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e^*-HOLLOW-LIFTING and COFINITELY e^*-LIFTING MODULES

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Abstract. The novel ideas in module M over a ring R are introduced in this study. The first one, a generalization of the e^*-lifting module, is known as e^*-hollow-lifting. The second idea, an inference of e^*-lifting, is known as a cofinite e^*-lifting module. We shall demonstrate some of these ideas' properties.

Key words and Phrases: e^*-Lifting modules, Lifting modules, Hollow-lifting-modules, Cofinitely lifting module, e^*-Cofinitely lifting modules

1. INTRODUCTION

In this work M is a right module over a ring R with identity. $E(M)$ is the injective envelope of M. When $S + T = M$ implies $T = M$ for each $T \leq M$, S is called a small submodule of M, symbolized by $(S \ll M)$. See [8]. If $S \cap T \neq \{0\}$ for each $0 \neq T \leq M$, then S is called an essential submodule of M. See [8] and [7]. In [11], Özcan introduced a new type of submodules which defined as $Z^*(M) = \{a \in M | aR \text{ small in } E(M)\}$. If $Z^*(M) = M$, then M is called cosingular. In [2], Baanoon and Khalid introduced a class of submodules called e^*-essential. If $S \cap T \neq \{0\}$ for each cosingular T where $0 \neq T \leq M$, S is called an e^*-essential submodule of M, symbolized by $S \leq_{e^*} M$. In [3], Baanoon and Khalid used e^*-essential submodules to present a new class of submodules, a generalization of a small submodule, called e^*-essential small. If $S + T = M$ implies $T = M$ for each $T \leq_{e^*} M$, S is called an e^*-essential small submodule of M symbolized by $(S \ll_{e^*} M)$. The generalization of the radical submodule was introduced in [3], which is called e^*-radical denoted by, $\underset{e^*}{Rad}(M)$ and defined as the intersection of all

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e^*-essential maximal submodule of a module M. Equivalently, $\underset{e^*}{Rad}(M) = \sum_{N \ll_{e^*} M} N$. If each proper submodule of M is e^*-essential small, then M is anointed e^*-hollow, where M is a nonzero module. See [3]. If module M has a direct summand T such that $T \leq S$ and $\frac{S}{T} \ll \frac{M}{T}$ for each submodules S, it is said to be lifting. See [5]. Generalization of the lifting module introduced in [4], which is called e^*-lifting, defined as a module M is called e^*-lifting if for any submodule S of M there exists $T \leq S$ with $M = T \oplus T^{'}$ for some $T^{'} \leq M$ and $S \cap T^{'} \ll_{e^*} M$.

As in [14], [10], [9] and [13] we will use e^*-essential and e^*-essential small submodules to present a new generalization of lifting module and e^*-lifting. Namely e^*-hollow-lifting and cofinitely e^*-lifting modules. We will prove the main properties of these concepts.

2. e^*-Hollow-lifting modules

This section introduces generalizations for the e^*-lifting module with specific properties. The characteristics of e^*-essential small are listed below that appeared in [3].

Lemma 2.1. *Suppose M is a module.*

(1) If M is a simple module, then $M \ll_{e^*} M$.
(2) If $S \ll_{e^*} M$ and $f : M \to U$ is an R-homomorphism, then $f(S) \ll_{e^*} U$.
(3) The direct sum of two e^*-essential small submosules is e^*-essential small.

The following gives the properties of an e^*-lifting modules which appeared in [4].

Lemma 2.2. *The following are similar for a module M.*

(1) M is e^*-lifting.
(2) There is a decomposition $A = A_1 \oplus A_2$ such that A_1 is a direct summand of M and $A_2 \ll_{e^*} M$ for any submodule A of M.
(3) There exists $A_1 \leq A$ such that $M = A_1 \oplus A_2$, for some $A_2 \leq M$, and $\frac{A}{A_1} \ll_{e^*} \frac{M}{A_1}$ for every submodule A of M.

Definition 2.3. *For a module M. If every submodule S of M with $\frac{M}{S}$ is e^*-hollow, there exists $S_1 \leq S$ such that $M = S_1 \oplus S_2$ for some $S_2 \leq M$ and $S \cap S_2 \ll_{e^*} M$, then M is called e^*-hollow-lifting.*

The following describes e^*-hollow-lifting modules in considerable detail.

Theorem 2.4. *The following are similar for a module M.*

(1) M is e^*-hollow-lifting.

(2) *There is a decomposition* $A = A_1 \oplus A_2$ *such that* A_1 *is a direct summand of* M *and* $A_2 \ll_{e^*} M$ *for any submodule* A *of* M *with* $\frac{M}{A}$ *is* e^**-hollow.*
(3) *There is a submodule* $A_1 \leq A$ *which a direct summand of* M *such that* $\frac{A}{A_1} \ll_{e^*} \frac{M}{A_1}$ *for any submodule* A *of* M *with* $\frac{M}{A}$ *is* e^**-hollow.*

PROOF. $1 \Rightarrow 2$) Suppose that $A \leq M$ with $\frac{M}{A}$ is e^*-hollow. So there exists a submodule $A_1 \leq A$, such that $A_1 \oplus U = M$ for some submodule U of M and $A \cap U \ll_{e^*} M$. So $A = A \cap M = A \cap (A_1 \oplus U) = A_1 \oplus (A \cap U)$. Therefore, there exists a decomposition $A = A_1 \oplus A_2$ such that A_1 is a direct summand of M and $A \cap U = A_2 \ll_{e^*} M$.

$2 \Rightarrow 3$) By the hypothesis, there exists $A_1 \leq A$, such that $M = A_1 \oplus U$ for some submodule U of M, $A = A_1 \oplus A_2$, and $A_2 \ll_{e^*} M$. Let $\frac{W}{A_1} \leq_{e^*} \frac{M}{A_1}$ such that $\frac{M}{A_1} = \frac{W}{A_1} + \frac{A}{A_1}$, so $M = W + A = W + A_1 + A_2$. Then by Proposition 2 in [2] $W \leq_{e^*} M$. Also from Proposition 1 in [2], we have $W + A_1 \leq_{e^*} M$ since $W \leq (W + A_1) \leq M$. Now, since $A_2 \ll_{e^*} M$, so $M = W + A_1 = W$. Therefore, $\frac{M}{A_1} = \frac{W}{A_1}$ and $\frac{A}{A_1} \ll_{e^*} \frac{M}{A_1}$.

$3 \Rightarrow 1$) By the hypothesis, there exists $A_1 \leq A$ such that $A_1 \oplus U = M$ for some submodule U of M, and $\frac{A}{A_1} \ll_{e^*} \frac{M}{A_1}$. So $A = A \cap M = A \cap (A_1 \oplus U) = A_1 \oplus (A \cap U)$, $\frac{M}{A_1} = \frac{A_1 \oplus U}{A_1} \simeq \frac{U}{A_1 \cap U} \simeq U$, and $\frac{A}{A_1} \simeq A \cap U$. So $A \cap U \ll_{e^*} U \leq M$. Therefore, $A \cap U \ll_{e^*} M$ and M is e^*-hollow-lifting. □

Examples and Remarks 2.5.
(1) *Every module that doesn't have an* e^**-hollow factor is* e^**-hollow-lifting.*
(2) *Each* e^**-lifting module is* e^**-hollow-lifting. The opposite, however, need not always be true. For instance, let* M *be a non-zero indecomposable module with no* e^**-hollow factor. Hence* M *is* e^**-hollow-lifting. Declare that* M *is not* e^**-lifting. Assume* M *is* e^**-lifting and* A *is a proper submodule of* M*. Hence there exists a submodule* K *of* M *such that* $M = K \oplus W$ *for some a submodule* W *of* M *and* $\frac{A}{K} \ll_{e^*} \frac{M}{K}$*. As* M *is indecomposable,* $K = 0$ *and hence* $A \ll_{e^*} M$*. Thus* M *is* e^**-hollow which implies* $\frac{M}{A}$ *is* e^**-hollow, which is a contradiction; therefore,* M *is not* e^**-lifting.*
(3) *The* $\mathbb{Z}$*-module* $\mathbb{Z}$ *is not* e^**-hollow-lifting. For a submodule* $4\mathbb{Z}$*, since* $\frac{\mathbb{Z}}{4\mathbb{Z}} \simeq \mathbb{Z}_4$ *is* e^**-hollow, and the only direct summand contains in* $4\mathbb{Z}$ *is* $\{0\}$*.* $4\mathbb{Z}$ *is not* e^**-essential small in* $\mathbb{Z}$*. See* [3].
(4) *Every semisimple module is* e^**-hollow-lifting and every* e^**-hollow module is* e^**-hollow-lifting.*

Next, we will see when the e^*-lifting and e^*-hollow-lifting coincide.

Proposition 2.6. *Let* M_1 *and* M_2 *be* e^**-hollow modules. Then the following are equivalent for the module* $M = M_1 \oplus M_2$*.*

i. *M is an e^*-lifting module.*

ii. M is an e^*-hollow-lifting module.

Proof. $i \Rightarrow ii$) Obvious.
$ii \Rightarrow i$) Let $A \leq M$. Consider the natural projection homomorphism $\rho_1 : M \to M_1$ and $\rho_2 : M \to M_2$. We have two cases.
Case I: If $\rho_1(A) \neq M_1$ and $\rho_2(A) \neq M_2$, then $\rho_1(A) \ll_{e^*} M_1$ and $\rho_2(A) \ll_{e^*} M_2$ (since M_1 and M_2 are e^*-hollow modules). So by Lemma 2.1, $\rho_1(A) \oplus \rho_2(A) \ll_{e^*} M_1 \oplus M_2 = M$. Claim that $A \subseteq \rho_1(A) \oplus \rho_2(A)$. Let $a \in A$. Then $a \in M = M_1 \oplus M_2$, so $a = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Hence, $\rho_1(a) = m_1$ and $\rho_2(a) = m_2$ implies $a \in \rho_1(A) \oplus \rho_2(A)$. Hence $A \ll_{e^*} M$ by Proposition 1, in [3]. Thus M is e^*-lifting module.

Case II: If $\rho_1(A) = M_1$, then $M = A + M_2$ and $\frac{M}{A} = \frac{A+M_2}{A} = \frac{M_2}{A \cap M_2}$. Since M_2 is e^*-hollow. Hence $\frac{M_2}{A \cap M_2}$ is e^*-hollow see [[3],Corollary 3], so $\frac{M}{A}$ is e^*-hollow see [[3],Proposition 7]. But M is e^*-hollow-lifting, there exists $X \leq A$ such that $M = X \oplus X'$ for some $X' \leq M$, and $\frac{A}{X} \ll_{e^*} \frac{M}{X}$. Thus M is e^*-lifting module. □

Recall that if $f(S) \leq S$ for any endomorphism f of a module M, the submodule S of that module is said to be fully invariant. For more details about fully invariant submodule. See [1].
Now, to prove the next proposition, we need the following lemma.

Lemma 2.7. [9] *For a module M, if $M = M_1 \oplus M_2$, then $\frac{M}{S} = \frac{S+M_1}{S} \oplus \frac{S+M_2}{S}$ for any fully invariant S of M.*

Proposition 2.8. *If M is e*-hollow-lifting, then $\frac{M}{S}$ is an e*-hollow-lifting module for any fully invariant $S \leq M$.*

Proof. Let $\frac{L}{S} \leq \frac{M}{S}$ such that $\frac{\frac{M}{S}}{\frac{L}{S}} = \frac{M}{L}$ is e^*-hollow. From hypotheses M is e^*-hollow-lifting, there exist $S_1 \leq L$ such that $M = S_1 \oplus S_2$ for some $S_2 \leq M$, and $\frac{L}{S_1} \ll_{e^*} \frac{M}{S_1}$. By lemma 2.7, we have $\frac{M}{S} = \frac{S_1+S}{S} \oplus \frac{S_2+S}{S}$. Clearly $S_1 + S \leq L$ and $\frac{S_1+S}{S} \leq \frac{L}{S}$. So $\frac{\frac{L}{S}}{\frac{S_1+S}{S}} \simeq \frac{L}{S_1+S}$ and $\frac{\frac{M}{S}}{\frac{S_1+S}{S}} \simeq \frac{M}{S_1+S}$. Let $f : \frac{M}{S_1} \to \frac{M}{S_1+S}$ be an R-epimorphism defined by $f(m + S_1) = m + S_1 + S$ for each $m \in M$. Hence $f(\frac{L}{S_1}) = \frac{L}{S_1+S}$. Since $\frac{L}{S_1} \ll_{e^*} \frac{M}{S_1}$ implies that $\frac{L}{S_1+S} \ll_{e^*} \frac{M}{S_1+S}$ [Proposition 7, [3]]. Therefore $\frac{M}{S}$ is e^*-hollow-lifting. □

A module M is considered a duo if each submodule is fully invariant. See [12].
The following proposition gives a specific condition to make the direct sum of two e^*-hollow-lifting is an e^*-hollow-lifting module.

Proposition 2.9. *Assume that $M = S_1 \oplus S_2$ such that S_1 and S_2 are e*-hollow modules. If M is a duo module, then M is e*-hollow-lifting.*

Proof. Suppose that $U \leq M$ such that $\frac{M}{U}$ is e^*-hollow. Since M is duo module, then U is fully invariant and so $U = U \cap M = U \cap (S_1 \oplus S_2) = (U \cap S_1) \oplus (U \cap S_2)$.

Since S_1 and S_2 are e^*-hollow modules, then S_1 and S_2 are e^*-hollow-lifting modules and $\frac{S_1}{U\cap S_1}$, $\frac{S_2}{U\cap S_2}$ are e^*-hollow modules. See Proposition 7, in [3]. Hence there exists $K \leq (U\cap S_1) \leq U$ such that $K\oplus K_1 = S_1$ for some $K_1 \leq S_1$, and $(U\cap S_1)\cap K_1 \ll_{e^*} S_1$. Also, there exists $K^{'} \leq (U\cap S_2)$ such that $K^{'} \oplus K_1^{'} = S_2$ for some $K_1^{'} \leq S_2$, and $(U\cap S_2)\cap K_1^{'} \ll_{e^*} S_2$. So $M = S_1 \oplus S_2 = K\oplus K_1 \oplus K^{'} \oplus K_1^{'} = K\oplus K^{'} \oplus K_1 \oplus K_1^{'}$. It follows that $K\oplus K^{'} \leq U$. By corollary 1 in [3], $U\cap(K_1 \oplus K_1^{'}) = (U\cap S_1)\cap K_1 \oplus (U\cap S_2)\cap K_1^{'} \ll_{e^*} S_1 \oplus S_2 = M$. Thus, M is e^*-hollow-lifting. □

3. Cofinitely e^*-lifting modules

This section is devoted to introduce another generalizations for the e^*-lifting module with some properties. Remember that if the factor module $\frac{M}{S}$ is finitely generated, a submodule S of M is said to be cofinite in M. See [5].

Definition 3.1. *For a module M. If every cofinite submodule S of M has a direct summand $S_1 \leq S$ such that $M = S_1 \oplus S_2$ for some $S_2 \leq M$ and $S\cap S_2 \ll_{e^*} M$, then M is said to be cofinitely e^*-lifting.*

Examples and Remarks 3.2.

(1) *Eech e^*-lifting is a cofinitely e^*-lifting module. The opposite, however, need not always be true. For instance, the $\mathbb{Z}$-module $\mathbb{Q}$ is cofinitely e^*-lifting because $\mathbb{Q}$ is the only cofinite submodule, but not e^*-lifting* [4].

(2) *The $\mathbb{Z}$-module $\mathbb{Z}$ is not cofinitely e^*-lifting. Because $\frac{\mathbb{Z}}{4\mathbb{Z}} \simeq \mathbb{Z}_4$, so $4\mathbb{Z}$ is cofinite, but the only direct summand of $4\mathbb{Z}$ is $\{0\}$ and $4\mathbb{Z}$ is not e^*-essential small* [3].

(3) *From (1) and (2), we see the submodule of a cofinitely e^*-lifting module need not be cofinitely e^*-lifting. The $\mathbb{Z}$-module $\mathbb{Q}$ is cofinitely e^*-lifting, but the submodule $\mathbb{Z}$ is not cofinitely e^*-lifting.*

Theorem 3.3. *The following are similar for a module M.*

(1) *M is cofinitely e^*-lifting.*

(2) *There is a decomposition $A = A_1 \oplus A_2$ such that A_1 is a direct summand of M and $A_2 \ll_{e^*} M$ for any cofinite submodule A of M.*

(3) *There is a submodule $A_1 \leq A$ such that $M = A_1 \oplus A_2$ for some $A_2 \leq M$, and $\frac{A}{A_1} \ll_{e^*} \frac{M}{A_1}$ for every cofinite submodule A of M.*

PROOF. As in Theorem 2.4. □

Next, We think about the following issue: Whenever the submodule, direct summand, and factor modules inherit the cofinitely e^*-lifting condition?

Proposition 3.4. *Every cofinite direct summand e^*-essential of a cofinitely e^*-lifting module is cofinitely e^*-lifting.*

Proof. Suppose U is the cofinite direct summand e^*-essential of a module M where M is cofinitely e^*-lifting. Then $M = U \oplus T$, for some $T \leq M$, $U \leq_{e^*} M$ and $\frac{M}{U}$ is finitely generated. Let S be a cofinite submodule of U. Then $\frac{U}{S}$ is finitely generated. Now $M = U + (T + S)$ and by the modular law $U \cap (T \oplus S) = S$. Hence $\frac{M}{S} = \frac{U}{S} \oplus \frac{T \oplus S}{S}$. By the first isomorphism theorem we have $\frac{T\oplus S}{S} \simeq \frac{T}{T \cap S} = \frac{T}{\{0\}} \simeq T$ and $\frac{M}{U} = \frac{U+T}{U} \simeq \frac{T}{U \cap T} \simeq T$. So $\frac{M}{U} \simeq \frac{T \oplus S}{S}$ and $\frac{T\oplus S}{S}$ is finitely generated. Hence $\frac{M}{S}$ is finitely generated and S is a cofinite submodule in M. Since M is cofinitely e^*-lifting, there exists $Y \leq S$ such that $M = Y \oplus Y^{'}$ for some $Y^{'} \leq M$ and $S \cap Y^{'} \ll_{e^*} M$. So $U = U \cap M = U \cap (Y \oplus Y^{'}) = Y \oplus (Y^{'} \cap U)$. Claim that $S \cap Y^{'} \cap U \ll_{e^*} U$, since $S \cap Y^{'} \cap U = S \cap Y^{'} \ll_{e^*} M$, $S \cap Y^{'} \cap U \leq U \leq M$, and U is an e^*-essential direct summand of M by proposition 2 in [3], we have $S \cap Y^{'} \cap U \ll_{e^*} U$. Therefore, U is a cofinitely e^*-lifting module. □

Theorem 3.5. *If M is a cofinitely e^*-lifting module, then $\frac{M}{S}$ is cofinitely e^*-lifting for any fully invariant submodule S of M.*

Proof. Let $\frac{U}{S}$ be the cofinite submodule of $\frac{M}{S}$. Then U is a cofinite submdule of M since $\frac{\frac{M}{S}}{\frac{U}{S}} \simeq \frac{M}{U}$ is finitely generated. Since M is cofinitely e^*-lifting, there exist $Y \leq U$ such that $M = Y \oplus Y^{'}$ for some $Y^{'} \leq M$, and $\frac{U}{Y} \ll_{e^*} \frac{M}{Y}$. Now, by Lemma 2.7. $\frac{M}{S} = \frac{Y+S}{S} \oplus \frac{Y^{'}+S}{S}$ with $Y + S \leq U$ and $\frac{Y+S}{S} \leq \frac{U}{S}$. So $\frac{\frac{U}{S}}{\frac{Y+S}{S}} \simeq \frac{U}{Y+S}$ and $\frac{\frac{M}{S}}{\frac{Y+S}{S}} \simeq \frac{M}{Y+S}$, since $\frac{U}{Y} \ll_{e^*} \frac{M}{Y}$ by Proposition 7, in [3] implies $\frac{U}{Y+S} \ll_{e^*} \frac{M}{Y+S}$. Therefore, $\frac{M}{S}$ is cofinitely e^*-lifting. □

Corollary 3.6. *Let M be a cofinitely e^*-lifting module. Then $\frac{M}{Rad_{e^*}(M)}$ is cofinitely e^*-lifting module.*

If the total of two direct summands of a module M is likewise a direct summand of that module, that module is said to have the summand sum property. See [6].

Proposition 3.7. *Let S be a direct summand of M and M be a cofinitely e^*-lifting module. If M possesses the summand sum property, then $\frac{M}{S}$ is cofinitely e^*-lifting.*

Proof. Assume that $\frac{U}{S}$ is the cofinite submodule of $\frac{M}{S}$. So U is a cofinite submodule of M since $\frac{\frac{M}{S}}{\frac{U}{S}} \simeq \frac{M}{U}$ is finitely generated. Since M is cofinitely e^*-lifting, there exists $B \leq U$ such that $M = B \oplus B^{'}$, for some $B^{'} \leq M$, and $\frac{U}{B} \ll_{e^*} \frac{M}{B}$. Since M has the summand sum property, $B + S$ is a direct summand of M and $\frac{B+S}{S}$ is a direct summand of $\frac{M}{S}$ with $\frac{B+S}{S} \leq \frac{U}{S}$. So $\frac{\frac{U}{S}}{\frac{B+S}{S}} \simeq \frac{U}{B+S}$ and $\frac{\frac{M}{S}}{\frac{B+S}{S}} \simeq \frac{M}{B+S}$, since $\frac{U}{B} \ll_{e^*} \frac{M}{B}$, by Proposition 7, in [3] implies $\frac{U}{B+S} \ll_{e^*} \frac{M}{B+S}$. Therefore, $\frac{M}{S}$ is cofinitely e^*-lifting. □

We now look at whether direct sums inherit the cofinitely e^*-lifting property.

Theorem 3.8. *If $M = S_1 \oplus S_2$ is a duo module; if S_1 and S_2 are cofinitely e^*-lifting modules, then M is also a cofinitely e^*-lifting module.*

PROOF. Suppose that $U \leq M$ is cofinite. So $\frac{M}{U}$ is finitely generated and $U = U \cap M = U \cap (S_1 \oplus S_2)$. Because M is a duo module, $U = (U \cap S_1) \oplus (U \cap S_2)$. Also by Lemma 2.7, $\frac{M}{U} = \frac{S_1+U}{U} \oplus \frac{S_2+U}{U}$. Then $\frac{S_1+U}{U} \simeq \frac{S_1}{S_1 \cap U}$ and $\frac{S_2+U}{U} \simeq \frac{S_2}{S_2 \cap U}$ are finitely generated. Hence $U \cap S_i$ is a cofinite submodule of S_i, Since S_i is cofinitely e^*-lifting for $i = 1, 2$. Then there exists $D_i \leq U \cap S_i \leq U$ such that $S_i = D_i \oplus D_i^{'}$ for some $D_i^{'} \leq S_i$ and $U \cap S_i \cap D_i^{'} = U \cap D_i^{'} \ll_{e^*} S_i$, for $i = 1, 2$. Hence $M = (D_1 \oplus D_1^{'}) \oplus (D_2 \oplus D_2^{'}) = (D_1 \oplus D_2) \oplus (D_1^{'} \oplus D_2^{'})$, $D_1 \oplus D_2 \leq U$. $U \cap (D_1^{'} \oplus D_2^{'}) = (U \cap D_1^{'}) \oplus (U \cap D_2^{'}) \ll_{e^*} S_1 \oplus S_2 = M$. Therefore, M is cofinitely e^*-lifting. □

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