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THE MISSILE GUIDANCE ESTIMATION USING EXTENDED KALMAN FILTER-UNKNOWN INPUT-WITHOUT DIRECT FEEDTHROUGH (EKF-UI-WDF) METHOD

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Abstract. This paper consider the estimation of the optimal missile guidance which the objective is to minimize the interception time and the energy expenditure. The proposed Extended Kalman Filter-Unknown Input-Without Direct Feedthrough (EKF-UI-WDF) approach is to estimate the optimal missile guidance and the target acceleration as unknown input to the missile-target interception model. Unknown input is any type of signals without prior information from a given state model or a measurement. The computational for the EKF-UI-WDF method and optimal missile guidance show the closest range to the missile-target is smaller than using the EKF. However the Mean Squared Error (MSE) of estimating the optimal missile guidance using EKF method is smaller than using EKF-UI-WDF method.

Key words and Phrases:EKF-UI-WDF, unknown input, optimal control, missile, target.

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Abstrak. Makalah ini mengkaji estimasi panduan optimal peluru kendali. Fungsi tujuannya adalah meminimumkan waktu tembak dan energi yang digunakan peluru kendali. Metode yang digunakan dalam estimasi panduan optimal peluru kendali dan percepatan target adalah metode *Extended Kalman Filter-Unknown Input-Without Direct Feedthrough* (EKF-UI-WDF), dengan percepatan target sebagai input yang tidak diketahui. Input yang tidak diketahui merupakan semua tipe sinyal yang tidak ada informasi sebelumnya dari state model yang diberikan atau pengukuran. Hasil simulasi dari penerapan metode EKF-UI-WDF dan panduan optimal peluru kendali menunjukkan bahwa jarak terpendek antara peluru kendali dan target yang diperoleh lebih kecil dibandingkan dengan *Extended Kalman Filter* (EKF). Tetapi nilai *Mean Squared Error* (MSE) estimasi panduan optimal peluru kendali menggunakan metode EKF lebih kecil daripada dengan metode EKF-UI-WDF.

Kata kunci: EKF-UI-WDF, input tidak diketahui, kendali optimal, peluru kendali, target.

1. INTRODUCTION

The objective of optimal control problem is to obtain a controller (input signal) subject to dynamic systems and satisfy some constraints by minimizing or maximizing an objective function [6]. In most practical scenarios, there is a need to construct the estimates of state variables which are not available by a direct measurement, especially when they are used in the applications such as the implementation of state feedback controllers.

Extended Kalman Filter (EKF) can be applied to estimate a nonlinear dynamic model [10]. EKF method is based on linearizing the nonlinear model and the measurement equations with the first order Taylor series expansion. Generally, nonlinear filtering (e.g. EKF method) assume that all inputs are measurable and are not able to jointly estimate unknown inputs and the states [3]. In some cases to get a better estimation it is necessary to estimate the unknown inputs which can be any type of signals without prior information from a given state model. The method which developed from EKF method approach is called Extended Kalman Filter-Unknown Input-Without Direct Feedthrough (EKF-UI-WDF). This method can simultaneously estimates the states and unknown input for nonlinear stochastic discrete-time systems without direct feedthrough from unknown inputs to outputs. A recursive analytical approach of EKF-UI-WDF is derived with the weighted least squares estimation for an extended state vector including the states and unknown inputs.

In the previous research, Augmented Proportional Navigation (APN) guidance law to the missile-target interception model has been estimated with the EKF-UI-WDF approach and target acceleration is unknown input in EKF-UI-WDF approach. EKF-UI-WDF approach is more effective than the EKF method for estimate the missile-target interception control system [7]. The optimal missile guidance which the objective function is to minimize the interception time and the control energy expenditure and based on the design concept of decreasing the acceleration requirement commanded in the final phase of engagement is more effective than APN guidance law with $N = 3$ [9]. We estimate the optimal missile guidance with EKF-UI-WDF method and target acceleration is unknown input. For this discussion, the missile-target interception model is explained

2. Missile-Target Interception Model

The 2D missile-target engagement model can be expressed by the following equations [5]:

$$\dot{\lambda} = \frac{V_T \sin(\theta_T - \lambda) - V_M \sin(\theta_M - \lambda)}{R} \tag{1}$$

$$\dot{R} = V_T \cos(\theta_T - \lambda) - V_M \cos(\theta_M - \lambda) \tag{2}$$

$$\dot{\theta}_T = \frac{a_T}{V_T} \tag{3}$$

$$\dot{\theta}_M = \frac{a_M}{V_M} \tag{4}$$

and Table 1 shows the variables and parameters of the 2D missile-target engagement model.

Table 1. Variable and Parameter

Symbols	
R	the relative distance between the missile and target (range)
λ	the line-of-sight angle
V_T	the tangential velocity of the target
V_M	the tangential velocity of the missile
θ_T	flight path angle of the target
θ_M	flight path angle of the target and the missile
a_T	the normal acceleration of the target
a_M	the normal acceleration of the missile
V_M	the variations, $\dot{V}_M = \tilde{a}_M$
V_T	the variations, $\dot{V}_T = \tilde{a}_T$
$\tilde{a}_M$	the tangential accelerations of the missile
$\tilde{a}_T$	the tangential accelerations of the target

3. EKF-UI-WDF Method

EKF-UI-WDF method is developed based on EKF method. This method can estimate the states and unknown inputs for the nonlinear stochastic discrete-time systems, without direct feedthrough from unknown input to outputs. The unknown input is any type of signals without prior information from a given state model or a measurement.

TABLE 2. Extended Kalman Filter-Unknown Input-Without Direct Feedthrough Algorithm

System model and Measurement model
$\mathbf{Z}_k = \mathbf{g}\left(\mathbf{Z}_{k-1}, \mathbf{u}_{k-1}, \mathbf{u}^*_{k-1}, k-1\right) + \mathbf{w}_{k-1}$ $\mathbf{y}_k = \mathbf{h}\left(\mathbf{Z}_k, \mathbf{u}_k, k\right) + \mathbf{v}_k$ $\mathbf{Z}_0 \sim N\left(\bar{\mathbf{Z}}_0, \mathbf{P}_{\mathbf{Z}_0}\right); \mathbf{w}_k \sim N\left(0, \mathbf{Q}_k\right); \mathbf{v}_k \sim N\left(0, \mathbf{R}_k\right)$
Initialization
$\hat{\mathbf{Z}}_0 = \bar{\mathbf{Z}}_0$ $\mathbf{P}_0 = \mathbf{P}_{\mathbf{Z}_0}$
Prediction
Estimation : $\hat{\mathbf{Z}}_{k\|k-1} = \mathbf{g}\left(\hat{\mathbf{Z}}_{k-1\|k-1}, \mathbf{u}_{k-1}, \mathbf{u}^*_{k-2\|k-1}, k-1\right)$ **Error Covariance :** $\mathbf{P}_{\mathbf{Z},k\|k-1} = \mathbf{G}_{k-1\|k-1}\mathbf{P}_{\mathbf{Z},k-1\|k-1}\mathbf{G}^T_{k-1\|k-1} + \mathbf{Q}_{k-1}$
Update
Kalman Gain : $\mathbf{K}_{\mathbf{Z},k} = \mathbf{P}_{\mathbf{Z},k\|k-1}\mathbf{H}^T_{k\|k-1}\left(\mathbf{H}_{k\|k-1}\mathbf{P}^-_{\mathbf{Z},k\|k-1}\mathbf{H}^T_{k\|k-1} + \mathbf{R}_k\right)^{-1}$ $\mathbf{S}_k = \left(\mathbf{B}^{*T}_{k-1\|k-1}\mathbf{H}^T_{k\|k-1}\mathbf{R}^{-1}_k\left(\mathbf{I}_p - \mathbf{H}_{k\|k-1}\mathbf{K}_{\mathbf{Z},k}\right)\mathbf{H}_{k\|k-1}\mathbf{B}^*_{k-1\|k-1}\right)^{-1}$ **Estimation :** $\hat{\mathbf{Z}}_k = \hat{\mathbf{Z}}_{k\|k-1} + \mathbf{K}_{\mathbf{Z},k}\left(\mathbf{y}_k - \mathbf{h}\left(\hat{\mathbf{Z}}_{k\|k-1}, \mathbf{u}_k, k\right)\right)$ $\hat{\mathbf{u}}^*_{k-1\|k} = \mathbf{S}_k\mathbf{B}^{*T}_{k-1\|k-1}\mathbf{H}^T_{k\|k-1}\mathbf{R}^{-1}_k\left(\mathbf{I}_p - \mathbf{H}_{k\|k-1}\mathbf{K}_{\mathbf{Z},k}\right)$ $\left[\mathbf{y}_k - \mathbf{h}\left(\hat{\mathbf{Z}}_{k\|k-1}, \mathbf{u}_k, k\right) + \mathbf{H}_{k\|k-1}\mathbf{B}^*_{k-1\|k-1}\hat{\mathbf{u}}^*_{k-2\|k-1}\right]$ **Error Covariance :** $\mathbf{P}_{\mathbf{Z},k-1\|k-1} = \left(\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k-1}\mathbf{H}_{k-1\|k-2}\right)$ $\left[\mathbf{P}_{\mathbf{Z},k-1\|k-2} + \mathbf{B}^*_{k-2\|k-2}\mathbf{S}_{k-1}\mathbf{B}^*_{k-1\|k-1}\left(\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k-1}\mathbf{H}_{k-1\|k-2}\right)^T\right]$ **where :** $\mathbf{G}_{k-1\|k-1} = \left.\frac{\partial \mathbf{g}_{k-1}}{\partial \mathbf{Z}_{k-1}}\right\|_{\mathbf{Z}_{k-1}=\hat{\mathbf{Z}}_{k-1\|k-1}, \mathbf{u}^*_{k-1}=\hat{\mathbf{u}}^*_{k-2\|k-1}}$ $\mathbf{H}_{k\|k-1} = \left.\frac{\partial \mathbf{h}_k}{\partial \mathbf{Z}_k}\right\|_{\mathbf{Z}_k=\hat{\mathbf{Z}}_{k\|k-1}}$ $\mathbf{B}^*_{k-1\|k-1} = \left.\frac{\partial \mathbf{g}_{k-1}}{\partial \mathbf{u}^*_{k-1}}\right\|_{\mathbf{Z}_{k-1}=\hat{\mathbf{Z}}_{k-1\|k-1}, \mathbf{u}^*=\hat{\mathbf{u}}^*_{k-2\|k-1}}$

The following discrete nonlinear state and observation equations can be obtained :

$$\begin{aligned}\mathbf{Z}_k &= \mathbf{g}_{k-1}\left(\mathbf{Z}_{k-1}, \mathbf{u}_{k-1}, \mathbf{u}^*_{k-1}\right) + \mathbf{w}_{k-1}\\ \mathbf{y}_k &= \mathbf{h}_k\left(\mathbf{Z}_k, \mathbf{u}_k\right) + \mathbf{v}_k\end{aligned}$$

where

$\mathbf{Z}_k$: the states at time k and the value of initialization of the states estimation is $\bar{\mathbf{Z}}_0$ and have covariance $\mathbf{P}_{\mathbf{Z}_0}$
$\mathbf{u}_k$: the known deterministic inputs at time k
$\mathbf{u}^*_{k-1}$: the unknown inputs at time k
$\mathbf{w}_{k-1}$: the model noise (uncertainty) have mean $\bar{\mathbf{w}}_k = 0$ and covariance $\mathbf{Q}_k$
$\mathbf{y}_k$: the measurement vector
$\mathbf{v}_k$: the measurement noise have $\bar{\mathbf{v}}_k = 0$ and covariance $\mathbf{R}_k$

Table 2 shows EKF-UI-WDF algorithm which has four parts to estimate $\mathbf{Z}_k$ and $\mathbf{u}^*_{k-1}$ at $t = k\,\Delta t$ which are denoted as $\hat{\mathbf{Z}}_{k|k}$ and $\hat{\mathbf{u}}^*_{k-1|k}$; respectively. If unknown inputs are known then EKF-UI-WDF approach becomes EKF method.

4. Optimal Control Solution

Optimal control system of the missile-target interception model based on the design concept of decreasing the acceleration requirement commanded in the final phase of engagement. It is assumed that a perfect knowledge (information) of the target motion is available to the missile (i.e. V_T and a_T are known). The objective function is to minimize the interception time and the control energy expenditure, which can be written as follow

$$J = t_f + \rho \int_0^{t_f} {a_M}^2 dt.$$

The first term in J is a measure of the interception time, the second term is the required acceleration command, ρ is the weighting factor reflecting the relative importance of the commanded acceleration with respect to the interception time. In general, for short ranges where the time is paramount, the value of ρ has to be zero, for long ranges where the control energy should be saved so a larger ρ can be chosen.

The solution is derived analytically from the time-varying linear state equations which composed the line-of-sight (LOS) angle and line-of-sight rate. By differentiating Eq. (1), we obtain the time-varying linear differential equation as

$$\ddot{\lambda} = A(t)\,\lambda + B(t)\,a_M + C(t)$$

where $A = -2\,\frac{\dot{R}}{R}$, $B = -\frac{\cos(\lambda-\theta_M)}{R}$ and $C = \frac{a_T \cos(\lambda-\theta_T)}{R}$.
Defining the states $\lambda = x_1$ and $\dot{\lambda} = x_2$, then the equations can be defined

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= A(t)\,x_2 + B(t)\,a_M + C(t) \end{aligned} \tag{5}$$

The terminal condition is achieved by

$$x_2(t_f) = 0$$

and the initial value of states :

$$\begin{aligned} x_1(0) &= x_{1_0} \\ x_2(0) &= x_{2_0} \end{aligned}$$

Here is the procedure to solve the optimal control problem :

i. Define the Hamiltonian equation : $H = 1 + \rho\, {a_M}^2 + \nu_1 x_2 + \nu_2 \left(Ax_2 + Ba_M + C\right)$ where ν_i is Lagrangian multiplier.

ii. Minimize H
$\frac{\partial H}{\partial a_M} = 0$ and obtained $a_M = -\frac{B\nu_2}{2\rho}$.

iii. Using the results a_M from (ii) into (i), we obtain the optimal $H^* = 1 + \frac{B^2{\nu_2}^2}{4\rho} + \nu_1 x_2 + \nu_2 \left(Ax_2 + -\frac{B^2\nu_2}{2\rho} + C\right)$.

iv. Solve the differential equations
$\dot{\mathbf{x}}\,(t) = \frac{\partial}{\partial \nu} H^* \left(\mathbf{x}, \nu, t\right)$
and co-state equation : $\dot{\nu_1} = -\frac{\partial H}{\partial x_1} = 0$, $\dot{\nu_2} = -\frac{\partial H}{\partial x_2} = -\nu_1 - \nu_2 A$. The boundary condition is the initial and final conditions, and it is called the transversality condition [8]. Generally, the boundary conditions for this system are

$$\begin{aligned} \nu_1\left(t_f\right) &= 0 \\ \nu_2\left(t_f\right) &= \psi \end{aligned}$$

where ψ is an unknown constant. And we obtain

$$\begin{aligned} \nu_1 &= 0 \\ \nu_2 &= \psi \mathrm{e}^{\int_t^{t_f} A dt} \end{aligned}$$

Defining $f(t) = \mathrm{e}^{\int_t^{t_f} A dt}$, we then have

$$\nu_2 = \psi f(t) \tag{6}$$

Thus, Eq. (5) becomes

$$\dot{x_2} - Ax_2 = C - \frac{\psi B^2 f\left(t\right)}{2\rho} \tag{7}$$

Equation (7) is a nonhomegenous differential equation and this solution viewed from the homogen solution and the particular [1], we obtain the following solution

$$x_2\left(t\right) = \frac{f\left(0\right)}{f\left(t\right)} \left(\int_0^t \left(C - \frac{\psi\, B^2 f\left(t\right)}{2\rho} \right) \frac{f\left(t\right)}{f\left(0\right)} dt + x_{2_0} \right)$$

v. Substitute the solutions x^*, ν^* from (iv) into the expression for the optimal control a_M from (ii) and substitute the variable from 0 to t. We obtain

$$a_M = \frac{R^3\dot{\lambda}\cos\left(\theta_M - \lambda\right) + R\cos\left(\theta_M - \lambda\right)\int_t^{t_f} R a_T \cos\left(\theta_T - \lambda\right) dt}{\int_t^{t_f} R^2 \cos^2\left(\theta_M - \lambda\right) dt} \tag{8}$$

Eq. (8) shows that the weighting factor ρ is not affect explicitly to the controller obtained.

For guidance implementation, the values of the time to go (t_{go}) in Eq. (8) is approximate from the following equation

$$t_{go} = t_f - t \cong -\frac{R(t)}{\dot{R}(t)} \tag{9}$$

We simplify Eq. (8) with the following assumptions :

i. The tangential velocity of the missile (V_M) and the tangential velocity of the target (V_T) are assumed to be constant
ii. The flight path angles of the target and the missile are identical and equal to the LOS angle ($\theta_M = \theta_T = \lambda$).

The relative closing velocity V_c is constant ($V_c = -\dot{R}$), Eq. (8) becomes

$$\begin{aligned} a_M &= \frac{{V_c}^3 (t_f - t)^3 \dot{\lambda} + V_c (t_f - t) \int_t^{t_f} V_c (t_f - t) a_T dt}{\int_t^{t_f} {V_c}^2 (t_f - t)^2 dt} \\ &= -3 \dot{R}\dot{\lambda} + \frac{3}{2} a_T \end{aligned} \tag{10}$$

Eq. (10) is the augmented proportional navigation law with a unitless gain is equal to 3. Therefore, the augmented proportional navigation law is the simplified version of the optimal guidance which we obtain in Eq. (8). After we get the optimal guidance for the missile then we estimate the variables of the control equation in Eq. (8).

5. The EKF-UI-WDF Design

To compute the control in Eq. (8), the estimates of $R, \dot{R}, \lambda, \dot{\lambda}, \theta_M, \theta_T$ and a_T should be obtained by using an appropriate nonlinear filter. For this purpose, the dynamic model in Eqs. (1)-(4) will be transformed into a state space model whose state and unknown inputs include $R, \dot{R}, \lambda, \dot{\lambda}, \theta_M, \theta_T$ and a_T. Taking the derivatives of $\dot{R}$ and $\dot{\lambda}$ in Eqs (2) and (1), and denoting the states as well as the known and unknown input as $x_1 = R$, $x_2 = \dot{R}$, $x_3 = \lambda$, $x_4 = \dot{\lambda}$, $x_5 = \theta_M$, $x_6 = \theta_T$, $u = a_M$ and $u^* = a_T$, respectively. We then have the state space model with model noise used to design of the nonlinear filter as

$$\dot{x}_1 = x_2 + w_1 \tag{11}$$

$$\dot{x}_2 = {x_4}^2 x_1 + u \sin(x_5 - x_3) - u^* \sin(x_6 - x_3) + w_2 \tag{12}$$

$$\dot{x}_3 = x_4 + w_3 \tag{13}$$

$$\dot{x}_4 = \frac{-2\, x_4 x_2 + u^* \cos(x_6 - x_3) - u \cos(x_5 - x_3)}{x_1} + w_4 \tag{14}$$

$$\dot{x}_5 = \frac{u}{V_M} + w_5 \tag{15}$$

$$\dot{x}_6 = \frac{u^*}{V_T} + w_6 \qquad (16)$$

The variations of V_M and V_T are $\dot{V}_M = 0.8$ and $\dot{V}_T = 0.4$.

Substituting $(d\mathbf{x}/dt)_{t=k\Delta t} = (\mathbf{x}_k - \mathbf{x}_{k-1}/\Delta t)$ [2] into Eqs. (11)-(16) where $\mathbf{x}_k = \mathbf{x}\,(k\,\Delta t)$ and Δt is the sampling time interval. We can obtain discrete nonlinear state equation as follows

$$\mathbf{x}_k = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{u}^*_{k-1}\right) + \mathbf{w}_{k-1} \qquad (17)$$

where

$$\mathbf{f}_{k-1} = [f_{1,k-1},\ f_{2,k-1},\ f_{3,k-1},\ f_{4,k-1},\ f_{5,k-1},\ f_{6,k-1}]^T$$

and

$$\begin{aligned}
f_{1,k-1} &= x_{2,k-1}\Delta t + x_{1,k-1}\\
f_{2,k-1} &= \left({x_{4,k-1}}^2 x_{1,k-1} + u_{k-1}\sin\left(x_{5,k-1} - x_{3,k-1}\right)\right)\Delta t\\
&\quad -{u^*}_{k-1}\sin\left(x_{6,k-1} - x_{3,k-1}\right)\Delta t + x_{2,k-1}\\
f_{3,k-1} &= x_{4,k-1}\Delta t + x_{3,k-1}\\
f_{4,k-1} &= \frac{-2x_{4,k-1}x_{2,k-1} + {u^*}_{k-1}\cos\left(x_{6,k-1} - x_{3,k-1}\right)}{x_{1,k-1}}\Delta t\\
&\quad - \frac{u_{k-1}\cos\left(x_{5,k-1} - x_{3,k-1}\right)}{x_{1,k-1}}\Delta t + x_{4,k-1}\\
f_{5,k-1} &= \frac{u_{k-1}}{V_{M,k-1}}\Delta t + x_{5,k-1}\\
f_{6,k-1} &= \frac{{u^*}_{k-1}}{V_{T,k-1}}\Delta t + x_{6,k-1}
\end{aligned}$$

as well as $\mathbf{w}_{k-1} = [w_{1,k-1}\,, w_{2,k-1}\,, w_{3,k-1}\,, w_{4,k-1}\,, w_{5,k-1}\,, w_{6,k-1}]^T$ and $\mathbf{w}_{k-1}$ is the model noise vector (Gaussian white noise, $\mathbf{w}_{k-1} \sim N\left(0, \mathbf{Q}_k\right)$).

The measurements for the process model refers to IRS (Inertial Reference System) of the missile are $R, \dot{R}, \lambda, \dot{\lambda}, \theta_M$ and θ_T, and given by the following equations

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k \qquad (18)$$

where $\mathbf{v}_k = [v_{1,k}\,, v_{2,k}\,, v_{3,k}\,, v_{4,k}\,, v_{5,k}\,, v_{6,k}]^T$ and $\mathbf{v}_k$ is the measurement noise vector (Gaussian white noise, $\mathbf{v}_k \sim N\left(0, \mathbf{R}_k\right)$) and the matrix $\mathbf{H} = \mathbf{I}_{6\times 6}$.

The estimates of unknown input $(\mathbf{u}^*_{k-1})$ and state vectors $(\mathbf{x}_k)$ at $t = k\,\Delta t$ which are denoted as $\hat{\mathbf{u}}^*_{k-1|k}$ and $\hat{\mathbf{x}}_{k|k}$, respectively. Since the equation (17) are nonlinear, a first order approximation [4] for the systems dynamic matrix is obtained on Table 2, thus $\mathbf{G}_{k-1|k-1}$ are given as follows

$$\mathbf{G}_{k-1|k-1} = \frac{\partial \mathbf{f}_{k-1}}{\partial \mathbf{x}_{k-1}} = \begin{bmatrix} 1 & \Delta t & 0 & 0 & 0 & 0 \\ {x_{4,k-1}}^2 \Delta t & 1 & G_1 & 2\Delta t x_{1,k-1} x_{4,k-1} & G_2 & G_3 \\ 0 & 0 & 1 & \Delta t & 0 & 0 \\ G_4 & -2\Delta t \frac{x_{4,k-1}}{x_{1,k-1}} & G_5 & G_6 & G_7 & G_8 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\begin{aligned}
G_1 &= \left(-u_{k-1} \cos\left(x_{5,k-1} - x_{3,k-1}\right) + u^*_{k-1} \cos\left(x_{6,k-1} - x_{3,k-1}\right)\right) \Delta t \\
G_2 &= u_{k-1} \Delta t \cos\left(x_{5,k-1} - x_{3,k-1}\right) \\
G_3 &= -u^*_{k-1} \Delta t \cos\left(x_{6,k-1} - x_{3,k-1}\right) \\
G_4 &= \frac{\left(2x_{4,k-1} x_{2,k-1} - u^*_{k-1} \cos\left(x_{6,k-1} - x_{3,k-1}\right) + u_{k-1} \cos\left(x_{5,k-1} - x_{3,k-1}\right)\right) \Delta t}{{x_{1,k-1}}^2} \\
G_5 &= \frac{\left(u^*_{k-1} \sin\left(x_{6,k-1} - x_{3,k-1}\right) - u_{k-1} \sin\left(x_{5,k-1} - x_{3,k-1}\right)\right) \Delta t}{x_{1,k-1}} \\
G_6 &= -2\Delta t \frac{x_{2,k-1}}{x_{1,k-1}} + 1 \\
G_7 &= \frac{u_{k-1} \Delta t \sin\left(x_{5,k-1} - x_{3,k-1}\right)}{x_{1,k-1}} \\
G_8 &= \frac{-u^*_{k-1} \Delta t \sin\left(x_{6,k-1} - x_{3,k-1}\right)}{x_{1,k-1}}
\end{aligned}$$

and

$$\mathbf{B}^*_{k-1|k-1} = \frac{\partial \mathbf{f}_{k-1}}{\partial u^*_{k-1}} = \begin{bmatrix} 0 \\ -\Delta t \sin\left(x_{6,k-1} - x_{3,k-1}\right) \\ 0 \\ \frac{\Delta t \cos\left(x_{6,k-1} - x_{3,k-1}\right)}{x_{1,k-1}} \\ 0 \\ \frac{\Delta t}{V_{T,k-1}} \end{bmatrix}$$

.

Substituting t_f from Eq. (9) to Eq. (8), we obtain the input for discrete nonlinear filtering as follows

$$u_{k-1} = \frac{\hat{x}^3_{1,k-1|k-1}\hat{x}_{4,k-1|k-1}\cos\left(\hat{x}_{5,k-1|k-1}-\hat{x}_{3,k-1|k-1}\right)}{\int_t^{t-\frac{\hat{x}_{1,k-1|k-1}}{\hat{x}_{2,k-1|k-1}}}\hat{x}^2_{1,k-1|k-1}\cos^2\left(\hat{x}_{5,k-1|k-1}-\hat{x}_{3,k-1|k-1}\right)dt}$$
$$+\hat{x}_{1,k-1|k-1}\cos\left(\hat{x}_{5,k-1|k-1}-\hat{x}_{3,k-1|k-1}\right)\left(\frac{\int_t^{t-\frac{\hat{x}_{1,k-1|k-1}}{\hat{x}_{2,k-1|k-1}}}\hat{x}_{1,k-1|k-1}\hat{u}^*_{k-2|k-1}\cos\left(\hat{x}_{6,k-1|k-1}-\hat{x}_{3,k-1|k-1}\right)dt}{\int_t^{t-\frac{\hat{x}_{1,k-1|k-1}}{\hat{x}_{2,k-1|k-1}}}\hat{x}^2_{1,k-1|k-1}\cos^2\left(\hat{x}_{5,k-1|k-1}-\hat{x}_{3,k-1|k-1}\right)dt}\right) \tag{19}$$

6. The EKF Design

For comparison, the EKF approach is used in the missile guidance unlike the proposed EKF-UI-WDF approach, because EKF method can not estimate the joint of states and unknown input. The target acceleration a_T estimated with EKF method has to be treated as a state with assumed dynamics. Thus, the state equations for the design of EKF can be obtained by adding an extra state equation for the a_T to Eqs. (11)-(16) as follow :

$$\frac{dx_7}{dt} = -x_7 + w_T$$

in which $x_7 = a_T$ and w_T is a Gaussian white noise and replacing u^* by x_7 in Eqs. (11)-(16).

7. Computational Result

This section presents five simulations to show the performances of the EKF-UI-WDF and the EKF for the implementation of optimal missile guidance. The total time for the simulation run was $2.5\,s$ and a time step of $\Delta t = 0.05\,s$ was used.

EKF-UI-WDF Design Parameter. The initial conditions chosen for the model in EKF-UI-WDF design are

$$\hat{\mathbf{x}}_{0|0} = [90,\ -41,\ 0,\ 0,\ 0,\ \pi]^T,\ \hat{u}^*_{-1|0} = 0$$

The initial error covariance matrix is
$\mathbf{P}_{\mathbf{x},0|0} = diag\left\{10^{-1}, 10^{-1}, 10^{-2}, 10^{-2}, 10^{-2}, 10^{-2}\right\}$.
The covariance matrix for the model noise $\mathbf{w}_{k-1}$ is chosen to be
$\mathbf{Q}_{k-1} = diag\left\{10^{-4}, 10^{-2}, 10^{-4}, 10^{-2}, 10^{-4}, 10^{-4}\right\}$, bigger values are chosen for 2nd and 4th equations.

The covariance matrix for the measurement noise $\mathbf{v}_k$ is chosen to be $\mathbf{R}_k = diag\left\{10^{-8}, 10^{-4}, 10^{-8}, 10^{-4}, 10^{-8}, 10^{-8}\right\}$.

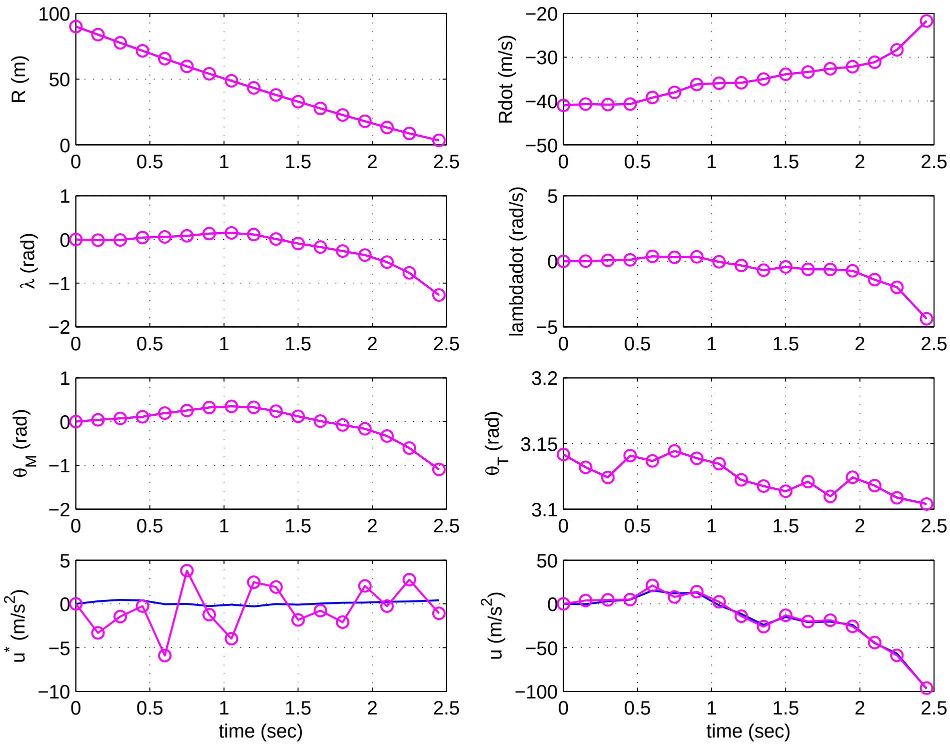

FIGURE 1. The estimated states and unknown input for the missile guidance with EKF-UI-WDF (blue curves for real and purple curves for estimation)

TABLE 3. The MSE (*Mean Squared Error*) for The Missile Guidance Estimation with EKF-UI-WDF Method

Simulation	R	$\dot{R}$	λ	$\dot{\lambda}$
1	$9,54.10^{-9}$	$7,83.10^{-5}$	$7,17.10^{-9}$	$8,79.10^{-5}$
2	$9,38.10^{-9}$	$6,3.10^{-5}$	$1,08.10^{--8}$	$7,82.10^{-5}$
3	$1,17.10^{-8}$	$1,03.10^{-4}$	$9,14.10^{-9}$	$7,79.10^{-5}$
4	$7,86.10^{-9}$	$8,87.10^{-5}$	$1,05.10^{-8}$	$1,09.10^{-4}$
5	$1,02.10^{-8}$	$1,4.10^{-4}$	$1,39.10^{-8}$	$6,47.10^{-5}$
Average	$9,74.10^{-9}$	$9,46.10^{-5}$	$1,03.10^{-8}$	$8,35.10^{-5}$

TABLE 4. The MSE (*Mean Squared Error*) for The Missile Guidance Estimation with EKF-UI-WDF Method

Simulation	θ_M	θ_T	u^*
1	$1,04.10^{-8}$	$8,42.10^{-9}$	$8,2161$
2	$8,98.10^{-9}$	$1,08.10^{-8}$	$7,7323$
3	$9,77.10^{-9}$	$1,22.10^{-8}$	$15,8603$
4	$1,23.10^{-8}$	$1,04.10^{-8}$	$12,7958$
5	$8,34.10^{-9}$	$1,21.10^{-8}$	$7,8962$
Average	1.10^{-8}	$1,08.10^{-8}$	$10,5$

TABLE 5. The Closest Range Missile-Target with EKF-UI-WDF Method

Simulation	The Closest Range Missile-Target (m)	Time (s)
1	0	$2,45$
2	$3,338$	$2,5$
3	$1,745$	$2,25$
4	$0,5487$	$2,35$
5	$0,0521$	$2,5$
Average	$1,14$	$2,41$

EKF Design Parameter. The initial conditions chosen for EKF design are

$$\hat{\mathbf{x}}_{0|0} = [90,\ -41,\ 0,\ 0,\ 0,\ \pi,\ 0]^T$$

The initial error covariance matrix is
$\mathbf{P}_{\mathbf{Z},0|0} = diag\left\{10^{-1}, 10^{-1}, 10^{-2}, 10^{-2}, 10^{-2}, 10^{-2}, 10^{-2}\right\}$.
The covariance matrix for the model noise $\mathbf{w}_{k-1}$ is chosen to be
$\mathbf{Q}_{k-1} = diag\left\{10^{-4}, 10^{-2}, 10^{-4}, 10^{-2}, 10^{-4}, 10^{-4}, 10^{-2}\right\}$. The covariance matrix for the measurement noise is chosen the same for EKF-UI-WDF design.

TABLE 6. The MSE (*Mean Squared Error*) for The Missile Guidance Estimation with EKF Method

Simulation	R	$\dot{R}$	λ	$\dot{\lambda}$
1	$9,38.10^{-9}$	$1,07.10^{-4}$	$9,62.10^{-9}$	$8,65.10^{-5}$
2	6.10^{-9}	$1,25.10^{-4}$	$1,49.10^{-8}$	$1,04.10^{-4}$
3	$8,28.10^{-9}$	$9,51.10^{-5}$	$9,3.10^{-9}$	$6,35.10^{-5}$
4	$1,33.10^{-8}$	$9,39.10^{-4}$	$6,86.10^{-9}$	$1,29.10^{-4}$
5	1.10^{-8}	$8,62.10^{-5}$	$6,02.10^{-9}$	$8,05.10^{-5}$
Average	$9,39.10^{-9}$	$2,7.10^{-4}$	$9,34.10^{-9}$	$9,27.10^{-5}$

It can be observed from Figures 1-2 that the optimal controller obtained in Eq. (10) with the EKF-UI-WDF method has better interception performance than the one with the EKF method (the closest missile-target ranges $R_{closest} = 3.338\,m$ at $t = 2.5\,s$ and $R_{closest} = 10.36\,m$ at $t = 2.5\,s$ for EKF-UI-WDF and EKF,

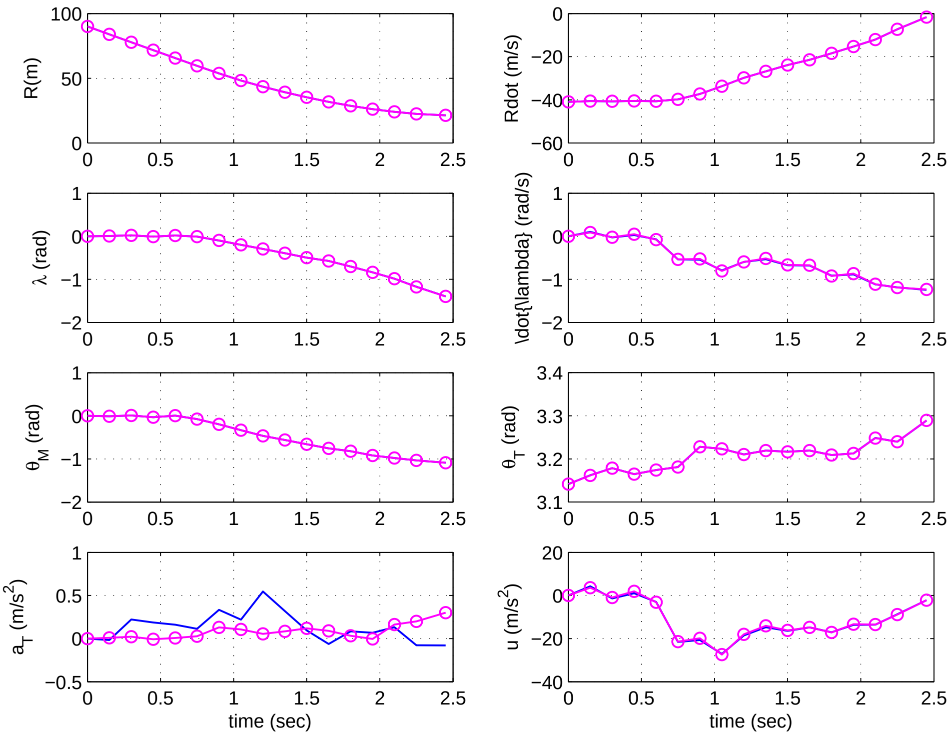

FIGURE 2. The estimated states and target acceleration for the missile guidance with EKF (blue curves for real and purple curves for estimation)

TABLE 7. The MSE (*Mean Squared Error*) for The Missile Guidance Estimation with EKF Method

Simulation	θ_M	θ_T	a_T
1	$1,15.10^{-8}$	$9,6.10^{-9}$	$0,0168$
2	$9,41.10^{-9}$	$6,44.10^{-9}$	$0,0974$
3	$7,69.10^{-9}$	$9,44.10^{-9}$	$0,13$
4	$1,12.10^{-8}$	$1,05.10^{-8}$	$5,2694$
5	$8,19.10^{-9}$	$1,02.10^{-8}$	$0,1169$
Average	$9,6.10^{-9}$	$9,24.10^{-9}$	$1,13$

respectively). The MSE of estimating target acceleration (unknown input in EKF-UI-WDF design) with EKF method is smaller than with EKF-UI-WDF method, because of the value Kalman gain for unknown input in EKF-UI-WDF method. Tables 2-6 show the computational results.

TABLE 8. The Closest Range Missile-Target with EKF Method

Simulation	The Closest Range Missile-Target (m)	Time (s)
1	$0,3401$	$2,35$
2	$10,36$	$2,5$
3	$0,2543$	$2,35$
4	$6,775$	$2,4$
5	$1,68$	$2,35$
Average	$3,882$	$2,4$

8. CONCLUDING REMARKS

The solution of the optimal missile guidance is derived analytically from time-varying linear state equations which composed of the line-of-sight angle and rate. The augmented proportional navigation law is the simplified version of the resulting optimal missile guidance which a unitless gain is 3. The computational results of the EKF-UI-WDF method to estimate the optimal missile guidance shows that the range to the missile-target is smaller than using the EKF. However the MSE (Mean Squared Error) for the optimal missile guidance estimation using EKF method is smaller than using EKF-UI-WDF method.

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