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ENERGY OF COMPLEMENT OF STARS

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Abstract. The concept of energy of a graph was put forward by I. Gutman in 1978 [2]. The characteristic polynomial of a graph G with p vertices is defined as $\phi(G:\lambda) = det(\lambda I - A(G)$, where $A(G)$ is the adjacency matrix of G and I is the unit matrix. The roots of the characteristic equation $\phi(G:\lambda) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G. The energy $E = E(G)$ of a graph G is defined as

$$E(G) = \sum_{i=1}^{p} |\lambda_i|$$

The graphs with large number of edges are referred as graph represenation of inorganic clusters, called as cluster graphs. In this paper we obtain the characteristic polynomial and energy of class of cluster graphs which are termed as complement of stars.

Key words and Phrases: Spectra of graphs, energy of graphs, stars, cluster graphs.

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Abstrak. Konsep energi dari sebuah graf dikemukakan oleh I. Gutman in 1978 [2]. Polinom karakteristik dari sebuah graf G dengan p buah titik didefinisikan sebagai $\phi(G:\lambda) = det(\lambda I - A(G)$, dengan $A(G)$ adalah matriks ketetanggaan dari G dan I adalah matriks satuan. Akar dari persamaan karakteristik $\phi(G:\lambda) = 0$, dinotasikan dengan $\lambda_1, \lambda_2, \ldots, \lambda_p$ adalah nilai eigen dari G. Energi $E = E(G)$ dari sebuah graf G didefinisikan sebagai

$$E(G) = \sum_{i=1}^{p} |\lambda_i|.$$

Graf dengan sisi yang banyak yang dinyatakan sebagai penyajian graf dari kluster-kluster anorganik, disebut sebagai graf-graf kluster. Pada paper ini kami mendapatkan polinom karakteristik dan energi dari kelas graf-graf kluster yang diistilahkan dengan komplemen dari graf bintang.

Kata kunci: Spektrum dari graf-graf, energi of graf-graf, graf bintang, graf kluster.

1. Introduction

Let G be a simple undirected graph with p vertices and q edges. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of G. The adjacency matrix of G is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The characteristic polynomial of G is $\phi(G:\lambda) = det(\lambda I - A(G))$,where I is unit matrix of order p. The roots of the equation $\phi(G:\lambda) = 0$ denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G and their collection is the spectrum of G [2]. The energy [2] of G is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i|$. The spectrum of the complete graph K_p is $\{p-1, \text{ and } -1(p-1) \text{ times}\}$. Consequently $E(K_p) = 2(p-1)$. It was conjectured some years ago that, among all graphs with p vertices the complete graph has the greatest energy [2]. But this is not true [3]. There are graphs having energy greater than $E(K_p)$. The graph G with $E(G) > E(K_p)$ is referred as hyperenergetic graph, otherwise known as non-hypernergetic [3].

2. Some Edge Deleted Cluster Graphs

I. Gutman and L. Pavlović [4] introduced four classes of graps obtained from compete graph by deleting edges. For the sake of continuity we recall these here.

Definition 2.1. [4] *Let v be a vertex of the complete graph K_p, $p \geq 3$ and let e_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq p-1$, be its disdinct edges, all being incident to v. The graph $Ka_p(k)$ (or $Cl(p,k)$) is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from K_p. In addition $Ka_p(0) \cong K_p$.*

Definition 2.2. [4] *Let f_i, $i = 1, 2, \ldots, k$, $i \leq k \leq \lfloor p/2 \rfloor$ be independent edges of the complete graph K_p, $p \geq 3$. The graph $Kb_p(k)$ is obtained by deleting edges f_i, $i = 1, 2, \ldots, k$ from K_p. The graph $Kb_p(0) \cong K_p$.*

Definition 2.3. [4] *Let complete graph K_k be the induced subgraph of K_p, $2 \leq k \leq p$, $p \geq 3$. The graph $Kc_p(k)$ is obtained by deleting from K_p all the edges of K_k. In addition $Kc_p(0) \cong Kc_p(1) \cong K_p$.*

Definition 2.4. [4] *Let $3 \leq k \leq p$, $p \geq 3$. The graph $Kd_p(k)$ obtained from K_p, by deleting the edges belonging to a k-membered cycle.*

Theorem 2.5. [4] *For $0 \leq k \leq p-1$,*

$$\phi(Cl(p,k):\lambda) = (\lambda+1)^{p-3}[\lambda^3 - (p-3)\lambda^2 - (2p-k-3)\lambda + (k-1)(p-1-k)] \quad (1)$$

Theorem 2.6. [4] *For $p \geq 3$ and $0 \leq k \leq \lfloor p/2 \rfloor$,*

$$\phi(Kb_p(k):\lambda) = \lambda^k(\lambda+1)^{p-2k-1}(\lambda+2)^{k-1}[\lambda^2 - (p-3)\lambda - 2(p-k-1)] \quad (2)$$

Theorem 2.7. [4] *For $p \geq 3$ and $2 \leq k \leq p$,*

$$\phi(Kc_p(k):\lambda) = \lambda^{k-1}(\lambda+1)^{p-k-1}[\lambda^2 - (p-k-1)\lambda - k(p-k)] \quad (3)$$

Theorem 2.8. [4] *For $p \geq 3$ and $3 \leq k \leq p$,*

$$\phi(Kd_p(k):\lambda) = (\lambda+1)^{p-k-1}[\lambda^2-(p-4)\lambda-(3p-2k-3)] \prod_{i=1}^{k-1} \left(\lambda + 2cos\left(\frac{2\pi i}{k}\right) + 1\right) \quad (4)$$

We introduce here another class of graphs obtained from K_p and we denote it by $Cl(p,m,k)$. Two subgraphs G_1 and G_2 of graph G are independent subgraphs if $V(G_1) \cap V(G_2)$ is an empty set.

Definition 2.9. *Let $K_{1,m-1}$ be a star graph. Let K_p be a complete graph on p vertices. The graph obatined by deleting k independent stars $(K_{1,m-1})$ from K_p will be denoted by $Cl(p,m,k)$. In addition for $k=1$, $Cl(p,m,k) = Cl(p,m)$.*

Definition 2.10. *Let K_{1,m_1-1} and K_{1,m_2-1} be star graphs. The graph obatined by deleting the edges of k_1 independent stars (K_{1,m_1-1}) and edges of k_2 independent stars (K_{1,m_2-1}) from K_p denoted by $Cl(p,(m_1,k_1),(m_2,k_2))$. In addition for $k_2 = 0$, $m_1 = m$, $k_1 = k$, $Cl(p,(m_1,k_1),(m_2,k_2)) \cong Cl(p,m,k)$.*

Now we proceed to prove the main result in the form of Theorem 2.11 and Theorem 2.12.

Theorem 2.11. *Let p, m, k be positive integers. Let $mk \leq p$. Then the characteristic polynomial of $Cl(p, m, k)$ is*

$$
\begin{aligned}
\phi(Cl(p,m,k) : \lambda) \quad = \quad & (\lambda+1)^{p-2k-1}[(\lambda+1)^2-(m-1)]^{k-1} \\
& [\lambda^3-(p-3)\lambda^2+(2mk-2p-2k-m+4)\lambda \\
& +(p-mk)(m-2)].
\end{aligned}
$$

Proof. Without loss of generality let k independent stars of order m of $Cl(p, m, k)$ be situated as follows. For $j = 0, 1, \ldots, k-1$, on each of the vertices v_{jm+1}, $m-1$ vertices v_{jm+i}, $i = 2, 3, \ldots, m$ are incident. Let the adjacency matrix of $Cl(p, m, k)$ be A. The characteristic polynomial of $Cl(p, m, k)$ is $det(\lambda I - A)$. For convinience, we compute $det(A-\lambda I)$ and then multiply by $(-1)^p$ to be the final result. Therefore the characteristic polynomial of $Cl(p, m, k)$ is the following determinant.

$$
\begin{array}{c|ccccccccccccccccc|}
v_1 & -\lambda & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
 & 0 & -\lambda & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
\vdots & \vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
v_m & 0 & 1 & \dots & -\lambda & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
v_{m+1} & 1 & 1 & \dots & 1 & -\lambda & 0 & \dots & 0 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
 & 1 & 1 & \dots & 1 & 0 & -\lambda & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
\vdots & \vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
v_{2m} & 1 & 1 & \dots & 1 & 0 & 1 & \dots & -\lambda & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
\vdots & \vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & -\lambda & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\
 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 0 & -\lambda & \dots & 1 & 1 & 1 & \dots & 1 \\
\vdots & \vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 0 & 1 & \dots & -\lambda & 1 & 1 & \dots & 1 \\
v_{mk+1} & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & -\lambda & 1 & \dots & 1 \\
 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & -\lambda & \dots & 1 \\
\vdots & \vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
v_p & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & -\lambda
\end{array}
$$

Subtracting first column from all other columns and setting $\lambda + 1 = X$, in the above determinant, we obtain (5)

$$\begin{vmatrix}
-\lambda & -\lambda & \dots & \lambda & X & X & \dots & X & \dots & X & X & \dots & X & X & X & \dots & X \\
0 & -\lambda & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
0 & 1 & \dots & -\lambda & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
1 & 0 & \dots & 0 & -X & -1 & \dots & -1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
1 & 0 & \dots & 0 & -1 & -X & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
1 & 0 & \dots & 0 & -1 & 0 & \dots & -X & \dots & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & -X & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & -1 & -X & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \ddots & & & \vdots & & & & \ddots & \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & -1 & 0 & \dots & -X & 0 & 0 & \dots & 0 \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & -X & 0 & \dots & 0 \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & -X & \dots & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & \vdots & & & & \vdots & \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -X
\end{vmatrix} \tag{5}$$

Then performing the following sequence of operations on (5)
1. $R_i - R_m$ for $i = 2, 3, \dots, m-1$ and $R_1 - (\lambda+1)R_m$.
2. $C_1 + \frac{C_i}{X}$ for $i = mk+1, mk+2, \dots, p$, we get

$$Cl(p, m, k) = \begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q| \tag{6}$$

as P is zero block.
3. Simplifying $|Q|$ of (6) we obtain

$$\phi(Cl(p, m, k) : \lambda) = (-X)^{p-mk}|M|$$

Again performing next set of operations on $|M|$ sequentially,
4. Adding columns C_{jm+1}, for $j = 1, 2, \dots, k-1$ to the first column C_1.
5. Multiplying X to columns C_{jm+1}, $j = 1, 2, \dots, k-1$ and dividing by X^{k-1}.
6. Performing $C_{im+1} - \sum_{j=2}^{k-1} C_{im+j}$, $i = 1, 2, \dots, k-1$.
7. Performing $C_1 + \sum_{i=1}^{k-1} \frac{C_{im+1}\lambda}{(m-1)-X^2}$

Putting $X = \lambda + 1$, multiplying by $(-1)^p$ and simplifying, we get the result of Theorem 2.11. □

Performing similar set of opeartions on $det(\lambda I - Cl(p, (m_1, k_1), (m_2, k_2)))$ we obtain the characteristic polynomial of $Cl(p, (m_1, k_1), (m_2, k_2))$. We state this formally in the following theorem.

Theorem 2.12. *Let p, m_1, k_1, m_2, k_2 be positive integers. Let $m_1k_1 + m_2k_2 \leq p$. Then the characteristic polynomial of $Cl(p, (m_1, k_1), (m_2, k_2))$ is*

$$\phi(Cl(p,(m_1,k_1),(m_2,k_2)):\lambda) = X^{p-2(k_1+k_2)}[(m_1-1)-X^2]^{k_1-1}[(m_2-1)-X^2]^{k_2}D$$

where $D = \lambda[\lambda-(m_1-2)] - Z[\lambda^2+2\lambda-(m_1-2)]$

where

$$Z = k_1+k_2-1+\frac{p-(m_1k_1+m_2k_2)}{X}+\frac{(k_1-1)\lambda[X-(m_1-1)]}{(m_1-1)-X^2}+\frac{k_2\lambda[X-(m_2-1)]}{(m_2-1)-X^2}.$$

Here we give some examples in Table 1 and Table 2.

TABLE 1. Characteristic polynomial of $Cl(p,(m_1,k_1),(m_2,k_2))$

$(p,(m_1,k_1),(m_2,k_2))$	Characteristic equation of $Cl(p,(m_1,K_1),(m_2,k_2))$
(4, (2, 1), (2, 1))	$\lambda^4-2\lambda^2=0$
(6, (3, 1), (1, 1))	$(\lambda+1)^3(\lambda^3-3\lambda^2-5\lambda+3)=0$
(7, (3, 1), (2, 2))	$(\lambda^4+3\lambda^3+2\lambda^2)(\lambda^3-3\lambda^2-10\lambda+4)=0$
(8, (4, 1), (2, 2))	$(\lambda^5+4\lambda^4+5\lambda^3+2\lambda^2)(\lambda^3-4\lambda^2-12\lambda+8)=0$
(10, (3, 2), (2, 2))	$(\lambda^6-4\lambda^5-27\lambda^4-24\lambda^3+12\lambda^2)(\lambda^4+\lambda^3+4\lambda^2-1)=0$

TABLE 2. Energy of $Cl(p,(m_1,k_1),(m_2,k_2))$

$(p,(m_1,k_1),(m_2,k_2))$	Spectra of $G=Cl(p,(m_1,k_1),(m_2,k_2))$	Energy $E(G)$
(4, (2, 1), (2, 1))	$-2, 0, 0, 2$	4
(6, (3, 1), (1, 1))	$-1.5341, -1, -1, -1, 0.4827, 4.0514$	9.0709
(7, (3, 1), (2, 2))	$-2.2458, -2, -1, 0, 0, 0.3649, 4.8809$	10.4916
(8, (4, 1), (2, 2))	$-2.3974, -2, -1, -1, 0, 0, 0.5729, 5.8245$	12.7948
(10, (3, 2), (2, 2))	$-2.4142, -2.1731, -2, -1, -1, 0, 0, 0.3531, 0.4142, 7.8200$	17.1728

Remarks:

1. For $m_1 = m$, $k_1, = k$, $k_2 = 0$, $Cl(p,(m_1,k_1),(m_2,k_2) \cong Cl(p,m,k)$
2. For $m_1 - 1 = k$, $k_1, = 1$, $k_2 = 0$, $Cl(p,(m_1,k_1),(m_2,k_2) \cong Cl(p,k)$
3. For $m_1 = 2$, $k_1, = k$, $k_2 = 0$, $Cl(p,(m_1,k_1),(m_2,k_2) \cong Kb_p(k)$
4. For $m_1 = 1$, $k_1, = 0$, $k_2 = 0$, $Cl(p,(m_1,k_1),(m_2,k_2) \cong K_p$

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References

[1] Cvetković, D.M., Doob, M. and Sachs, H. *Spectra of Graphs*, Academic Press, New York, 1980.

[2] Gutman, I., "The Energy of a Graph", *Ber. Math. Stat. Sekt. Forschungszentrum Graz*, **103** (1978), 1 - 22.

[3] Gutman, I. "Hyperenergetic Molecular Graphs", *J. Serb. Chem. Soc.*, **64(3)** (1999), 199 - 205.

[4] Gutman, I. and Pavlović, L., "The Energy of Some Graphs with Large Number of Edges", *Bull. Acad. Serbe. Sci. Arts (Cl. Math. Natur)*, **118** (1999), 35 - 50.

[5] Gutman, I. and Polansky, O.E., *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986.