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n-BOUNDEDNESS AND n-CONTINUITY OF LINEAR OPERATORS

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Abstract. The concept of n-bounded and n-continuous operators is discussed as an extension of the concept introduced in [12]. The equivalence of three statements on n-continuity and n-boundedness of a linear operator from a normed space into an n-normed space is also proved. This newly introduced concept is proved to be identical to one type of n-continuity introduced in [12].

Key words and Phrases: n-normed space, n-bounded operator, n-continuous operator.

1. INTRODUCTION

Let X be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions:

$(2N_1)$ $\|x, y\| = 0$ if and only if x and y are linearly dependent.
$(2N_2)$ $\|x, y\| = \|y, x\|$.
$(2N_3)$ $\|\alpha x, y\| = |\alpha|\|x, y\| \ \forall\, x, y \in X$ and $\alpha \in \mathbb{R}$.
$(2N_4)$ $\|x + y, z\| \leq \|x, z\| + \|y, z\| \ \forall\, x, y, z \in X$.

Then, $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. 2-norms are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$.

The concept of 2-normed spaces was initially investigated and developed by Gähler in 1960s and has been extensively developed by Diminnie, Gähler, White and many others[1, 2, 13].

Let X be a real vector space with $\dim X \geq n$ where n is a positive integer. A real valued function $\|., ..., .\| : X^n \to \mathbb{R}$ is called an n-norm on X if the following conditions hold:

(1) $\|x_1, \ldots, x_n\| = 0$ iff $x_1, \ldots, x_n$ are linearly dependent.

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(2) $\|x_1, \ldots, x_n\|$ remains invariant under permutations of $x_1, \ldots, x_n$.
(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\| \, \forall x_1, \ldots, x_n \in X$ and $\alpha \in \mathbb{R}$.
(4) $\|x_0 + x_1, x_2, \ldots, x_n\| \leq \|x_0, \ldots, x_n\| + \|x_1, \ldots, x_n\|$ for all $x_0, x_1, \ldots, x_n \in X$.

The pair $(X, \|., \ldots, .\|)$ is called an n-normed space.

Let X be a real vector space with dim$X \geq n$, n is a poitive integer and be equipped with an inner product $\langle ., . \rangle$.Then the standard n-norm on X is given by

$$\|x_1, \ldots, x_n\|^{\mathrm{S}} = \sqrt{\det[\langle x_i, x_j \rangle]}.$$

A standard example of an n-normed space is $X = \mathbb{R}^n$ equipped with the Euclidean n-norm:

$$\|x_1, \ldots, x_n\|^E = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, ..., n$.

Note that the value of $\|x_1, ..., x_n\|^{\mathrm{S}}$ represents the volume of n-dimensional parallelepiped spanned by $x_1, ..., x_n$.

Gähler was the first to develop theories of n-normed spaces in 1960s [3, 4, 5] and later, Misiak [10] developed the theory more extensively . Notion of boundedness in 2-normed space was then introduced by White [13].

Gozali et al. also introduced the notion of bounded n-linear functionals in n-normed spaces in [6]. Zofia Lewandowska introduced notions of 2-linear operators on 2-normed sets in [9]. Soenjaya then introduced the notions of continuity and boundedness of n-linear operators in [12].

2. PRELIMINARIES

From the work of Soenjaya in [12], we have the following definitions and theorem.

Let $(X, \|.\|)$ and $(X, \|., ..., .\|)$ be respectively a normed space and an n-normed space.

Definition 2.1. *An operator $T : (X, \|.\|) \to (X, \|., ..., .\|)$ is n-bounded of type-A if there is a constant K such that for all $x_1, x_2, ..., x_n \in X$,*

$$\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\| \leq K\|x_1\|...\|x_n\|.$$

Definition 2.2. *If T is an n-bounded operator of type-A, define $\|T\|_n^A$ by*

$$\|T\|_n^A = \sup\{\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1\|...\|x_n\|} : x_1, x_2, ..., x_n \in X, \|x_1\|...\|x_n\| \neq 0\}$$

Definition 2.3. *An operator $T : (X, \|.\|) \to (X, \|., ..., .\|)$ is n-continuous of type-A at $x \in X$ if for all $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\|Tx_1 - Tx, x_2 - x, ..., x_n - x\| + \|x_1 - x, Tx_2 - Tx, ..., x_n - x\| + ... + \|x_1 - x, x_2 - x, ..., Tx_n - Tx\| < \epsilon$$

whenever $\|x_1 - x\|\|x_2 - x\|...\|x_n - x\| < \delta$, where $x_1, x_2, ..., x_n \in X$.

T is n-continuous of type-A if it is n-continuous of type-A at each $x \in X$.

Let $(X, \|., ..., .\|)$ and $(Y, \|., ..., .\|)$ be n-normed spaces.

Definition 2.4. *An operator $T : (X, \|., ..., .\|) \to (Y, \|., ..., .\|)$ is n-bounded of type-B if there is a constant K such that for all $x_1, \cdots, x_n \in X$,*

$$\|Tx_1, \cdots, Tx_n\| \leq K\|x_1, \cdots, x_n\|.$$

Definition 2.5. *If T is an n-bounded of type-B, define $\|T\|_n^B$ by*

$$\|T\|_n^B = \sup_{\|x_1, \cdots, x_n\| \neq 0} \frac{\|Tx_1, \cdots, Tx_n\|}{\|x_1, \cdots, x_n\|}$$

Definition 2.6. *Let $T : X \to Y$ be an operator. T is n-continuous of type-B at $x \in X$ if for $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\|Tx_1 - Tx, Tx_2 - Tx, \cdots, Tx_n - Tx\| < \epsilon$$

whenever

$$\|x_1 - x, x_2 - x, \cdots, x_n - x\| < \delta$$

.

T is n-continuous of type-B on X if it is n-continuous of type-B at each $x \in X$.

When $n = 1$, it is reduced to usual notion of continuity in normed space.

Definition 2.7. *An operator $T : (X, \|., ..., .\|) \to (X, \|., ..., .\|)$ is n-bounded of type-C if there is a constant K such that for all $x_1, x_2, ..., x_n \in X$,*

$$\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\| \leq K\|x_1, ..., x_n\|.$$

Definition 2.8. *T is an n-bounded operator, define $\|T\|_n^C$ by*

$$\|T\|_n^C = \sup\{\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1, ..., x_n\|} : x_1, x_2, ..., x_n \in X, \|x_1, ..., x_n\| \neq 0\}$$

Definition 2.9. *An operator $T : (X, \|., ..., .\|) \to (X, \|., ..., .\|)$ is n-continuous of type C at $x \in X$ if for all $\epsilon > 0$, there is a $\delta > 0$ such that*
$\|Tx_1 - Tx, x_2 - x, ..., x_n - x\| + \|x_1 - x, Tx_2 - Tx, ..., x_n - x\| + ... + \|x_1 - x, x_2 - x, ..., Tx_n - Tx\| < \epsilon$
whenever $\|x_1 - x, x_2 - x, ..., x_n - x\| < \delta$, where $x_1, x_2, ..., x_n \in X$.
T is n-continuous of type-C if it is n-continuous of type-C at each $x \in X$.

Using this concept, we extend the following works on n-boundedness and n continuity.

3. MAIN RESULTS

In this work, we discuss the notion of n-boundedness and n-continuity of linear operators as an extension of the work of Soenjaya in [12].
We insert a new type of n-continuity by defining an n-bounded operator from a normed space into an n-normed space and duscuss its relationship with the previously defined n-bounded operators in [12].

Let $(X, \|.\|)$ and $(Y, \|., ..., .\|)$ be respectively a normed space and an n-normed space.

Definition 3.1. *An operator $T : (X, \|.\|) \to (Y, \|., ..., .\|)$ is n-bounded of type-D if there is a constant K such that for all $x_1, \cdots, x_n \in X$,*

$$\|Tx_1, \cdots, Tx_n\| \leq K\|x_1\| \cdots \|x_n\|.$$

Definition 3.2. *If T is n-bounded of type-D, define $\|T\|_n^D$ by*

$$\|T\|_n^D = \sup_{x_i \in X, \|x_i\| \neq 0} \frac{\|Tx_1, \cdots, Tx_n\|}{\|x_1\| \cdots \|x_n\|}.$$

Example 3.3. *Let X be an inner product space equipped with standard n-norm $\|., ..., .\|^S$ and $T : (X, \|.\|) \to (X, \|., ..., .\|^S)$be an operator such that $Tx = cx \; \forall \; x \in X$ and $c \in \mathbb{R}$.*
Then T is n-bounded of type-D.

Example 3.4. *Let $X = \mathbb{R}^2$ be a normed space equipped with Euclidean 2-norm $\|.,.\|^E$ and $T : (X, \|.\|) \to (X, \|.,.\|^E)$ be an operator such that $Tx_i = (x_{i2}, x_{i1})$, where $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2$ for $i = 1, 2, ...$ and $\|x_i\| = \sqrt{x_{i1}^2 + x_{i2}^2}$. Then, T is 2-bounded of type-D.*

Definition 3.5. *$T : X \to Y$ be an operator. T is n-continuous of type-D at $x \in X$ if for given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\|Tx_1 - Tx, Tx_2 - Tx, \cdots, Tx_n - Tx\| < \epsilon$$

whenever $\|x_1 - x\|\|x_2 - x\| \cdots \|x_n - x\| < \delta$, where $x_1, \cdots, x_n \in X$.

T is n-continuous of type-D if it is n-continuous at each $x \in X$.

When $n = 1$, this notion of n-continuity of type-D becomes the notion of continuity in a normed space.

Example 3.6. *The operator T in example 3.3 is n-continuous of type-D*

Example 3.7. *The operator T in example 3.4 is 2-continuous of type-D*

Theorem 3.8. *Let $T : X \to Y$ be a linear operator. Then, the following statements are equivalent.*
(1) T is n-continuous of type-D.
(2) T is n-continuous of type-D at $0 \in X$.
(3) T is n-bounded of type-D.

Proof. It is obvious that (1) implies (2).

(2) $\implies$ (3) : Suppose T is n-continuous of type-D at $0 \in X$. By definition, there is a $\delta > 0$ such that

$$\|Tu_1, ..., Tu_2\| < 1$$

whenever

$$\|u_1\|\|u_2\| \cdots \|u_n\| < \delta.$$

Let $(x_1, ..., x_n) \in X^n$.

If $\|x_1\|\|x_2\|...\|x_n\| = 0$, at least one of $x_1, ..., x_n$ is 0. Then by linearity of T, at least one of $Tx_1, Tx_2, ..., Tx_n$ is 0. It implies that $Tx_1, Tx_2, ..., Tx_n$ are linearly dependent. Hence, $\|Tx_1, ..., Tx_n\| = 0$.

If $\|x_1\|\|x_2\|...\|x_n\| \neq 0$, let $u_i = (\frac{\delta}{4})^{\frac{1}{n}} \frac{x_i}{\|x_i\|}$, $i = 1, 2, ..., n$.

Clearly,

$$\|u_1\|\|u_2\|...\|u_n\| = \frac{\delta}{4} < \delta.$$

Then, we have

$$\begin{aligned}
\|Tu_1, Tu_2, ..., Tu_n\| &= \|T(\frac{\delta}{4})^{\frac{1}{n}}\frac{x_1}{\|x_1\|}, T(\frac{\delta}{4})^{\frac{1}{n}}\frac{x_2}{\|x_2\|}, ..., T(\frac{\delta}{4})^{\frac{1}{n}}\frac{x_n}{\|x_n\|}\| \\
&= \frac{\delta}{4} \cdot \frac{1}{\|x_1\|...\|x_n\|} \cdot \|Tx_1, Tx_2, ..., Tx_n\| \\
\implies \|Tx_1, Tx_2, ..., Tx_n\| &= \frac{4}{\delta}\|x_1\|...\|x_n\|\|Tu_1, Tu_2, ..., Tu_n\| \\
&< \frac{4}{\delta}\|x_1\|...\|x_n\| \cdot 1
\end{aligned}$$

$$\implies T \text{ is } n\text{-bounded of type-}D.$$

(3) $\implies$ (1) : Suppose T is n-bounded of type-D.

Then for $x \in X$,

$$\|Tx_1 - Tx, Tx_2 - Tx, ..., Tx_n - Tx\| \leq \|T\|_n^D \|x_1 - x\|...\|x_n - x\|.$$

Let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{1+\|T\|_n^D}$ with $\|x_1 - x\|\|x_2 - x\|...\|x_n - x\| < \delta$.
Then,

$$\begin{aligned}
\|Tx_1 - Tx, Tx_2 - Tx, ..., Tx_n - Tx\| &\leq \|T\|_n^D \|x_1 - x\|...\|x_n - x\| \\
&< \|T\|_n^D.\delta \\
&= \|T\|_n^D.\frac{\epsilon}{1 + \|T\|_n^D} \\
&< \epsilon.
\end{aligned}$$

Thus, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|Tx_1 - Tx, Tx_2 - Tx, \cdots, Tx_n - Tx\| < \epsilon$$

whenever $\|x_1 - x\|\|x_2 - x\| \cdots \|x_n - x\| < \delta$, where $x_1, \cdots, x_n \in X$.

Therefore, T is n-continuous of type-D.
This completes the proof. □

Proposition 3.9. *Let X be a real vector space with dimension $\geq n$, n being a positive integer and be equipped with a norm $\|.\|$ and an n-norm $\|., ..., .\|$. Also, Let $(Y, \|., ..., .\|)$ be an n-normed space and $T : X \to Y$ be a linear operator.*
If T is n-bounded of both types-B and D, then $\|T\|_n^B = \|T\|_n^D$.

PROOF. If T is n-bounded of type B,

$$\|T\|_n^B = \sup_{\|x_1,\cdots,x_n\|\neq 0} \frac{\|Tx_1, \cdots, Tx_n\|}{\|x_1, \cdots, x_n\|}.$$

If T is n-bounded of type D,

$$\|T\|_n^D = \sup_{x_i\in X, \|x_i\|\neq 0} \frac{\|Tx_1, \cdots, Tx_n\|}{\|x_1\| \cdots \|x_n\|}.$$

Let $x_i \in X$ with $\|x_i\| \neq 0$ for $i = 1, 2, ..., n$.

Define

$$x_i = \frac{\|x_i\|y_i}{\sqrt[n]{\|y_1, ..., y_n\|}}, \quad y_i \in X \text{ and } \|y_1, ..., y_n\| \neq 0$$

$$= \frac{\|x_i\|y_i}{\alpha}, \quad \alpha = \sqrt[n]{\|y_1, ..., y_n\|}.$$

Now,

$$\|Tx_1, ..., Tx_n\| = \|T(\frac{\|x_1\|y_1}{\alpha}), ..., T(\frac{\|x_n\|y_n}{\alpha})\|$$
$$= \frac{\|x_1\|...\|x_n\|}{\alpha^n}\|Ty_1, ..., Ty_n\|$$
$$\implies \frac{\|Tx_1, ..., Tx_n\|}{\|x_1\|...\|x_n\|} = \frac{\|Ty_1, ..., Ty_n\|}{\|y_1, ..., y_n\|} \tag{3.9.1}$$

Taking supremum of the right side of (3.9.1) over $\{(y_1, ..., y_n) \in X^n : \|y_1, ..., y_n\| \neq 0\}$, we have

$$\frac{\|Tx_1, ..., Tx_n\|}{\|x_1\|...\|x_n\|} \leq \|T\|_n^B.$$

It is true for all $(x_1, ..., x_n) \in X^n$ and each $x_i \neq 0$.
Therefore,

$$\sup_{x_i\in X, \|x_i\|\neq 0} \frac{\|Tx_1, ..., Tx_n\|}{\|x_1\|...\|x_n\|} \leq \|T\|_n^B$$
$$\implies \|T\|_n^D \leq \|T\|_n^B.$$

Again, Taking supremum of the left side of (3.9.1) over $\{(x_1, ..., x_n) \in X^n : \|x_i\| \neq 0, i = 1, 2, ..., n\}$, we have

$$\|T\|_n^D \geq \frac{\|Ty_1, ..., Ty_n\|}{\|y_1, ..., y_n\|}.$$

It is true for all $(y_1, ..., y_n) \in X^n$ and $\|y_1, ..., y_n\| \neq 0$.
Therefore,

$$\|T\|_n^D \geq \sup_{x_i \in X, \|x_i\| \neq 0} \frac{\|Ty_1, ..., Ty_n\|}{\|y_1, ..., y_n\|}.$$

$$\implies \|T\|_n^D \geq \|T\|_n^B.$$

This completes the proof. □

Proposition 3.10. *Let X be an inner product space equipped with standard n-norm $\|., ..., .\|^{\mathrm{S}}$ and $(Y, \|., ..., .\|)$ be an n-normed space. If $T : X \to Y$ is n-bounded of type-B, then T is n-bounded of type-D.*

PROOF. Since T is n-bounded of type-B,

$$\|Tx_1, ..., Tx_n\| \leq K\|x_1, ..., x_n\|^{\mathrm{S}}.$$

But,

$$\begin{aligned}\|x_1, ..., x_n\|^{\mathrm{S}} &= \sqrt{\det\langle x_i, x_j\rangle} \\ &\leq \sqrt{\|x_1\|^2\|x_2\|^2...\|x_n\|^2} \\ &\text{(Hadamard's determinant theorem)} \\ &= \|x_1\|\|x_2\|...\|x_n\|.\end{aligned}$$

Therefore,

$$\|Tx_1, ..., Tx_n\| \leq K\|x_1\|\|x_2\|...\|x_n\|$$

$$\implies T \text{ is n- bounded of type-D.}$$

This completes the proof. □

Proposition 3.11. *Let X be a real vector space equipped with a norm $\|.\|$ and an n-norm $\|., ..., .\|$. Also, Let $T : X \to X$ be a linear operator.*
If T is n-bounded of both types-A and C, then $\|T\|_n^A = \|T\|_n^C$.

PROOF. If T is n-bounded of type-A,

$$\|T\|_n^A = \sup\{\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1\|...\|x_n\|} : x_1, x_2, ..., x_n \in X, \|x_1\|...\|x_n\| \neq 0\}$$

And, if T is n-bounded of type-C,

$$\|T\|_n^C = \sup\{\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1, ..., x_n\|} : x_1, x_2, ..., x_n \in X, \|x_1, ..., x_n\| \neq 0\}$$

Let $x_i \in X$ with $\|x_i\| \neq 0$ for $i = 1, 2, ..., n$.

Define

$$x_i = \frac{\|x_i\| y_i}{\sqrt[n]{\|y_1, ..., y_n\|}}, y_i \in X \text{ and } \|y_1, ..., y_n\| \neq 0$$
$$= \frac{\|x_i\| y_i}{\alpha}, \ \alpha = \sqrt[n]{\|y_1, ..., y_n\|}$$

Now,

$$\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|$$
$$= \|T(\frac{\|x_1\|}{\alpha} y_1), \frac{\|x_2\|}{\alpha} y_2, ..., \frac{\|x_n\|}{\alpha} y_n\| + \cdots + \|\frac{\|x_1\|}{\alpha} y_1, \frac{\|x_2\|}{\alpha} y_2, ..., T(\frac{\|x_n\|}{\alpha} y_n)\|$$
$$= \frac{\|x_1\|\|x_2\|...\|x_n\|}{\alpha^n}(\|Ty_1, y_2, ..., y_n\| + \|y_1, Ty_2, ..., y_n\| + \cdots + \|y_1, y_2, ..., Ty_n\|)$$

Therefore,

$$\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1\|\|x_2\|...\|x_n\|} = \frac{\|Ty_1, y_2, ..., y_n\| + \|y_1, Ty_2, ..., y_n\| + \cdots + \|y_1, y_2, ..., Ty_n\|}{\|y_1, ..., y_n\|}$$

Consequently,

$$\|T\|_n^{\mathrm{A}} \geq \frac{\|Ty_1, y_2, ..., y_n\| + \|y_1, Ty_2, ..., y_n\| + \cdots + \|y_1, y_2, ..., Ty_n\|}{\|y_1, ..., y_n\|}$$

It is true for all $y_1, y_2, ..., y_n \in X$ with $\|y_1, y_2, ..., y_n\| \neq 0$.

Therefore,

$$\|T\|_n^{\mathrm{A}} \geq \sup\{\frac{\|Ty_1, y_2, ..., y_n\| + \|y_1, Ty_2, ..., y_n\| + \cdots + \|y_1, y_2, ..., Ty_n\|}{\|y_1, ..., y_n\|} : y_1, y_2, ..., y_n \in X, \|y_1, ..., y_n\| \neq 0\}$$

$$\implies \|T\|_n^{\mathrm{A}} \geq \|T\|_n^{\mathrm{C}}.$$

Also,

$$\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1\|\|x_2\|...\|x_n\|} \leq \|T\|_n^{\mathrm{C}}$$

It is true for all $x_1, x_2, ..., x_n \in X$ with $\|x_1\|\|x_2\|...\|x_n\| \neq 0$.

Therefore,

$$\sup\{\frac{\|Tx_1, x_2, ..., x_n\| + \|x_1, Tx_2, ..., x_n\| + \cdots + \|x_1, x_2, ..., Tx_n\|}{\|x_1\|...\|x_n\|}$$

$$: x_1, x_2, ..., x_n \in X, \|x_1\|...\|x_n\| \neq 0\} \leq \|T\|_n^{\mathrm{C}}$$

$$\implies \|T\|_n^{\mathrm{A}} \leq \|T\|_n^{\mathrm{C}}.$$

This completes the proof. □

Proposition 3.12. *Let X be an inner product space equipped with standard n-norm $\|., ..., .\|^S$. If $T : X \to X$ is n-bounded of type-C, then T is n-bounded of type-A.*

PROOF. $T : X \to X$ is n-bounded of type-C. It implies that there exists a constant K such that
for all $x_1, x_2, ..., x_n \in X$,

$$\|Tx_1, x_2, ..., x_n\|^S + \|x_1, Tx_2, ..., x_n\|^S + \cdots + \|x_1, x_2, ..., Tx_n\|^S \leq K\|x_1, ..., x_n\|^S.$$

But,

$$\|x_1, ..., x_n\|^S = \sqrt{\det\langle x_i, x_j\rangle}$$

Applying Hadamard inequality,

$$\|x_1, ..., x_n\|^S \leq \|x_1\|\|x_2\|...\|x_n\|$$

Therefore, for all $x_1, x_2, ..., x_n \in X$,

$$\|Tx_1, x_2, ..., x_n\|^S + \|x_1, Tx_2, ..., x_n\|^S + \cdots + \|x_1, x_2, ..., Tx_n\|^S \leq K\|x_1\|...\|x_n\|.$$

It implies T is n-bounded of type-A. This completes the proof. □

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