S-PRIME IDEALS IN PRINCIPAL DOMAIN

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Abstract. Let $R$ be a commutative ring and $S$ be a multiplicative subset of $R$. The $S$-prime ideal is a generalization of the concept of prime ideal. In this paper, we completely determine all $S$-prime and $S$-maximal ideals of a principal domain. It is shown that the intersection of any descending chain of $S$-prime ideals in a principal domain is an $S$-prime ideal, also the $S$-radical is investigated.

Key words and Phrases: Principal domain, $S$-prime ideal, $S$-maximal ideal, $S$-radical.

1. Introduction

Throughout this paper all rings are commutative with identity $\neq 0$. Let $R$ be a commutative ring and $S$ be a multiplicative subset of $R$. Recently, Sevim et al. [11], studied the concept of $S$-prime ideal which is a generalization of prime ideal and used it to characterize integral domains, certain prime ideals, fields and $S$-Noetherian rings. An ideal $P$ with $P \cap S = \emptyset$ is said to be $S$-prime ideal if there exists an element $s \in S$ such that, whenever $a, b \in R$, if $ab \in P$ then $sa \in P$ or $sb \in P$. Note that if $S$ consist of units of $R$, then the notions of $S$-prime ideal and prime ideal coincide. Recall from [4] that an ideal $P$ of $R$ is said to be $S$-maximal ideal if $P \cap S = \emptyset$ and there exists $s \in S$ such that whenever $P \subseteq Q$ for some ideal $Q$ of $R$, then either $sQ \subseteq P$ or $Q \cap S \neq \emptyset$. The $S$-radical of an ideal $I$ is defined by $\sqrt{S} = \{a \in R / sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$. In this paper we study the concept of $S$-prime ideal in a principal ideal domain, for instance, we completely determine all $S$-prime ideals of a principal ideal domain. In [4], the author showed that any $S$-maximal ideal is $S$-prime. If $R$ is a principal ideal domain, we show that every non-zero $S$-prime ideal is $S$-maximal. Also the $S$-radical of an ideal is given.

Recall from [5] that a multiplicative subset $S$ of $R$ is said to be strongly multiplicative if for each family $(s_\alpha)_{\alpha \in \Lambda}$ we have $\cap_{\alpha \in \Lambda} (s_\alpha R) \cap S \neq \emptyset$. In [5], the

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author showed that if \( S \) is a strongly multiplicative subset, then the intersection of any chain of \( S \)-prime ideals is an \( S \)-prime ideal, and in particular, any ideal disjoint with \( S \) is contained in a minimal \( S \)-prime ideal. Then the author asked the following question;

**Question:** Is the assumption “\( S \) strongly multiplicative subset” necessary for the theorem?

As part of our study, we give a negative answer to this question.

Her, we fix some notations that will be used throughout this paper. If \( R \) is a principal ideal domain. The set of all irreducible (prime) elements of \( R \) is denoted by \( \mathbb{P} \). For a multiplicative subset \( S \) the set \( \mathbb{P}_S \) is defined by \( \mathbb{P}_S = \{ p \in \mathbb{P} / \mathbb{P} \cap S \neq \emptyset \} \), that is, \( \mathbb{P}_S \) is the set of all irreducible elements of \( R \) that belong to some element of \( S \). An irreducible element \( p \) is in \( \mathbb{P}_S \) if and only if there exists \( s \in S \) and \( b \in R \) such that \( s = bp \). Note that if \( S = R \setminus \{ 0 \} \), then \( \mathbb{P}_S = \mathbb{P} \).

### 2. \( S \)-prime ideal in principal domain

We start this section by recalling the concept of \( S \)-prime ideals of a commutative ring \( R \) in order to give the form of all \( S \)-prime ideals in principal ideal domain.

**Definition 2.1.** Let \( R \) be a commutative ring, \( S \) be a multiplicative subset of \( R \) and \( P \) be an ideal of \( R \) disjoint with \( S \). Then \( P \) is said to be \( S \)-prime ideal if there exists an \( s \in S \) such that for all \( a, b \in R \) with \( ab \in P \), we have \( sa \in P \) or \( sb \in P \).

The following result will be frequently used and can be found in [5].

**Proposition 2.2.** Let \( R \) be a commutative ring and \( S \) be a multiplicative subset of \( R \). Let \( P \) be an ideal of \( R \). The following statements are equivalent

1. \( P \) is an \( S \)-prime ideal of \( R \).
2. There exists \( s \in S \) such that \( (P : s) \) is a prime ideal of \( R \).

The \( S \)-prime ideals of a principal ideal domain are completely determined in the following result.

**Theorem 2.3.** Let \( R \) be a principal ideal domain and \( S \) be a multiplicative subset of \( R \) and let \( I \) be an ideal of \( R \). The following statements are equivalent:

1. \( I \) is an \( S \)-prime ideal of \( R \),
2. \( I = (0) \) or \( I = (vp) \) for some \( p \in \mathbb{P} - \mathbb{P}_S \) and \( v \in R \) such that \( (v) \cap S \neq \emptyset \).

**Proof.** (2) \( \Rightarrow \) (1). If \( I = (0) \), then \( I \) is an \( S \)-prime ideal since it is a prime ideal. Now, let \( I = (vp) \) where \( p \in \mathbb{P} - \mathbb{P}_S \) and \( (v) \cap S \neq \emptyset \). There exists an \( s_0 \in S \) and \( v' \in R \) such that \( s_0 = vv' \). Let \( x \in (I : s_0) \), then \( xs_0 \in I \) so \( xs_0 = \alpha vp \) for some \( \alpha \in R \), therefore \( s_0x \in (p) \), hence \( x \in (p) \) since \( s_0 \notin (p) \). It follows that \( (I : s_0) \subseteq (p) \). On the other hand, we have \( ps_0 = v'vp \in I \), that is \( p \in (I : s_0) \), so that \( (I : s_0) = (p) \) is a prime ideal of \( R \). Thus \( I \) is an \( S \)-prime ideal of \( R \).

(1) \( \Rightarrow \) (2). Let \( I = (a) \) be a non-zero \( S \)-prime ideal of \( R \). Let \( s_0 \in S \) such that \( (I : s_0) \) is a prime ideal of \( R \). Since \( (0) \neq I \subseteq (I : s_0) \) there exists an irreducible
element \( p \) of \( R \) such that \((I : s_0) = (p)\). As \( ps_0 \in I \), we have \( ps_0 = a'a \) for some \( a' \in R \), in particular \( a'a \in (p) \) so \( a' \in (p) \) or \( a \in (p) \). If \( a' \in (p) \), then \( a' = a''p \) where \( a'' \in R \), that is \( ps_0 = a'a = ad''p \), so that \( s_0 = a'a \in (a) \cap S \), a contradiction. Thus \( a \in (p) \), hence \( a = vp \) where \( v \in R \). Now \( ps_0 = a'a = a'vp \), so \( s_0 = a'v \), that is \((v) \cap S \neq \emptyset \). It follows that \( I = (vp) \) and \((v) \cap S \neq \emptyset \) and \( p \in \mathbb{P} - \mathbb{P}_S \); in fact if \( p \in \mathbb{P}_S \) then \((p) \cap S \neq \emptyset \), so there is an element \( s \in S \) such that \( s = cp \) where \( c \in R \). Then clearly \( ss_0 = cs_0p \in I \), which is not compatible with the fact that \( I \cap S = \emptyset \).

\[ \square \]

**Remark 2.4.**  
(1) If \( S \) is a multiplicative subset of a commutative ring \( R \), then there exists a saturated multiplicative subset \( S' \) of \( R \) such that \( \text{Spec}_{S} R = \text{Spec}_{S'} R \) (see the appendix).

(2) If \( S \) is a saturated multiplicative subset of a principal ideal domain \( R \). Then an ideal \( P \) is \( S \)-prime if and only if \( P \) is the zero ideal or \( P = (sP) \) where \( s \in S \) and \( p \in \mathbb{P} - \mathbb{P}_S \).

**Example 2.5.** Let \( R = \mathbb{Z} \) and \( S = \{2^k : k \in \mathbb{N}\} \). Note that \( \mathbb{P}_S = \{2\} \). Let \( I \) be a non-zero \( S \)-prime ideal of \( \mathbb{Z} \). Then \( I = (vp) \) where \( p \) is a prime integer and \( v \in \mathbb{Z} \) such that \( p \neq 2 \) and \((v) \cap S \neq \emptyset \) that is \( mv = 2^k \) for some \( m \in \mathbb{Z} \) and \( k \in \mathbb{N} \). Thus \( v = \pm 2^l \) for some \( l \in \mathbb{N} \). It follows that the \( S \)-prime ideals of \( \mathbb{Z} \) are the zero ideal and the ideals of the form \((2^l p) \) where \( p \neq 2 \) is a prime integer and \( l \in \mathbb{N} \).

**Lemma 2.6.** Let \( R \) be a principal ideal domain. Let \((I_n)_{n \in \mathbb{N}} \) be a descending chain of ideals of \( R \). Then \( I_n \) stabilize or \( \bigcap_n I_n = (0) \).

**Proof.** Let \( I = (a) = \bigcap_n I_n \) and assume that \( a \neq 0 \). If \( a \) is invertible, then the chain stabilize. If \( a \) is not invertible, consider the commutative ring \( R' = R/(a) \). Then \( R' \) is Noetherian and \( \dim R' = 0 \), so \( R' \) is an Artinian ring. Thus \( \gamma_n \) stabilize (in \( R' \)). There exists \( N \) such that for all \( n \geq N \), \( \gamma_n = \gamma_N \), so \( I_n = I_N \).

\[ \square \]

**Proposition 2.7.** Let \( R \) be a principal ideal domain. If \((Q_n)_{n \in \mathbb{N}} \) is a descending chain of \( S \)-prime ideals of \( R \), then \( \bigcap_n Q_n \) is an \( S \)-prime ideal of \( R \).

**Proof.** This follows from the previous lemma.

\[ \square \]

**Theorem 2.8.** Let \( R \) be a principal ideal domain. Then every ideal which is disjoint with \( S \) is contained in a minimal \( S \)-prime ideal.

**Proof.** Let \( I \) be an ideal of \( R \) with \( I \cap S = \emptyset \). Let

\[ \Gamma = \{Q /Q \text{ is an } S \text{-prime ideal and } I \subseteq Q\} \]

Note that \( \Gamma \) is not empty since \( I \subseteq P \) for some prime ideal \( P \) of \( R \) with \( P \cap S \neq \emptyset \), which is an \( S \)-prime ideal of \( R \). If \((Q_n)_{n} \) is a descending chain of \( S \)-prime ideals of \( R \) containing \( I \), then by the previous Proposition, \( Q = \bigcap_n Q_n \) is an \( S \)-prime ideal containing \( I \). By applying the Zorn’s lemma, we get the desired results.

\[ \square \]

**Proposition 2.9.** Let \( R \) be a principal ideal domain and \( S \) be a multiplicative subset of \( R \). The following statements are equivalent.

(1) \( S \) is a strongly multiplicative subset of \( R \).
(2) $S \subseteq U(R)$, where $U(R)$ is the set of invertible elements of $R$.

Proof. If $S \subseteq U(R)$, then $S$ is clearly a strongly multiplicative subset since for any $s \in S$ we have $sR = R$. Now, assume that $S \not\subseteq U(R)$. Then there exists a nonzero element $s \in S$ which is not invertible. Let $p \in P$ such that $(s) \subseteq (p)$, then for any $n \in \mathbb{N}$, $(s^n) \subseteq (p^n)$, thus $\cap_{n\in\mathbb{N}}(s^n) \subseteq \cap_{n\in\mathbb{N}}(p^n) = (0)$, in particular $\cap_{n\in\mathbb{N}}(s^n) \cap S = \emptyset$. Thus $S$ is not a strongly multiplicative subset of $R$. \hfill $\square$

Example 2.10. Let $p$ be an irreducible element of a principal ideal domain $R$ and $S = \{p^n : n \in \mathbb{N}\}$. Then $S$ is not a strongly multiplicative subset since $\cap_{n\in\mathbb{N}}(p^nR) \cap S = \emptyset$. But the intersection of a chain of $S$-prime ideals $pR$ is an $S$-prime ideal of $R$.

3. $S$-maximal ideal in principal domain

Definition 3.1. Let $R$ be a commutative ring and $S$ be a multiplicative subset. Let $P$ be an ideal of $R$ with $P \cap S = \emptyset$. Then $P$ is said to be an $S$-maximal ideal of $R$ if there exists $s \in S$ such that whenever $P \subseteq Q$ for some ideal $Q$ of $R$ then either $sQ \subseteq P$ or $Q \cap S \neq \emptyset$.

Remark 3.2. Every $S$-maximal ideal of $R$ is an $S$-prime ideal of $R$ (see [4]).

Lemma 3.3. Let $R$ be a principal ideal domain and $S$ be a multiplicative subset of $R$. Then $(0)$ is an $S$-maximal ideal of $R$ if and only if $P_S = P$.

Proof. If $(0)$ is an $S$-maximal ideal of $R$ and $p \in P$ then $(p) \cap S \neq \emptyset$ since $s(p) \not\subseteq (0)$, so $p \in P_S$. Conversely, assume that $P_S = P$. Let $Q = (a)$ be an ideal of $R$. If $a = 0$ then $1(Q) = (0) \subseteq (0)$. If $a \neq 0$. Then either $Q = R$, in this case $Q \cap S \neq \emptyset$, or $Q \neq R$, in this case $a = p_1^{n_1} \cdots p_m^{n_m}$ where $p_1, \cdots, p_m \in P$ and $n_1, \cdots, n_m$ are positive integers. Since $p_i \in P = P_S$ there exists $\alpha_i \in R$ such that $\alpha_i p_i \in S$, so $\prod_{i=1}^{m}(\alpha_i p_i)^{n_i} \in (a) \cap S$. Thus $Q \cap S \neq \emptyset$. \hfill $\square$

Classically, in a principal ideal domain every non-zero prime ideal is a maximal ideal, it’s $S$-version is the following result.

Theorem 3.4. Let $R$ be a principal ideal domain. Then every non-zero $S$-prime ideal is an $S$-maximal ideal.

Proof. Let $P$ be a non-zero $S$-prime ideal of $R$. Then $P = (vp)$ for some $p \in P - P_S$ and $v \in R$ with $(v) \cap S \neq \emptyset$. Let $Q = (a)$ be an ideal of $R$ with $P \subseteq Q$. Since $vp \in (a)$, $vp = ba$ for some $b \in R$. In particular $ab \in (p)$, so $a \in (p)$ or $b \in (p)$.

First case, if $a \in (p)$, then $a = a'p$ for some $a' \in R$, so that $vp = ba'p$, thus $v = ba'$. As $(v) \cap S \neq \emptyset$, there exists $t \in R$ such that $s = tv = tba' \in S$. Therefore $sa = twa'p \in (vp)$. It follows that $sQ \subseteq P$.

Second case, if $a \not\in (p)$, then $b \in (p)$. So $b = b'p$ for some $b' \in R$. Hence $v = b'a$ since $vp = ba = b'ap$. Thus $\emptyset \neq (v) \cap S \subseteq (a) \cap S$. It follows that $Q \cap S \neq \emptyset$. \hfill $\square$
4. S-Radical in Principal Domain

Definition 4.1. Let $R$ be a commutative ring and $S$ be a multiplicative subset of $R$. The $S$-radical of an ideal $I$ is defined by

$$\sqrt{S}I = \{a \in R \mid sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$$

Theorem 4.2. Let $R$ be a principal ideal domain and $S$ be a multiplicative subset of $R$. Let $I = (a)$ be a proper ideal of $R$ write $a = \prod_{j=1}^{m} q_j^{m_j} \prod_{i=1}^{d} p_i^{n_i}$ where $q_j \in \mathbb{P}_S$ and $p_i \in \mathbb{P} - \mathbb{P}_S$. Then $\sqrt{S}I = (\prod_{i=1}^{d} p_i)$.

Proof. Since $q_j \in \mathbb{P}_S$, there exists $\alpha_j \in R$ such that $\alpha_j q_j \in S$. Let $n = \max(n_i)$, then $\prod_{j=1}^{n} (\alpha_j q_j)^{n_j} (\prod_{i=1}^{d} p_i)^{n_i} \in I$, thus $\prod_{i=1}^{d} p_i \in \sqrt{S}I$, that is $(\prod_{i=1}^{d} p_i) \subseteq \sqrt{S}I$.

Conversely, let $x \in \sqrt{S}I$, then $sx^n \in I$ for some $s \in S$ and $n \in \mathbb{N}$. Let $b \in R$ such that $sx^n = b \prod_{j=1}^{m} q_j^{m_j} \prod_{i=1}^{d} p_i^{n_i}$. Then for each $1 \leq i \leq d$, $sx^n \in (p_i)$, since $(p_i) \cap N = 0$ and $(p_i)$ is a prime ideal of $R$, we have $x^n \in (p_i)$, so $x \in (p_i)$. Thus $x \in \cap_{i=1}^{d} (p_i) = (\prod_{i=1}^{d} p_i)$. It follows that $\sqrt{S}I = (\prod_{i=1}^{d} p_i)$. \hfill \Box

5. Appendix

Here we show, to studying the concept of $S$-prime ideal, we can always assume that the multiplicative subset $S$ is saturated. So, for a multiplicative subset $S$ of a commutative ring $R$, denote $S'$ the set defined by $S' = \{a \in R \mid (a) \cap S \neq \emptyset\}$.

Proposition 5.1. With the previous notations, we have

1. $S \subseteq S'$ and $S'$ is a saturated multiplicative subset.
2. If $I$ is an ideal of $R$, then $I \cap S = \emptyset$ if and only if $I \cap S' = \emptyset$.
3. If $P$ is an ideal of $R$, then $P$ is $S'$-prime if and only if $P$ is $S'$-prime.
4. If $P$ is an ideal of $R$, then $P$ is $S'$-maximal if and only if $P$ is $S'$-maximal.
5. If $I$ is an ideal of $R$, then $\sqrt{S}I = \sqrt{S}'.

Proof. (1) Clearly $S \subseteq S'$, $0 \notin S'$ and $1 \in S'$. If $a, b \in S'$, then $aa' \in S$ and $bb' \in S'$ for some $a', b' \in R$, so $(a'b')(ab) \in S$, that is $ab \in S'$. If $ab \in S'$, then $ab \in S$ for some $t \in R$, so $a, b \in S'$.

(2) Clearly, if $I \cap S = \emptyset$, then $I \cap S' = \emptyset$. If $I \cap S' = \emptyset$, then there exists $i \in I$ such that $(i) \cap S' = \emptyset$, so $ia \in S$ for some $a \in R$. Thus $ia \in I \cap S$.

(3) If $P$ is an $S'$-prime ideal of $R$, then it is easy to see that $P$ is also an $S'$-prime ideal of $R$. Conversely, assume that $P$ is an $S'$-prime ideal of $R$. Then $(P : s')$ is a prime ideal for some $s' \in S'$. We have $ts' \in S$ for some $t \in R$. Now we show that $(P : ts') = (P : s')$. If $x \in (P : ts')$, then $xts' \in P$, so $xt \in (P : s')$, since $t \notin (P : s')$, we have $x \in (P : s')$, hence $(P : ts') \subseteq (P : s')$. If $x \in (P : s')$, then $xts' \in P$, so $xt \in (P : s')$. It follows that $(P : ts')$ is a prime ideal of $R$, therefore $P$ is an $S'$-prime ideal of $R$.

(4) If $P$ is an $S'$-maximal ideal. We fix an element $s \in S$ as in the definition, in particular $s \in S'$. If $P \subseteq Q$ and $Q \cap S' = \emptyset$, then $Q \cap S = \emptyset$, so $sQ \subseteq P$. It follows that $P$ is a $S'$-maximal ideal of $R$. Now, assume that $Q$ is an
$S'$-maximal ideal and fix $s' \in S'$ as in the definition. There exits $t \in R$ such that $ts' \in S$. If $P \subseteq Q$ with $Q \cap S = \emptyset$, then $Q \cap S' = \emptyset$, so $s'Q \subseteq P$, thus $stQ \subseteq tP \subseteq P$.

(5) From the definition we have $\sqrt{I} \subseteq \sqrt{I'}$. Let $x \in \sqrt{I}$, then $s'x^n \in I$ for some $s' \in S'$ and $n \in \mathbb{N}$. There exists $t \in R$ such that $ts' \in S$, so $ts'x^n \in I$, thus $x \in \sqrt{I}$.

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