1) + \rho\int_{[0,1]}^{(R)} u(t,1)d(-g(t,1)) + \rho\int_{[0,1]}^{(R)} u(1,s)d(-g(1,s))\\ &+\rho\int_{\mathbf{I}}^{(R)} u(t,s)dg(t,s) - \frac{1}{2}\rho^2\left\|\varphi\right\|^2_{\mathcal{H}_{B_{f_1,F_0}}}\Bigg\}.\end{aligned}$$

Corollary 2.11. *If u and φ satisfy the conditions of Theorem 2.10, we get*

$$\begin{aligned}&\mathbb{P}\{\exists(t,s) \in \mathbf{I} :\ \rho\varphi(t,s) + B_{f_1,F_0}(t,s) \ge u(t,s)\}\\ &\ge 1 - m^*\mathbb{P}\{\forall(t,s) \in \mathbf{I} :\ B_{f_1,F_0}(t,s) < u(t,s)\}\\ &= 1 - m^* + m^*\mathbb{P}\{\exists(t,s) \in \mathbf{I} :\ B_{f_1,F_0}(t,s) \ge u(t,s)\}.\end{aligned}$$

Corollary 2.12. *For $\rho > 0$, let*

$$\begin{aligned}m^*_{B_{f_1,F_0}} :=& \int_{[0,1]}^{(R)} \rho B_{f_1,F_0}(t,1)d(-g(t,1)) + \int_{[0,1]}^{(R)} \rho B_{f_1,F_0}(1,s)d(-g(1,s))\\ &+\int_{\mathbf{I}}^{(R)} \rho B_{f_1,F_0}(t,s)dg(t,s).\end{aligned}$$

Then by Proposition 2.6, integration by parts and Jensen's inequality, we get

$$\begin{aligned}&\mathbb{P}\left\{\forall(t,s) \in \mathbf{I} : \rho\varphi(t,s) + B_{f_1,F_0}(t,s) < u(t,s)\right\}\\ &\ge \exp\left\{\mathbb{E}\left(m^*_{B_{f_1,F_0}}\mathbf{1}\{\forall(t,s) \in \mathbf{I} : B_{f_1,F_0}(t,s) < u(t,s)\}\right) - \frac{1}{2}\rho^2\left\|\varphi\right\|^2_{B_{f_1,F_0}}\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbb{P}\left\{\exists(t,s) \in \mathbf{I} : \rho\varphi(t,s) + B_{f_1,F_0}(t,s) \ge u(t,s)\right\}\\ &\le 1 - \exp\left\{\mathbb{E}\left(m^*_{B_{f_1,F_0}}\mathbf{1}_{\{\forall(t,s)\in\mathbf{I}: B_{f_1,F_0}(t,s)<u(t,s)\}}\right) - \frac{1}{2}\rho^2\left\|\varphi\right\|^2_{B_{f_1,F_0}}\right\}.\end{aligned}$$

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REFERENCES

[1] Arnold, S.F., *The Theory of Linear Models and Multivarite Analysis*, John Wiley & Sons, Inc., New York, 1981.

[2] Billingsley, P. *Convergence of Probability Measures* (2nd. edition), John Wiley & Sons, Inc., New York, 1999.

[3] Bischoff, W., "A Functional Central Limit Theorem for Regression Models", *Ann. Stat.* **26** (4) (1998), 1398 - 1410.

[4] Bischoff, W. and Miller, F., "Asymptotically Optimal Tests and Optimal Designs for Testing the Mean in Regression Models with Applications to Change-Point Problems", *Ann. Inst. Statist. Math.* **52** (2000), 658 - 679.

[5] Bischoff, W., "The Structure of Residual Partial Sums Limit Processes of Linear Regression Models", *Theory of Stochastic Processes*, **2(24)** (2002), 23 - 28.

[6] Bischoff, W. and Somayasa, W. "The Limit of the Partial Sums Process of Spatial Least Squares Residuals", *J. Multivariate Analysis*, **100** (2009), 2167 - 2177.

[7] Chow, Y.S. and Teicher, H. *Probability Theory* (3rd. edition), Springer-Verlag New York, Inc., New York, 2003.

[8] Clarkson, J.A. and Adams, C.R., "On Definition of Bounded Variation for Functions of Two Variables", *Transactions of the American Mathematical Society*, **5(4)** (1933), 824 - 854.

[9] Harville, D.A., *Matrix Algebra from a Statistician's Perspective*, Springer-Verlag New York Inc., New York, 1997.

[10] Kuelbs,J., "The Invariance Principle for a Lattice of Random Variables", *The Ann. of Math. Stat.* **39(2)** (1968), 382 - 389.

[11] Lifshits, M.A., *Gaussian Random Function*, Kluwer Academic Publishers, Dordrecht, 1996.

[12] MacNeill, I.B. "Properties of Partial Sums of Polynomial Regression Residuals with Applications to Test for Change of Regression at Unknown Times", *Ann. Statist.* **6** (1978), 422 - 433.

[13] MacNeill, I.B., "Limit Processes for Sequences Partial Sums of Regression Residuals", *Ann. Probab.* **6** (1978), 695 - 698.

[14] MacNeill, I.B. and Jandhyala, V.K., *Change-Point Methods for Spatial Data, Multivariate Environmental Statistics eds. by G.P. Patil and C.R. Rao*, Elsevier Science Publishers B.V., (1993), 298 - 306.

[15] MacNeill, I.B., Mao, Y. and Xie, L., "Modeling Heteroscedastic Age-Period-Cohort Cancer Data", *The Canadian Journal of Statistics*, **22(4)** (1994), 529 - 539.

[16] Móricz, F., "Pointwise Behavior of Double Forier Series of Functions of Bounded Variation", *Monatsh. Math.* **148** (2006), 51 - 59.

[17] Park, W.J., "Weak Convergence of Probability Measures on the Function Space $\mathcal{C}([0,1]^2$", *J. of Multivariate Analysis*, **1** (1971), 433 - 444.

[18] Seber, G.A.F. and Lee, A.J. *Linear Regression Analysis (2nd edition)*, John Wiley & Sons, Inc., New Jersey, 2003.

[19] Somayasa, W., "On Set-Indexed Residual Partial Sum Limit Process of Spatial Linear Regression Models", *J. Indones. Math. Soc.* **17(2)** (2011), 73 - 83.

[20] Stroock, D.W. *A Concise Introduction to the Theory of Integration (2nd edition)*, Birkhäuser, Berlin, 1994.

[21] Van der Vaart, A.W., *Asymptotic Statistics*, Cambridge University Press, Cambridge, 1998.

[22] Xie, L. and MacNeill, I.B., "Spatial Residual Processes and Boundary Detection", *South African Statist. J.* **40(1)** (2006), 33 - 53.

[23] Yeh, J., "Wiener Measure in a Space of Functions of two variables", *Trans. Amer. Math. Soc.* **95** (1960), 433 - 450.

[24] Young, W.H., "On Multiple Integration by Parts and the Second Theorem of Mean", *The Ann. of Math. Stat.* **43(4)** (1917), 1235 - 1246.