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NUMERICAL INVARIANTS OF COPRIME GRAPH OF A GENERALIZED QUATERNION GROUP

Nurhabibah[1,a], I Gede Adhitya Wisnu Wardhana[1,b], and Ni Wayan Switrayni[1,c]

[1]Department of Mathematics, Faculty of Mathematics and Natural Science, University of Mataram. Email:
[a]habibahmtk05@gmail.com
[b]corresponding author: adhitya.wardhana@unram.ac.id
[c]niwayan.switrayni@unram.ac.id

Abstract. The coprime graph of a finite group was defined by Ma, denoted by Γ_G, is a graph with vertices that are all elements of group G and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) = 1$. In this study, we discuss the numerical invariants of a generalized quaternion group. The numerical invariant is a property of a graph in numerical value and that value is always the same on an isomorphic graph. This research is fundamental research and analysis based on patterns in some examples. Some results of this research are the independence number of $\Gamma_{Q_{4n}}$ is $4n - 1$ or $3n$ and its complement metric dimension is $4n - 2$ for each $n \geq 2$.

Key words: coprime graph, generalized quaternion group, numerical invariants

1. INTRODUCTION

The graph representation of an algebraic structure becomes a hot topic in recent years. For a finite group G, we can represent G in some simple graphs such as the coprime graph, the non-coprime graph, the power graph, the intersection graph, the commuting graph, and others. See [1] [2] [4] [11] [12] for details. In 2014, Ma et al [1] introduced the coprime graph of a finite group G, where the vertex set of the graph is G and two distinct vertices x and y are adjacent if and only if the order of x and the order of y are relative primes [1].

In 2021, Nurhabibah et al. [4] give some results on the shape of the coprime graph of a generalized quaternion group as a bipartite graph, tripartite graph, or multipartite graph. So, in this article, we would like to discuss numerical invariants

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of the graph as degree, radius, diameter, domination number, independence number, girth, metric dimension, and complement metric dimension. The numerical invariant is a property of a graph in numerical value and that value is always the same on an isomorphic graph.

2. DEFINITION AND SOME PROPERTIES

A generalized quaternion group is a special group with a definition as follows.

Definition 2.1. [5] *A Generalized quaternion group* (Q_{4n}) *with* $n \geq 2$ *is a group with a presentation*

$$< a, b | a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1} >$$

in this group, $a^k b = ba^{-k}$ *and the order of* $a^k b$ *is* 4.

For example, $D_6 = \{e, a, a^2, b, ab, a^2 b\}$ is a dihedral group with order six. In this study, we give a function f that defines the adjacency of two vertices as follows.

Definition 2.2. *Given* $H = (V, E)$ *is a graph with* $V \neq \emptyset$. *Define* f *as follows.*

$$f : V \times V \to \{0, 1\}$$

with

$$f\left((v_i, v_j)\right) \begin{cases} 1, & \textit{if } (v_i, v_j) \in E \\ 0, & \textit{else} \end{cases}$$

One graph that is associated with a finite group is the coprime graph. The definition of this graph is given in Definition 2.3.

Definition 2.3. [1] *Given* G *a finite group, the coprime graph of* G*, denoted by* Γ_G *is the graph with* $V(\Gamma_G) = G$ *and two distinct vertices* x *and* y *are adjacent if and only if* $(|x|, |y|) = 1$.

And now, we define some numerical invariants of the graph that we analyze in this study.

Definition 2.4. *Let* G *be a graph. The degree of vertex* v *in* G *is denoted as* $deg(v) = |A|$ *where* $A = \{v_i \in V | f\left((v, v_i)\right) = 1\}$.

Definition 2.5. [6] *Let* G *be a graph and* $c(v) = max\{d(v, x) | x \in V(G)\}$ *is the eccentricity of* G*,* $rad(G) = min\{c(v) | v \in V(G)\}$ *and* $diam(G) = maks\{c(v) | v \in V(G)\}$.

Definition 2.6. [7] *Let $G = (V, E)$ be a graph. The girth of G, denoted by $g(G)$ is defined as:*

$$g(G) = \begin{cases} min\{\,|C_G|\,|\,C_G \text{ cycle of } G\}, & G \text{ contains cycle} \\ 0, & else \end{cases}$$

Where $|C_G|$ is the length of the cycle C_G.

Definition 2.7. [8] *Let G be a graph, the domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of the domination set of G. The domination set is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D.*

Definition 2.8. [9] *Let G be a graph, the independence number of G, denoted by $\beta(G)$ is the maximum cardinality of the independence set of G. The independence set is a subset I of $V(G)$ such that no two vertices in the subset represent an edge of $V(G)$.*

Definition 2.9. [10] *Let G be a graph, the metric dimension of G, denoted by $dim(G)$ is the minimum cardinality of the resolving set of G. The resolving set is a subset W for a graph G if, for every two distinct vertices u and v of G, there is an element w in W that resolves u and v.*

Definition 2.10. [11] *Let G be a graph, complement metric dimension of G, denoted by $\overline{dim}(G)$ is the maximum cardinality of complement resolving a set of G.*

Definition 2.11. *Let Γ_G be the coprime graph of a finite group G and $x \in V(\Gamma_G)$. We can define A_x as $A_x = \{x_i \in V(\Gamma_G) | f\,((x, x_i)) = 1\}$.*

We will discuss numerical invariants of the generalized quaternion group that is represented in the coprime graph. Given the previous result of the coprime graph of a generalized quaternion group.

Theorem 2.12. [4] *Let Q_{4n} be a generalized quaternion group. If $n = 2^k$ then the coprime graph of Q_{4n} is a complete bipartite graph.*

Theorem 2.13. [4] *Let Q_{4n} be a generalized quaternion group. If n is prime, then the coprime graph of Q_{4n} is tripartite graph.*

Theorem 2.14. [4] *Let Q_{4n} be a generalized quaternion group. If $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_i \neq 2, p_i \neq p_j$ for each $i \neq j, p_i$ a prime number then the coprime graph of Q_{4n} is $m + 2$ partite graph.*

Theorem 2.15. [4] *Let Q_{4n} be a generalized quaternion group. If $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_1 = 2$, p_i are distinct prime numbers then the coprime graph of Q_{4n} is $m+1$ partite graph.*

We also give the following theorem that we need to prove some numerical invariants of the coprime graph of a finite group.

Theorem 2.16. *Let $(G, *)$ be a finite group, $|G| = n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$ and Γ_G be the coprime graph of G. For some $x, y \in V(\Gamma_G)$ with $x \neq y$, $1 \leq l_i, t_i \leq k_i$, $|x| = \prod_{i=1}^{j} p_i^{l_i}$ and $|y| = \prod_{i=1}^{j} p_i^{t_i}$ if and only if $A_x = A_y$.*

Proof. Take any $x_1 \in A_x$, it means $(|x_1|, |x|) = 1$. Note that $|x| = \prod_{i=1}^{j} p_i^{l_i}$ and $(|x_1|, |x|) = 1$, so $\nexists p_i$ with $1 \leq i \leq j$ such that $p_i||x_1|$. Since $|y| = \prod_{i=1}^{j} p_i^{t_i}$ and $\nexists p_i$ with $1 \leq i \leq j$ which resulted in $p_i||x_1|$ then $(|y|, |x_1|) = 1$ and hence $f((x_1, y)) = 1$. Thus, $x_1 \in A_y$, which means $A_x \subseteq A_y$. By a similar approach, we can prove that $A_y \subseteq A_x$, hence $A_x = A_y$

Let G be a finite group with $|G| = n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$ and Γ_G be the coprime graph of G. Define $A = \{1, 2, ..., m\}$ and $B, C \subseteq A$ with $B \neq C$. If $x, y \in G$, $|x| = \prod_{s \in B} p_s^{l_s}$ and $|y| = \prod_{r \in C} p_r^{l_r}$, we need to prove $A_x \neq A_y$. Choose $z \in A_x$ with $|z| = p_a$ where $a \in C$, but $a \notin B$. Consequently, $(|y|, |z|) = p_a \neq 1$. It means $z \notin A_y$. So, it is complete to prove $A_x \neq A_y$. $\square$

3. NUMERICAL INVARIANTS OF THE COPRIME GRAPH OF Q_{4n}

In this section, we analyze the numerical invariants of the coprime graph of Q_{4n}. The first numerical invariants obtained from this study are the degree of each vertex as stated in Theorem 3.1 and Theorem 3.2.

Theorem 3.1. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ with $k \geq 1$ then $deg(e) = 4n - 1$ and $deg(v) = 1$ for each $v \in Q_{4n} \backslash \{e\}$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take $n = 2^k$. By Theorem 2.12, $\Gamma_{Q_{4n}}$ is a complete bipartite graph with partitions $V_1 = \{e\}$ and $V_2 = Q_{4n} \backslash \{e\}$, see the proof in [4] for detail. Thus, $f((e, v)) = 1$, for each $v \in Q_{4n} \backslash \{e\}$. By definition, obtained $deg(e) = |V_2| = 4n - 1$ and $deg(v) = |V_1| = 1$. $\square$

And for n is an odd prime number we have

Theorem 3.2. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If n is an odd prime number, the degree of each vertex is $1, n, 2n + 2$, or $4n - 1$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take n for any odd prime number p. Let $S_1 = \{e\}$, $S_2 = \{a^p, b, ab, ..., a^{2p-1}b\}$, $S_3 = \{a^2, a^4, ..., a^{2p-2}\}$, and $S_4 = \{a, a^3, ..., a^{p-2}, a^{p+2}, ..., a^{2p-1}\}$. Note that $|e| = 1$, $|x| = 2$ or $|x| = 4$ for each $x \in S_2$, $|y| = p$ for each $y \in S_3$, and $|z| = 2p$ for each $z \in S_4$. Thus,

- $f((e, v_1)) = 1$ for each $v_1 \in A_1$ with $A_1 = Q_{4n} \backslash \{e\}$, so $deg(e) = |A_1| = 4n - 1$.
- For $x \in S_2$, we get $f((x, v_2)) = 1$ if and only if $v_2 \in A_2$ with $A_2 = S_1 \cup S_3$. So, $deg(x) = |A_2| = |S_1| + |S_3| = n$.
- For $y \in S_3$, we get $f((y, v_3)) = 1$ if and only if $v \in A_3$ with $A_3 = S_1 \cup S_2$. So, $deg(y) = |A_3| = |S_1| + |S_2| = 1 + 2n + 1 = 2n + 2$.
- For $z \in S_4$, we get $f((z, v_4)) = 1$ if and only if $v_4 \in A_4$ with $A_4 = S_1$. So, $deg(z) = |S_1| = 1$.

□

The next numerical invariants are radius and diameter.

Theorem 3.3. *If $\Gamma_{Q_{4n}}$ is the coprime graph of a generalized quaternion group with $n \geq 2$, then $rad\,(\Gamma_{Q_{4n}}) = 1$ and $diam\,(\Gamma_{Q_{4n}}) = 2$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take any $n \in \mathbb{N}$ and $n \geq 2$. From [4] we know that $1 \leq d(u, v) \leq 2$ for each $u, v \in V\,(\Gamma_{Q_{4n}})$. It means $c(v) = 1$ or $c(v) = 2$ and by definition we get $rad\,(\Gamma_{Q_{4n}}) = 1$ and $diam\,(\Gamma_{Q_{4n}}) = 2$. □

The next result obtained from this study is the girth of $\Gamma_{Q_{4n}}$ as stated in the two following theorem.

Theorem 3.4. *If $\Gamma_{Q_{4n}}$ is the coprime graph of a generalized quaternion group with $n = 2^k$ then $g\,(\Gamma_{Q_{4n}}) = 0$.*

Proof. According to Theorem 2.12, if $n = 2^k$ then $\Gamma_{Q_{4n}}$ is a complete bipartite graph. It means, there is no cycle in $\Gamma_{Q_{4n}}$. By definition, we get $g\,(\Gamma_{Q_{4n}}) = 0$. □

And for other cases we have

Theorem 3.5. *If $\Gamma_{Q_{4n}}$ is the coprime graph of a generalized quaternion group with $n \neq 2^k$ and $n > 2$ then $g\,(\Gamma_{Q_{4n}}) = 3$.*

Proof. Let ΓQ_{4n} is the coprime graph of a generalized quaternion group Q_{4n}. Take $n \in \mathbb{N}$ with $n > 2$ and $n \neq 2^k$. There are $v_1, v_2 \in V\,(\Gamma_{Q_{4n}})$ with $|v_1| = p$, p an odd prime number and $|v_2| = 2$. So, we get a cycle with length 3 which is $e - v_1 - v_2 - v_1$. It is complete to prove $g\,(\Gamma_{Q_{4n}}) = 3$. □

The domination number is one of the numerical invariants of the graph. The domination number of $\Gamma_{Q_{4n}}$ is always the same for all $n \geq 2$. This property is stated in Theorem 3.6.

Theorem 3.6. *If $\Gamma_{Q_{4n}}$ be the coprime graph of a generalized quaternion then $\gamma\,(\Gamma_{Q_{4n}}) = 1$ for each $n \geq 2$.*

Proof. Let $D = \{e\}$ and D be the domination set because e is adjacent to other vertices in $\Gamma_{Q_{4n}}$. So, $\gamma\,(\Gamma_{Q_{4n}}) = |D| = 1$. □

The next numerical invariant is the independence number of the graph. This property is stated in four following Theorem.

Theorem 3.7. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of a generalized quaternion group. If $n = 2^k$ then $\beta(\Gamma_{Q_{4n}}) = 4n - 1$.*

PROOF. Let $\Gamma_{Q_{4n}}$ be the coprime graph of a generalized quaternion group with $n = 2^k$. By Theorem 2.12, $\Gamma_{Q_{4n}}$ is a complete bipartite graph that is a star graph. It means there are two partitions and these partitions are independent sets, $V_1 = \{e\}$ and $V_2 = Q_{4n}\backslash\{e\}$. Now, suppose that there is independence set I with $|Q_{4n}| \geq |I| > |V_2|$. So, I must equal to Q_{4n} and $f((e,v)) = 1$ for any $v \in Q_{4n}\backslash\{e\}$. It is a contradiction and it means V_2 is an independence set with maximum cardinality. Finally, we get $\beta(Q_{4n}) = |V_2| = 4n - 1$. $\square$

And for case n is an odd prime we have

Theorem 3.8. *Let Q_{4n} be a generalized quaternion group, $n = p$, p an odd prime number, then $\beta(\Gamma_{Q_{4n}}) = 3n$.*

PROOF. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Where n is an arbitrary odd prime number. Define $I = \{a^j | j = 1, 3, ..., 2n-1\} \cup \{a^i b | i = 0, 1, 2, ..., 2n-1\}$. Note that $|v| = 2q, q \in \mathbb{N}, \forall v \in I$. Thus, $\forall v_i, v_j \in I$ it is valid that $(|v_i|, |v_j|) \neq 1$ which resulted in $f((v_i, v_j)) = 0$. So, I is the independence set of $\Gamma_{Q_{4n}}$. And we need to prove that there is no independence set of $\Gamma_{Q_{4n}}$ with condition $|I'| > |I| = 3n$. If there is I', we can find $v \in I'$, but $v \notin I$, it means $|v| = 1$ or $|v| = p$. Consequently, there is $u \in I \cap I'$ with condition $(|u|, |v|) = 1$ which means $f((u,v)) = 1$. It is a contradiction with I' independence set of $\Gamma_{Q_{4n}}$. So, $\beta(\Gamma_{Q_{4n}}) = |I| = 3n$. $\square$

And for a more general case, when n is an odd

Theorem 3.9. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_i \neq p_j$, $i \neq j$, p_i an odd prime number, then $\beta(\Gamma_{Q_{4n}}) = 3n$.*

PROOF. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_i \neq 2$, $p_i \neq p_j$, $i \neq j$, p_i an odd prime number. Define $I = I_1 \cup I_2$ with $I_1 = \{a^{2l-1} | l = 1, 2, ..., n\}$ and $I_2 = \{a^i b | i = 0, 1, 2, ..., 2n-1\}$. Take $a^{2l-1} \in I_1$, we get $|a^{2l-1}| = 2z_1$ with $z_1 \in \mathbb{N}$ and $\forall a^i b \in I_2$ we get $|a^i b| = 4$. Thus, $\forall u, v \in I$ we get $(|u|, |v|) = 2z_2 \neq 1$ which shows that $f((u,v)) = 0$. So, I is an independence set. Suppose that I' is an independence set with $|I'| > |I| = 3n$. We can find $v_1 \in I'$ and $v_1 \notin I$, it means $|v_1| \neq 2z_3$, $\forall z_3 \in \mathbb{Z}$. Because of $|I'| > 3n$ then $|I \cap I'| \geq 2n + 1$. Consequently, $\exists v_2 \in I \cap I'$ with $v_2 = a^i b$ with $i \in \{0, 1, ..., 2n-1\}$, it means $|v_2| = 4$. Thus, $(|v_1|, |v_2|) = 1$ or $f((v_1, v_2)) = 1$. It is a contradiction with I' is independence set. So, $\beta(\Gamma_{Q_{4n}}) = |I| = 3n$. $\square$

And for a more general result

Theorem 3.10. *If $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_1 = 2$, $p_i \neq p_j$, $i \neq j$ then $\beta(\Gamma_{Q_{4n}}) = 4n - \frac{n}{2^{k_1}}$.*

PROOF. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_1 = 2$, $p_i \neq p_j$, $i \neq j$. Define $I = I_1 \cup I_2 \cup I_3$ with $I_1 = \{a^{2l-1} | l = 1, 2, ..., n\}$, $I_2 = \{a^{2l} | l =$

$1,2,...,n-1\}\backslash\{a^{2^{k_1+1}q}|q=1,2,...,\frac{n}{2^{k_1}}-1\}$, and $I_3=\{a^ib|i=0,1,2,...,2n-1\}$. For any $v\in I$ we get $|v|=2z_1, z_1\in\mathbb{N}$. Consequently, $\forall u,v\in I$ we get $(|u|,|v|)=2z_2\neq 1$. Thus I is an independence set. And now, we need to prove that there is no I' as the independence set of $\Gamma_{Q_{4n}}$ with condition $|I'|>|I|$. Suppose that there is I' with this condition, then $\exists v_1\in I'$ and $v_1\notin I$, It means $|v_1|\neq 2z_3$, $\forall z_3\in Z$. Because of $|I'|>|I|$ then $|I\cap I'|\geq 4n-\frac{n}{2^{k_1-1}}+1>3n$. Consequently, $\exists v_2\in I\cap I'$ where $v_2=a^ib$ and $|v_2|=4$. Those, we get $(|v_1|,|v_2|)=1$ or $f((v_1,v_2))=1$. It is a contradiction. So, $\beta(\Gamma_{Q_{4n}})=|I|=|I_1|+|I_2|+|I_3|=n+(n-1)-\left(\frac{n}{2^{k_1}}-1\right)+2n=4n-\frac{n}{2^{k_1}}$. □

Two of the numerical invariants of the graph that are defined based on the metric concept are the metric dimension and the complement metric dimension. In this study, we find the metric dimension and complement metric dimension of $\Gamma_{Q_{4n}}$.

Theorem 3.11. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n=2^k$ then $dim\,(\Gamma_{Q_{4n}})=4n-2$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n=2^k$, $k\in\mathbb{N}$, then $\Gamma_{Q_{4n}}$ is star graph. Choose $W=Q_{4n}\backslash\{e,a\}=\{v_1,v_2,...,v_{4n-2}\},\forall v\in W$ obtained $r(v_i|W)=(d_1,d_2,...,d_{4n-2})$ with $d_i=0$ and $d_j=2$ for each $i\neq j$. So, each $v\in W$ has a distinct representation of W. And then $r(e|W)=(1,1,...,1)$ and $r(a|W)=(2,2,...,2)$. Thus, each vertex of $\Gamma_{Q_{4n}}$ has distinct representation to W. In other words, W is a resolving set of $\Gamma_{Q_{4n}}$.

Suppose that there is W' as a resolving set with condition $|W'|<4n-2$. There are two distinct vertices $v_1,v_2\in Q_{4n}\backslash\{e\}$ and $v_1,v_2\notin W'$ which resulted in $r(v_1|W')=r(v_2|W')$. Thus, W' is not resolving set and it is complete to $dim\,(\Gamma_{Q_{4n}})=|W|=4n-2$. □

And for n is an odd prime number

Theorem 3.12. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n=p$, p an odd prime number, then $dim\,(\Gamma_{Q_{4n}})=4n-4$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Let n be any odd prime number. Choose $W=Q_{4n}\backslash\{e,a,a^2,b\}$. Note that each $v_1,v_2\in W$ with $v_1\neq v_2$ obtained $r(v_1|W)\neq r(v_2|W)$. Because of $|e|\neq|a|\neq|a^2|\neq|b|$ and the set of prime numbers that divide the order of e,a,a^2, and b are not equal, so by Theorem 2.16 obtained $A_e\neq A_a\neq A_{a^2}\neq A_b$. So, $r(e|W)\neq r(a|W)\neq r(a^2|W)\neq r(b|W)$. Thus W is a resolving set of $\Gamma_{Q_{4n}}$.

Suppose that there is W' with condition $|W'|<|W|=4n-4$ as a resolving set of $\Gamma_{Q_{4n}}$. We can find two distinct vertices $u,v\notin W'$ with $|u|=|v|$, means $A_u=A_v$ and $r(u|W')=r(v|W')$. Thus, W is a resolving set with minimum cardinality. In other words, $|W"|\geq 4n-4$ for each $W"$ resolving set of $\Gamma_{Q_{4n}}$. So, $dim\,(\Gamma_{Q_{4n}})=|W|=4n-4$. □

And for n is an odd number

Theorem 3.13. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, p_i an odd prime number, $p_i \neq p_j$ for $i \neq j$ then $dim\,(\Gamma_{Q_{4n}}) = 4n - 2^{m+1}$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_i \neq p_j$ for $i \neq j$. Define $W = Q_{4n} \backslash S$ with $S = \{e\} \cup S_1 \cup ... \cup S_{m+1}$ and $S_t = \{v_{t1}, v_{t2}, ..., v_{tC_t^{m+1}}\}$ for each $1 \leq t \leq m+1$ with $|v_{ts}|$ has t distinct prime factor and there are no two distinct elements in S_t with t the same prime factor. Thus, $|S_t|$ is the total of ways to choose t prime numbers from $m+1$ distinct prime numbers or $|S_t| = C_t^{m+1}$ where $C_k^m = \frac{m!}{(m-k)!k!}$. Consequently,

$$|S| = 1 + C_1^{m+1} + C_2^{m+1} + ... + C_{m+1}^{m+1} = 2^{m+1}$$

And now, we need to prove $W = Q_{4n} \backslash S$ is the resolving set. Each $v_1, v_2 \in W$ with $v_1 \neq v_2$, obtained $r(v_i|W) = (d_1, d_2, ..., d_{|W|})$ with $d_i = 0$ and $d_j \neq 0$ for each $i \neq j$, thus $r(v_1|W) \neq r(v_2|W)$. By Theorem 2.16, each two distinct vertices $u_1, u_2 \in S$, we get $A_{u_1} \neq A_{u_2}$ which resulted in $r(u_1|W) \neq r(u_2|W)$. Thus, $r(u|W) \neq r(v|W)$ for each $u, v \in W$ with $u \neq v$. In other words, W is a resolving set of $\Gamma_{Q_{4n}}$.

Suppose that $W' = Q_{4n} \backslash S'$ with $|S'| > |S|$ is resolving set of $\Gamma_{Q_{4n}}$. We can find $v_5, v_6 \in S'$ with $v_5 \neq v_6$, but $|v_5|$ and $|v_6|$ have t the same prime factor. By Theorem 2.16, obtained $A_{v_5} = A_{v_6}$ and $r(v_5|W') = r(v_6|W')$. So, W' is not resolving set and W is resolving set $\Gamma_{Q_{4n}}$ with minimum cardinality. It is done to prove $dim\,(\Gamma_{Q_{4n}}) = |W| = 4n - 2^{m+1}$. □

And for a more general case, by Theorem 3.13 we have the following result.

Corollary 3.14. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_1 = 2$, $p_i \neq p_j$ for $i \neq j$, p_i prime number, then $dim\,(\Gamma_{Q_{4n}}) = 4n - 2^m$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Take $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$, $p_1 = 2$, $p_i \neq p_j$ for $i \neq j$, p_i prime number. Note that $p_1 = 2$, so the number of odd prime number p_j is $m - 1$. Consequently, Theorem 3.13 obtained $dim\,(\Gamma_{Q_{4n}}) = 4n - 2^{(m-1)+1} = 4n - 2^m$. □

The last theorem explains the complement metric dimension of $\Gamma_{Q_{4n}}$.

Theorem 3.15. *If $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n} with $n \geq 2$ then $\overline{dim}\,(\Gamma_{Q_{4n}}) = 4n - 2$.*

Proof. Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. Choose any natural number n with $n \geq 2$. Choose $S = Q_{4n} \backslash \{b, ab\}$, obtained $r(b|S) = r(ab|S)$. So, S is a complement to resolving a set of Q_{4n}. By definition, it is impossible to find a complement resolving set with cardinality greater than $|V(G)| - 2$. Thus, S is a complement resolving set with maximum cardinality. It is done to prove that $\overline{dim}\,(\Gamma_{Q_{4n}}) = |S| = 4n - 2$.

4. CONCLUSION

Some numerical invariants of the coprime graph of a generalized quaternion that were obtained from this study are the degree of each vertex is $1, n, 2n+2$, or $4n-1$, its radius is 1, its diameter is 2, its girth is 0 or 3, its independence number is $4n-1$, $3n$, or $4n-\frac{n}{2^{k_1}}$, its domination number is 1, its metric dimension is $4n-2, 4n-4, 4n-2^m$, or $4n-2^{m+1}$, and its complement metric dimension is $4n-2$.

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