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BALANCED INDEX SETS OF GRAPHS AND SEMIGRAPHS

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Abstract. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Graph labeling is an assignment of integers to the vertices or the edges, or both, subject to certain conditions. For a graph $G(V, E)$, a friendly labeling $f : V(G) \to \{0, 1\}$ is a binary mapping such that $|v_f(1) - v_f(0)| \leq 1$, where $v_f(1)$ and $v_f(0)$ represents number of vertices labeled by 1 and 0 respectively. A partial edge labeling f^* of G is a labeling of edges such that, an edge $uv \in E(G)$ is, $f^*(uv) = 0$ if $f(u) = f(v) = 0$; $f^*(uv) = 1$ if $f(u) = f(v) = 1$ and if $f(u) \neq f(v)$ then uv is not labeled by f^*. A graph G is said to be balanced graph if it admits a vertex labeling f that satisfies the conditions, $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$, where $e_f(0), e_f(1)$ are the number of edges labeled with 0 and 1 respectively. The balanced index set of the graph G is defined as, $\{|e_f(1) - e_f(0)|$: the vertex labeling f is friendly$\}$. A semigraph is a generalization of graph. The concept of semigraph was introduced by E. Sampath Kumar. Frank Harrary has defined an edge as a 2-tuple (a, b) of vertices of a graph satisfying, two edges (a, b) and (a', b') are equal if and only if either $a = a'$ and $b = b'$ or $a = b'$ and $b = a'$. Using this notion, E. Sampath Kumar defined semigraph as a pair (V, X) where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples called edges of G of distinct vertices, for various $n \geq 2$ satisfying the conditions: (i) Any two edges of G can have at most one vertex in common; and (ii) two edges $(a_1, a_2, a_3, ..., a_p)$ and $(b_1, b_2, b_3, ..., b_q)$ are said to be equal if and only if the number of vertices in both edges must be equal, i.e $p = q$, and either $a_i = b_i$ for $1 \leq i \leq p$ or $a_i = b_{p-i+1}$, $1 \leq i \leq p$. In this article, balance index set of $T(P_n)$, $T(W_n)$, $T(K_{m,n})$ and $T(S_n)$ determined, and the balance index set of semigraph is introduced. Additionally, the balanced index set of semigraph $C_{n,m}$, $K_{n,m}$ is determined.

Key words and Phrases: Balance Index set, Semigraph, Partial Edge Labeling

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1. Introduction

Consider a simple graph $G(V, E)$ consist of finite non- empty set V called vertex set and a set E of 2-element subsets of V called edges. For a binary vertex labeling f of a graph G, a partial edge labeling f^* of a graph G defined as: for all edge $uv \in E(G)$

$$f^*(uv) = \begin{cases} 0, & if\ f(u) = f(v) = 0, \\ 1, & if\ f(u) = f(v) = 1. \end{cases}$$

If $f(u) \neq f(v)$ then the edge uv is not labeled by f^*. Let $v_f(i)$ be the number of vertices of G that are labeled by i under f and $e_{f^*}(i)$ is the number of edges that are labeled by i under f^*, where $i = 0, 1$[1].

The vertex labeling f is called friendly if $|v_f(0) - v_f(1)| \leq 1$ and friendly labeling is called balanced if $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$.

It is clear that every friendly labeling is not balanced. Therefore, Lee et al. [2] introduced the balance index set of a graph G as,

$$BI(G) = \{|e_{f^*}(0) - e_{f^*}(1)|, where\ f^*\ is\ the\ partial\ edge\ labeling\ runs \\ over\ all\ friendly\ labelings\ f\ of\ G\}.$$

Our objective is to assign a binary labeling to some substructure of graphs G (e.g., the vertices) so that the assignment is balanced and induces a labeling on some other substructure (e.g., the edges). We then attempt to classify the degree of imbalance in the induced labeling of G. We hope that such an index set could form an invariant that in some way can distinguish classes of graphs. In this paper, we focus on the balanced index sets of various product graphs. The balance index set of some families of graph forms an arithmetic progression, but not every graph. Some balanced graphs are considered in citekong,Alhe,kim,tan. In general, it is difficult to determine the balance index set of a given graph. Most of the existing research work is focused on some special families of graph with simple structure. For the wonderful work one can see [2, 6, 7, 8].

Kwong and Shiu [9] developed an algebraic approach to find the balance index set. It shows that the balance index set depends on the degree sequence of the graph. It becomes a very powerful tool to deal with balance indices. Later, Lee et al. [10] proved the following lemma and corollary. In this article we have used **C.o.v** in the table, it stands for **'Compositions of vertices'**.

Lemma 1.1. [10] *For any graph G,*

(1) $2e_{f^*}(0) + e_{f^*}(X) = \sum_{v\in v(0)} deg(v)$.
(2) $2e_{f^*}(1) + e_{f^*}(X) = \sum_{v\in v(1)} deg(v)$.
(3) $2|E(G)| = \sum_{v\in v(G)} deg(v) = \sum_{v\in v(0)} deg(v) + \sum_{v\in v(1)} deg(v)$,

where $e_{f^*}(X)$ *is the subset of* $E(G)$ *containing all the unlabeled edges.*

Corollary 1.2. [10] *For any friendly labeling f, the balance index is*

$$\Big|e_{f^*}(0) - e_{f^*}(1)\Big| = \frac{1}{2}\Big|\sum_{v\in v(0)} deg(v) - \sum_{v\in v(1)} deg(v)\Big|$$

2. Balance index set of total graphs

In this section balance index set of total graph of P_n, $K_{m,n}$, W_n and S_n are discussed.

Theorem 2.1. *The Balance index set of $T(P_n)$ is*

$$BI(T(P_n)) = \begin{cases} \{0,1,2,3\}, & when\ n = 3; \\ \{0,1,2,3,4\}, & when\ n = 4; \\ \{0,1,2,3,4,5\}, & when\ n \geq 5. \end{cases}$$

Proof. Let $T(G)$ be a total graph of P_n. It consist of $2n-1$ vertices and $4n-5$ edges. Since $2n-1$ is odd, to satisfy friendly labeling, $v_f(0) = n$ and $v_f(1) = n-1$ or $v_f(0) = n-1$ and $v_f(1) = n$. The graph $T(P_n)$ has 2 vertices with degree 2, two vertices with degree 3 and remaining $2n-5$ vertices with degree 4.

For friendly labeling, composition of degree 2, degree 3 and degree 4 vertices are given in Table 2.

Table 1. Composition of vertices of degree 2, degree 3 and degree 4, for friendly labeling

Case	C.o.v of degree 2	C.o.v of degree 3	C.o.v of degree 4	
			when $(v_f(0) = n, v_f(1) = n-1)$	when $(v_f(0) = n-1, v(1) = n)$
1	(0,2)	(2-i,i)	(n-2+i,n-3-i)	(n-3+i,n-2-i)
2	(1,1)	(2-i,i)	(n-3+i,n-2-i)	(n-4+i,n-1-i)
3	(2,0)	(2-i,i)	(n-4+i,n-1-i)	(n-5+i,n-i)

1) If the composition of degree 2 vertices is (0,2), then the possible compositions of degree 3 and degree 4 vertices are given in Table 2.

Table 2. Compositions of degree 3 and degree 4 vertices, when composition of degree 2 vertices is $(0,2)$.

i	$(2-i,i)$	When $v_f(0) = n, v_f(1) = n-1$	When $v_f(0) = n-1, v_f(1) = n$
		(n-2+i,n-3-i)	(n-3+i,n-2-i)
i=0	(2,0)	(n-2,n-3)	(n-3,n-2)
i=1	(1,1)	(n-1,n-4),n $\geq$ 4	(n-3,n-2)
i=2	(0,2)	(n,n-5),n$\geq$5	(n-1,n-4),n$\geq$ 4

a) If compositions of degree 2, 3 and 4 are $(0,2)$, $(2-i,i)$, $(n-2+i,n-3-i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\Big|e_{f^*}(0)-e_{f^*}(1)\Big| =&\frac{1}{2}\Bigg|\Big[3(2-i)+4(n-2+i)\Big]-\Big[2(2)+3i+4(n-3-i)\Big]\Bigg|\\ =&\frac{1}{2}\Bigg|\Big[6-3i+4n-8+4i\Big]-\Big[4+3i+4n+12+4i\Big]\Bigg|\\ =&|3+i|, i=0,1,2.\end{aligned}$$

b) If compositions of degree 2, 3 and 4 are $(0,2)$, $(2-i,i)$, $(n-3+i,n-2-i)$ respectively, then by Corollary 1.2

$$\begin{aligned}\Big|e_{f^*}(0)-e_{f^*}(1)\Big| =&\frac{1}{2}\Bigg|\Big[3(2-i)+4(n-3+i)\Big]-\Big[2(2)+3i+4(n-2-i)\Big]\Bigg|\\ =&\frac{1}{2}\Bigg|\Big[6-3i+4n-12+4i\Big]-\Big[4+3i+4n-8-4i\Big]\Bigg|\\ =&|i-1|, i=0,1,2.\end{aligned}$$

2) If the compositions of degree 2 vertices is (1,1), then the compositions of degree 3 and degree 4 vertices are given in Table 3.

Table 3. Compositions of degree 3 and degree 4 vertices, when composition of degree 2 vertices is $(1,1)$.

i	$(2-i,i)$	When $v_f(0)=n, v_f(1)=n-1$	When $v_f(0)=n-1, v_f(1)=n$
		(n-3+i,n-2-i)	(n-4+i,n-1-i)
i=0	(2,0)	(n-3,n-2)	(n-4,n-1), n$\geq$ 4
i=1	(1,1)	(n-2,n-3)	(n-3,n-2)
i=2	(0,2)	(n-1,n-4),n$\geq$4	(n-2,n-3)

a) If compositions of degree 2, 3 and 4 are $(0,2)$, $(2-i,i)$, $(n-3+i,n-2-i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\Big|e_{f^*}(0)-e_{f^*}(1)\Big| =&\frac{1}{2}\Bigg|\Big[2+3(2-i)+4(n-3+i)\Big]-\Big[2+3i+4(n-2-i)\Big]\Bigg|\\ =&\frac{1}{2}\Bigg|\Big[2+6-3i+4n-12+4i\Big]-\Big[2+3i+4n-8-4i\Big]\Bigg|\\ =&|1+i|, i=0,1,2.\end{aligned}$$

b) If compositions of degree 2, 3 and 4 are $(1,1)$, $(2-i,i)$, $(n-4+i,n-1-i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\Big|e_{f^*}(0)-e_{f^*}(1)\Big| &=\frac{1}{2}\Bigg|\Big[(2+3(2-i)+4(n-4+i)\Big]-\Big[2+3i+4(n-1-i)\Big]\Bigg| \\ &=\frac{1}{2}\Bigg|\Big[2+6-3i+4n-16+4i\Big]-\Big[2+3i+4n-4-4i\Big]\Bigg| \\ &=|i-3|, i=0,1,2.\end{aligned}$$

3) If the composition of degree 2 vertices is (2, 0), then the compositions of degree 3 and degree 4 vertices are given in Table 4.

Table 4. Compositions of degree 3 and degree 4 vertices, when composition of degree 2 vertices is $(2,0)$.

i	$(2-i,i)$	When $V_f(0)=n, V_f(1)=n-1$	When $V_f(0)=n-1, V_f(1)=n$
		(n-4+i,n-1-i)	(n-5+i,n-i)
i=0	(2,0)	(n-4,n-1),n$\geq$ 4	(n-5,n), n$\geq$ 5
i=1	(1,1)	(n-3,n-2)	(n-4,n-1)n$\geq$ 4
i=2	(0,2)	(n-2,n-3)	(n-3,n-2)

a) If compositions of degree 2, 3 and 4 are $(2,0)$, $(2-i,i)$, $(n-4+i,n-1-i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\Big|e_{f^*}(0)-e_{f^*}(1)\Big| &=\frac{1}{2}\Bigg|\Big[(4+3(2-i)+4(n-4+i)\Big]-\Big[3i+4(n-1-i)\Big]\Bigg| \\ &=\frac{1}{2}\Bigg|\Big[4+6-3i+4n-16+4i\Big]-\Big[3i+4n-4-4i\Big]\Bigg| \\ &=|i-2|, i=0,1,2.\end{aligned}$$

b) If compositions of degree 2, 3 and 4 are $(2,0)$, $(2-i,i)$, $(n-5+i,n-i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\Big|e_{f^*}(0)-e_{f^*}(1)\Big| &=\frac{1}{2}\Bigg|\Big[4+3(2-i)+4(n-5+i)\Big]-\Big[3i+4(n-i)\Big]\Bigg| \\ &=\frac{1}{2}\Bigg|\Big[4+6-3i+4n-20+4i\Big]-\Big[3i+4n-4i\Big]\Bigg| \\ &=|i-5|, i=0,1,2.\end{aligned}$$

Therefore from above cases,

TABLE 5. Balance indices when $v_f(0) = n$ and $v_f(1) = 1$

$\left\|e_{f^*}(0) - e_{f^*}(1)\right\|$	i value when $n = 3$	i value when $n = 4$	i value when $n \geq 5$
$\|3 + i\|$	i=0	i=0,1	i=0,1,2
$\|1 + i\|$	i=0,1	i=0,1	i=0,1,2
$\|i - 2\|$	i=1,2	i=1,2	i=0,1,2

TABLE 6. Balance indices $v_f(0) = n - 1$ and $v_f(1) = n$

$\left\|e_{f^*}(0) - e_{f^*}(1)\right\|$	i value when $n = 3$	i value when $n = 4$	i value when $n \geq 5$
$\|i - 1\|$	i=0,1,2	i=0,1,2	i=0,1,2
$\|i - 3\|$	i=1,2	i=0,1,2	i=0,1,2
$\|1 - 5\|$	i=2	i=1,2	i=0,1,2

From Table 5 and 6,

$$BI(T(P_n)) = \begin{cases} \{0, 1, 2, 3\}, & when\ n = 3 \\ \{0, 1, 2, 3, 4\}, & when\ n = 4 \\ \{0, 1, 2, 3, 4, 5\}, & when\ n \geq 5 \end{cases}$$

□

Theorem 2.2. *Let W_n be a Wheel graph with n vertices, then*

$$BI\ (T(W_n)) = \begin{cases} \Bigg\{\left|\frac{1}{2}\Big(7n - n^2 + 2ni - 2i - 18\Big)\right|, \\ \left|\frac{1}{2}\Big(3n - n^2 + 2ni - 8i + 4\Big)\right| : i = 0, 1, ..., n - 1\Bigg\}, \\ when\ 3n - 2\ is\ even. \\ \Bigg\{\left|\frac{1}{2}\Big(3n - n^2 + 2ni - 8i - 2\Big)\right|, \left|\frac{1}{2}\Big(2ni - n - n^2 - 8i + 6\Big)\right|, \\ \left|\frac{1}{2}\Big(2ni - n - n^2 - 8i + 6\Big)\right|, \left|\frac{1}{2}\Big(2 - n - n^2 + 2ni - 8i\Big)\right|; \\ i = 0, 1, ..., n - 1\Bigg\}, \\ when\ 3n - 2\ is\ odd. \end{cases}$$

Proof. Let $T(W_n)$ be the total graph of wheel graph with $3n-2$ vertices. In which one vertex has degree $2(n-1)$, $n-1$ vertices has degree $n+2$ and remaining $2(n-1)$ vertices has degree 6. For friendly labeling the compositions of vertices of degree $2(n-1)$, $n+2$ and 6, when $3n-2$ is even and odd are given in Table 7 and Table 8 respectively. We use the same technique as in Theorem (2.1) to find the balanced index of $T(W_n)$.

Case 1) $3n-2$ is even.

TABLE 7. Compositions of vertices of degree $2(n-1)$, degree $(n+2)$ and degree 6, when $i = 0, 1, ..., n-1$

C.o.v of degree $2(n-1)$	C.o.v of degree $n-1$	C.o.v of degree 6
(1,0)	(i,n-1-i)	$\left(\frac{3n-4}{2} - i, \frac{n}{2} + i\right)$
(0,1)	(i,n-1-i)	$\left(\frac{3n-2}{2} - i, \frac{n-2}{2} + i\right)$

i) When compositions of vertices of degree $2(n-1)$,$(n+2)$ and 6 are $(1,0)$, $(i, n-1-i)$, $\left(\frac{3n-4}{2} - i, \frac{n}{2} + i\right)$ respectively, then $\Big|e_{f^*}(0) - e_{f^*}(1)\Big|$,

$$\Big|e_{f^*}(0) - e_{f^*}(1)\Big| = \frac{1}{2}\Bigg|\Bigg[2(n-1) + i(n+2) + 6\Big(\frac{3n-4}{2} - i\Big)\Bigg] - \Bigg[(n-1-i)(n+2) + \Big(\frac{n}{2} + i\Big)6\Bigg]\Bigg|$$

$$= \frac{1}{2}\Bigg|\Bigg[2n - 2 + ni + 2i + 9n - 12 - 6i\Bigg] - \Bigg[n^2 - n + 2n - 2 - 2i - ni + 3n - 3 + 6\Bigg]\Bigg|$$

$$= \Bigg|\frac{1}{2}\Big(7n - n^2 + 2ni - 2i - 18\Big)\Bigg|$$

ii) When compositions of vertices of degree $2(n-1)$,$(n+2)$ and 6 are $(0,1)$, $(i, n-1-i)$, $\left(\frac{3n-2}{2} - i, \frac{n-2}{2} + i\right)$ respectively, then $\Big|e_{f^*}(0) - e_{f^*}(1)\Big|$ is,

$$\Big|e_{f^*}(0) - e_{f^*}(1)\Big| = \frac{1}{2}\Bigg|\Bigg[i(n+2) + 6\Big(\frac{3n-2}{2} - i\Big)\Bigg] - \Bigg[2(n-1) + (n-1-i)(n+2) + 6\Big(\frac{n-2}{2} + i\Big)\Bigg]\Bigg|$$

$$= \frac{1}{2}\Bigg|\Bigg[ni + 2i + 9n - 6 - 6i\Bigg] - \Bigg[2n - 2 + n^2 + n - 2 - 2i - ni + 3n - 6 + 6i\Bigg]\Bigg|$$

$$=\left|\frac{1}{2}\Big(3n-n^2+2ni-8i+4\Big)\right|$$

Case 2) $3n-2$ is odd.

TABLE 8. Compositions of vertices of degree $2(n-1)$, degree $(n+2)$ and degree 6, when $i=0,1,...,n-1$

C.o.v of degree $2(n-1)$	C.o.v of degree $n-1$	C.o.v of degree 6
(1,0)	(i,n-1-i)	$\left(\frac{3(n-1)}{2}-i,\frac{n-1}{2}+i\right)$
		$\left(\frac{3n-5}{2}-i,\frac{n+1}{2}-i\right)$
(0,1)	(i,n-1-i)	$\left(\frac{3n-1}{2}-i,\frac{n-3}{2}+i\right)$
		$\left(\frac{3(n-1)}{2}-i,\frac{n-1}{2}-i\right)$

i) When compositions of vertices of degree $2(n-1)$,$(n+2)$ and 6 are $(0,1)$, $(i,n-1-i)$, $\left(\frac{3(n-1)}{2}-i,\frac{n-1}{2}+i\right)$ respectively, then $\Big|e_{f^*}(0)-e_{f^*}(1)\Big|$ is,

$$\Big|e_{f^*}(0)-e_{f^*}(1)\Big|=\frac{1}{2}\Bigg|\Big[i(n+2)+6\Big(\frac{3(n-1)}{2}-i\Big)\Big]-\Big[2(n-1)+(n-1-i)(n+2)+\Big(\frac{n-1}{2}+i\Big)\Big]\Bigg|$$

$$=\frac{1}{2}\Bigg|\Big[ni+2i+9n-9-6i\Big]-\Big[2n-2+n^2+n-2-2i-ni+3n-3+6i\Big]\Bigg|$$

$$=\left|\frac{1}{2}\Big(3n-n^2+2ni-8i-2\Big)\right|$$

ii) When compositions of vertices of degree $2(n-1)$,$(n+2)$ and 6 are $(1,0)$, $(i,n-1-i)$, $\left(\frac{3n-5}{2}-i,\frac{n+1}{2}+i\right)$ respectively, then $\Big|e_{f^*}(0)-e_{f^*}(1)\Big|$ is,

$$\Big|e_{f^*}(0)-e_{f^*}(1)\Big|=\frac{1}{2}\Bigg|\Big[2(n-1))+i(n+2)+6\Big(\frac{3n-5}{2}-i\Big)\Big]-\Big[(n-1-i)(n+2)+6\Big(\frac{n+1}{2}+i\Big)\Big]\Bigg|$$

$$=\left|\frac{1}{2}\Big(3n-n^2+2ni-8i-6\Big)\right|$$

iii) When compositions of vertices of degree $2(n-1)$,$(n+2)$ and 6 are $(1,0)$, $(i,n-1-i)$, $\left(\frac{3n-1}{2}-i,\frac{n-3}{2}+i\right)$ respectively, then $\left|e_{f^*}(0)-e_{f^*}(1)\right|$ is,

$$\left|e_{f^*}(0)-e_{f^*}(1)\right| =\frac{1}{2}\left|\left[2(n-1)+i(n+2)+6\Big(\frac{3n-1}{2}-i\Big)\right]-\right.$$
$$\left.\left[(n-1-i)*(n+2)+6\Big(\frac{n-3}{2}+i\Big)\right]\right|$$
$$=\frac{1}{2}\left|\Big(2ni-n-n^2-8i+6\Big)\right|$$

iv) When compositions of vertices of degree $2(n-1)$,$(n+2)$ and 6 are $(0,1)$, $(i,n-1-i)$, $\left(\frac{3(n-1)}{2}-i,\frac{n-1}{2}-i\right)$ respectively, then $\left|e_{f^*}(0)-e_{f^*}(1)\right|$ is,

$$\left|e_{f^*}(0)-e_{f^*}(1)\right| =\frac{1}{2}\left|\left[i(n+2)+6\Big(\frac{3(n-1)}{2}-i\Big)\right]-\right.$$
$$\left.\left[2(n-1)+(n-1-i)(n+2)+6\Big(\frac{n-1}{2}-i\Big)\right]\right|$$
$$=\frac{1}{2}\left|\Big(2-n-n^2+2ni-8i\Big)\right|$$

Therefore from Case 1 and Case 2 the balance index set is,

$$BI\ (T(W_n))=\begin{cases}\Big\{\left|\frac{1}{2}\Big(7n-n^2+2ni-2i-18\Big)\right|,\\ \left|\frac{1}{2}\Big(3n-n^2+2ni-8i+4\Big)\right| : i=0,1,2,...,n-1\Big\},\\ \textit{when } 3n-2 \textit{ is even.}\\ \Big\{\left|\frac{1}{2}\Big(3n-n^2+2ni-8i-2\Big)\right|,\left|\frac{1}{2}\Big(2ni-n-n^2-8i+6\Big)\right|,\\ \left|\frac{1}{2}\Big(2ni-n-n^2-8i+6\Big)\right|,\left|\frac{1}{2}\Big(2-n-n^2+2ni-8i\Big)\right| :\\ i=0,1,2,...,n-1\Big\},\ \textit{when } 3n-2 \textit{ is odd.}\end{cases}$$

□

Theorem 2.3. *Let $K_{m,n}$ be the complete Bipartite graph, then*

$$BI(T(K_{m,n})) = \begin{cases} \left\{ \frac{1}{2}\left|2ni + 2mj - 2mi - 2nj - 2mn + m^2 + n^2\right| \right\}, \\ when\ m+n+mn\ is\ even. \\ \left\{ \frac{1}{2}\left|2ni + 2mj - 2mi - 2nj - 2mn + m^2 + n^2 + m + n\right|, \right. \\ \left. \frac{1}{2}\left|2ni + 2mj - 2mi - 2nj - 2mn + m^2 + n^2 - m - n\right| \right\}, \\ when\ m+n+mn\ is\ odd. \end{cases}$$

Proof. Let $T(K_{m,n})$ be the total graph of complete bipartite graph with $m+n+mn$ number of vertices. In which m vertices are of degree $2n$, n vertices are of degree $2m$ and remaining mn vertices has degree $m+n$.

For friendly labeling, compositions of vertices of degree $2n$, degree $2m$ and degree $m+n$, when $m+n+mn$ is even and odd are given in Table 9 and Table 10 respectively.

1) $m+n+mn$ is even.

Table 9. Compositions of vertices of degree m, degree n and degree mn, when $i = 0, 1, ..., m;\ j = 0, 1, 2, ..., n$.

C.o.v of degree m	C.o.v of degree n	C.o.v of degree mn
(i,m-i)	(i,n-j)	$\frac{mn+m+n}{2} - (i+j)$, $\frac{mn-m-n}{2} + (i+j)$

If compositions of vertices of degree m, degree n and degree mn, are $(i, m-i)$, $(i, n-j)$, $\frac{mn+m+n}{2} - (i+j)$, $\frac{mn-m-n}{2} + (i+j)$ respectively, then by Corollary 1.2,

$$\begin{aligned} \left|e_{f^*}(0) - e_{f^*}(1)\right| =& \frac{1}{2}\Bigg|\left[2n + 2jm + \left(\frac{mn+m+n}{2} - (i+j)\right)(m+n)\right] - \\ & \left[(m-i)2n + (n-j)(2m) + \left(\frac{mn-m-n}{2} + (i+j)\right)(m+n)\right]\Bigg| \\ =& \frac{1}{2}\left|\left(2ni + 2mj - 2mi - 2nj - 2mn + m^2 + n^2\right)\right| \end{aligned}$$

2) $m+n+mn$ is odd.

Table 10. Compositions of vertices of degree m, degree n and degree mn, when $i = 0, 1, ..., m;\ j = 0, 1, 2, ..., n$.

C.o.v of degree m	C.o.v of degree n	C.o.v of degree mn
(i,m-i)	(i,n-j)	$\frac{mn+m+n+1}{2} - (i+j), \frac{mn-m-n-1}{2} + (i+j)$
		$\frac{mn+m+n-1}{2} - (i+j), \frac{mn-m-n+1}{2} + (i+j)$

(a) If compositions of vertices of degree m, degree n and degree mn, are $(i, m-i)$, $(i, n-j)$, $\frac{mn+m+n+1}{2} - (i+j)$, $\frac{mn-m-n-1}{2} + (i+j)$ respectively, then by Corollary 1.2,

$$\Big|e_{f^*}(0) - e_{f^*}(1)\Big| = \frac{1}{2}\Bigg|\Big[2n + 2jm + \Big(\frac{mn+m+n+1}{2} - (i+j)\Big)(m+n)\Big] - \Big[(m-i)2n + (n-j)(2m) + \Big(\frac{mn-m-n-1}{2} + (i+j)\Big)(m+n)\Big]\Bigg|$$

$$= \Big|\frac{1}{2}\Big(2ni + 2mj - 2mi - 2nj - 2mn + m^2 + n^2 + m + n\Big)\Big|$$

(b) If compositions of vertices of degree m, degree n and degree mn, are $(i, m-i)$, $(i, n-j)$, $\frac{mn+m+n-1}{2} - (i+j)$, $\frac{mn-m-n+1}{2} + (i+j)$ respectively, then by Corollary 1.2,

$$\Big|e_{f^*}(0) - e_{f^*}(1)\Big| = \frac{1}{2}\Bigg|\Big[2n + 2jm + \Big(\frac{mn+m+n-1}{2} - (i+j)\Big)(m+n)\Big] - \Big[(m-i)2n + (n-j)(2m) + \Big(\frac{mn-m-n+1}{2} + (i+j)\Big)(m+n)\Big]\Bigg|$$

$$= \frac{1}{2}\Bigg|\Big[2ni + 2mj + \frac{m^2n + m^2 + mn - m}{2} - m(i+j) + \frac{mn^2 + mn + n^2 + n}{2} - n(i+j)\Big] - \Big[2nm - 2ni + 2mn - 2mj + \frac{m^2n - m^2 - mn + m}{2} + m(i+j) + \frac{mn^2 - mn - n^2 + n}{2} + n(i+j)\Big]\Bigg|$$

$$=\left|\frac{1}{2}\Big(2ni+2mj-2mi-2nj-2mn+m^2+n^2-m-n\Big)\right|$$

Therefore from above cases,

$$BI(T(K_{m,n}))=\begin{cases}\left\{\frac{1}{2}\left|2ni+2mj-2mi-2nj-2mn+m^2+n^2\right|\right\},\\ when\ m+n+mn\ is\ even.\\ \left\{\frac{1}{2}\left|2ni+2mj-2mi-2nj-2mn+m^2+n^2+m+n\right|,\right.\\ \left.\frac{1}{2}\left|2ni+2mj-2mi-2nj-2mn+m^2+n^2-m-n\right|\right\},\\ when\ m+n+mn\ is\ odd.\end{cases}$$

□

Theorem 2.4. *If S_n is a Star graph with n vertices, then*

$$BI\ T(S_n)=\left\{\frac{1}{2}\left|4i-n-2ni+n^2\right|,\frac{1}{2}\left|2i-5n-2ni+n^2+4\right|:i=0,1,...,n-1\right\}$$

Proof. Consider a total graph of star graph $T(S_n)$ with $2n-1$ vertices. It has *one* vertex with degree $2(n-1)$, $n-1$ vertices with degree 2 and $n-1$ vertices with degree n. Since $2n-1$ is odd, the compositions of vertices are given in Table 11,

TABLE 11. Compositions of vertices of degree $2n-1$, degree 2 and degree n, when $i=0,1,...,n-1$.

C.o.v of degree $2n-$ 1	C.o.v of degree 2	C.o.v of degree n
(1,0)	(i,n-1-i)	$n-1-i,i$
(0,1)	(i,n-1-i)	$n-1-i,i$

Case 1) If compositions of vertices of degree $2n-1$, degree 2 and degree n are $(1,0)$, $(i,n-1-i)$, $(n-1-i,i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\left|e_{f^*}(0)-e_{f^*}(1)\right| &=\frac{1}{2}\left|\Big[2(n-1)+2i+(n-1-i)n\Big]-\Big[(n-1-i)*2+ni\Big]\right|\\ &=\frac{1}{2}\left|\Big[(2n-2+2i+n^2-n-ni)\Big]-\Big[(2n-2-2i+ni)\Big]\right|\\ &=\frac{1}{2}\left|4i-n-2ni+n^2\right|\end{aligned}$$

Case 2) If compositions of vertices of degree $2n-1$, degree 2 and degree n are $(0,1)$, $(i, n-1-i)$, $(n-1-i, i)$ respectively, then by Corollary 1.2,

$$\begin{aligned}\left|e_{f^*}(0)-e_{f^*}(1)\right| &= \frac{1}{2}\left|\Big[2i+(n-1-i)n\Big]-\Big[2(n-1)+(n-1-i)2+ni\Big]\right| \\ &= \frac{1}{2}\left|\Big[(2i+n^2-n-ni)\Big]-\Big[(2n-2+2n-2-2i+ni)\Big]\right| \\ &= \frac{1}{2}\left|2i-5n-2ni+n^2+4\right|\end{aligned}$$

Therefore BI set of $T(S_n)$ is,

$$BI\ T(S_n)=\left\{\frac{1}{2}\left|4i-n-2ni+n^2\right|, \frac{1}{2}\left|2i-5n-2ni+n^2+4\right| : i=0,1,...,n-1\right\}$$

□

3. Balance index set of some classes of semigraph

A semigraph is a generalization of graph (Given in Fig. 1). The concept of semigraph was introduced by E. Sampath Kumar[11]. If $E_1=(u_1,u_2,...,u_k)$ and $E_2=(u_k,u_{k-1},...,u_1)$ are two edges, then by $SG-II$, it is noted that $E_1=E_2$. The size of an edge is denoted by $|E|$, is the number of vertices in an edge E. Here we introduce balance index set of semigraph G.

For a semigraph $G(V,X)$, the binary labeling is a function $f_s : V(G) \to \{0,1\}$. If a binary labeling f_s satisfies $|v_{f_s}(1)-v_{f_s}(0)| \leq 1$, where $v_{f_s}(0)$ is number of vertices labeled by 0 and $v_{f_s}(1)$ is number of vertices labeled by 1 in a semigraph G, then it is called friendly labeling.

Example 3.1. *Consider a semigraph G with edges, $E_1=\{v_1,v_2,v_3,v_4\}$, $E_2=\{v_5,v_6,v_7,v_8\}$, $E_3=\{v_9,v_{10},v_{11},v_{12}\}$, $E_4=\{v_1,v_5,v_9\}$, $E_5=\{v_4,v_8,v_{12}\}$.*
End vertices: $\{V_1,V_4,V_9,V_{12}\}$, middle vertices: $\{V_2,V_3,V_6,V_7,V_{10},V_{11}\}$, and middle end vertices: $\{V_5,V_8\}$.

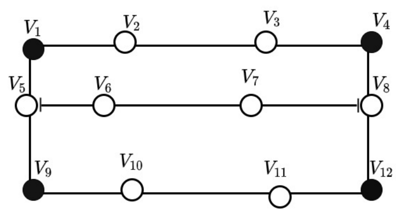

Figure 1. A Semigraph G.

For each binary vertex labeling f_s of a semigraph $G(V,X)$, the partial edge labeling f_s^* defined as: for an edge $E = v_i, v_{i+1}, v_{i+2}, ..., v_j \in X(G)$,

$$f_s^*(E) = \begin{cases} 1, & if\ v_{f_s}(1) > v_{f_s}(0)\ for\ the\ vertices\ v_k,\ i \leq k \leq j \\ 0, & if\ v_{f_s}(0) > v_{f_s}(1)\ for\ the\ vertices\ v_k,\ i \leq k \leq j. \end{cases}$$

If $v_{f_s}(0) = v_{f_s}(1)$, then E is not labeled.

In a semigraph G, f_s is a vertex labeling and f_s^* is a partial edge labeling then, a graph is said to be balanced if $|v_{f_s}(0) - v_{f_s}(1)| \leq 1$ and $|e_{f_s^*}(0) - e_{f_s^*}(1)| \leq 1$, where $e_{f_s^*}(0), e_{f_s^*}(1)$ are the number of edges labeled with 0 and 1 respectively. In a semigraph G, if $v_{f_s}(0) = v_{f_s}(1)\big(or\ e_{f_s^*}(0) = e_{f_s^*}(1)\big)$ then it is said to be strongly vertex (or edge) balanced semigraph. For a given semigraph G,

$$BI(G) = \{|e_{f_s^*}(0) - e_{f_s^*}(1)| :\ the\ vertex\ labeling\ f_s\ of\ G\ is\ friendly\}$$

is called the balance index set of semigraph G.

Theorem 3.2. *The balance index set of semigraph $K^c_{n,m}$ is*

$$BI(K^c_{n,m}) = \begin{cases} \{0,1,2,3,...,|E|\}, & when\ m\ is\ odd\ and\ |E|\ is\ even; \\ \{1,2,3,...,|E|\}, & when\ m\ is\ odd\ and\ |E|\ is\ odd; \\ \{0,1,2,3,...,\frac{n(n-2)}{2}\}, & when\ m\ is\ even\ and\ |E|\ is\ even; \\ \{1,2,3,...,\frac{(n-1)^2}{2}\}, & when\ m\ is\ even\ and\ |E|\ is\ odd. \end{cases}$$

Proof. Consider $(m+2) - uniform$ semigraph $K^c_{n,m}$ with $(\frac{n^2-n}{2})m + n$ vertices. Let $E_1, E_2, ..., E_m$ are the edges of $K^c_{n,m}$. It contains n edges of $(m+2) - uniform$ semigraph $C^c_{n,m}$ and $(\frac{n^2-3n}{2})$ edges joining any two end vertices.

Case 1) When m is odd, the vertices of $K^c_{n,m}$ can be labeled as follows.

Label end vertices of $K^c_{n,m}$ by 1 and one middle vertex in each edge of $C^c_{n,m}$ in $K^c_{n,m}$ by 0. Since m is odd, $m-1$ is even. To satisfy friendly labeling label $\frac{m-1}{2}$ vertices in each edge of $K^c_{n,m}$ by 1 and remaining $\frac{m-1}{2}$ vertices in each edge of $K^c_{n,m}$ by 0.

Remaining one vertex in each edge of $(\frac{n^2-3n}{2})$ edges joining any two end vertices can be labeled as follows.

(a) If $\frac{n^2-3n}{2}$ is odd, then $\frac{\frac{n^2-3n}{2}+1}{2}$ vertices are labeled by 1 and $\frac{\frac{n^2-3n}{2}-1}{2}$ vertices are labeled by 0 or vice versa. Thus it satisfies friendly labeling.

(b) If $\frac{n^2-3n}{2}$ is even, to satisfy friendly labeling, $\frac{n^2-3n}{4}$ vertices are labeled by 1 and $\frac{n^2-3n}{4}$ labeled by 0.

It can be observed that, in each edge $v_{f_s}(1) = \frac{m+3}{2}$ and $v_{f_s}(0) = \frac{m+1}{2}$ $\Big(v_{f_s}(1) = \frac{m+1}{2}$ and $v_{f_s}(0) = \frac{m+3}{2}\Big)$. Therefore $v_{f_s}(1) > v_{f_s}(0)$ $(v_{f_s}(0) > v_{f_s}(1))$ and highest balance index is $|E|$ with each edge label is 1. To find balance index set we use following steps.

(a) If the number of edges $\frac{n^2-n}{2}$ is even, then interchange the label of one of the vertex which is labeled by 1 in E_i with one of the vertex which is labeled by

0 in $E_{\frac{m}{2}+i}$. So that edge E_i takes the label 0 and label of edge $E_{\frac{m}{2}+i}$ remains 1. Repeat this process for $1 \leq i \leq \frac{m}{2}$. At each interchange, $e_{f_s^*}(1)$ decreases by one and $e_{f_s^*}(0)$ increases by one or vice versa. So we get balance index set as $\{0, 2, 4, ..., |E|\}$.

(b) If the number of edges $\frac{n^2-n}{2}$ is odd, then interchange the label of one of the vertex which is labeled by 1 in E_i with one of the vertex which is labeled by 0 in $E_{\frac{n^2-n+1}{2}+i}$. So that edge E_i takes the label 0 and label of an edge $E_{\frac{n^2-n+1}{2}+i}$ remains 1. Repeat this process for $1 \leq i \leq \frac{n^2-n-1}{2}$. At each interchange, $e_{f_s^*}(1)$ decreases by one and $e_{f_s^*}(0)$ increases by one or vice versa. So we get balance index set as $\{1, 3, 5, ..., |E|\}$.

Case 2) When m is even, the vertices of $K^c_{n,m}$ can be labeled as follows.

Since each edge has even number of middle vertices then label $\frac{m}{2}$ vertices of each edge in $(\frac{n^2-3n}{2})$ edges joining any two end vertices by 1 and remaining $\frac{m}{2}$ vertices of each edge of $(\frac{n^2-3n}{2})$ edges joining any two end vertices by 0. The end vertices of $K^c_{n,m}$ and first vertex of each edge in $C_{n,m}$ in $K^c_{n,m}$ are labeled by 1 and 0 respectively. Label the vertices v_i, $2 \leq i \leq m-1$ in each edge of $C_{n,m}$ in $K^c_{n,m}$ by alternative 0 and 1.The remaining one vertex in each edge of $C^c_{n,m}$ in $K^c_{n,m}$ are labeled as follows,

(a) If n is odd, then label $\frac{m+1}{2}$ vertices by 1 and $\frac{m-1}{2}$ vertices by 0 or vice versa. Therefore $v_{f_s}(1) = \frac{(\frac{n^2-n}{2})m+n+1}{2}$ and$v_{f_s}(0)\frac{(\frac{n^2-n}{2})m+n-1}{2}$ or vice versa. Therefore $|v_{f_s}(0)| - |v_{f_s}(1)| = 1$.

(b) When n is even, then label $\frac{m}{2}$ vertices by 1 and $\frac{m}{2}$ vertices by 0. Therefore $v_{f_s}(1) = v_{f_s}(0) = \frac{(\frac{n^2-n}{2})m+n}{2}$. Hence $|v_{f_s}(0)| - |v_{f_s}(1)| = 0$.

Since each edge in $(\frac{n^2-3n}{2})$ edges joining any two end vertices has $\frac{m+4}{2}$ vertices with label 1 and $\frac{m}{2}$ vertices with label 0 then each edge will get a label 1.

The edges of $C^c_{n,m}$ in $K^c_{n,m}$ takes the label as follows.

(a) If n is odd, then $\frac{n+1}{2}$ edges has $\frac{m+4}{2}$ vertices with label 1 and $\frac{m}{2}$ vertices with label 0. Also, $\frac{n-1}{2}$ edges has $\frac{m}{2}$ vertices with label 0 and $\frac{m}{2}$ vertices with label 1. Therefore $\frac{n+1}{2}$ edges will get label 1. Since $\frac{n-1}{2}$ edges has $|V_f(0)| = |V_f(1)|$, hence they are not labeled. Therefore highest balance index of $K^c_{n,m}$ when n is odd is, $\frac{n^2-3n}{2} + \frac{n+1}{2} = \frac{n^2-2n+1}{2} = \frac{(n-1)^2}{2}$.

(b) If n is even then $\frac{n}{2}$ edges has $|v_f(1)| = |v_f(0)|$ and $\frac{n}{2}$ edges has $|v_f(1)| = |v_f(0)|$. Therefore highest balance index of $K^c_{n,m}$ when n is even is, $\frac{n^2-3n}{2} + \frac{n}{2} = \frac{n^2-2n}{2} = \frac{n(n-2)}{2}$.

To get the balance index set, we need to decrease the number of edges label by 1 or increase the number of edges label by 0. In each edge we have $\frac{m}{2}$ zero's and $\frac{m}{2}$ once's. Therefore to decrease the number of edges label by 1, interchange two $1's$ of any edge with two $0's$ with any other edge. Continue this process until we get $\left||e_{f_s^*}(1)| - |e_{f_s^*}(0)|\right| \leq 1$. Therefore when $|E|$ is odd, we get balance index set as $\{1, 2, 3, ..., \frac{(n-1)^2}{2}\}$ and when $|E|$ is even, we get balance index set as $\{0, 1, 2, 3, ..., \frac{n(n-2)}{2}\}$.

□

Theorem 3.3. *The balance index set of $C^c_{n,m}$ is*

$$BI(C^c_{n,m}) = \begin{cases} \{1, 3, 5, ..., n\}, & \textit{when } m \textit{ and } n \textit{ are odd;} \\ \{0, 2, 4, ..., n\}, & \textit{when } m \textit{ is odd and } n \textit{ is even;} \\ \{1, 2, 3, ..., \frac{n+1}{2}\}, & \textit{when } m \textit{ is even and } n \textit{ is odd;} \\ \{0, 2, 4, ..., \frac{n}{2}\}, & \textit{when } m \textit{ and } n \textit{ are even.} \end{cases}$$

Proof. Let $C^c_{n,m}$ be a semigraph with $n(m+1)$ vertices and $E_1, E_2, E_3, ..., E_n$ be the edges.

Case 1) If m and n both are odd, then $n(m+1)$ is even. To satisfy friendly labeling $v_{f_s}(0) = v_{f_s}(1) = \frac{n(m+1)}{2}$. The vertex labeling is done as follows.

All n end vertices and all first vertices in each edge are labeled by 1 and 0 respectively. Label $\frac{m-1}{2}$ vertices in each edge by 1 and $\frac{m-1}{2}$ vertices in each edge by 0. Each edge contains $\frac{m+3}{2}$ vertices labeled by 1 and $\frac{m+1}{2}$ labeled by 0. Therefore highest balance index in $C^c_{n,m}$ is n.

To get balance index set of $C^c_{n,m}$, interchange the label of one of the vertex which is labeled by 1 in E_i with the label of one of the vertex which is labeled by 0 in $E_{\frac{n-1}{2}+i}$, $1 \leq i \leq \frac{n-1}{2}$. At each interchange, $v_{f_s}(1)$ is decreases by 1 and $v_{f_s}(0)$ increases by 1. Since n is odd we get balanced index set as $\{1, 3, 5, ..., |E|\}$.

Case 2) When m is odd and n is even, follow the labeling procedure when m and n are odd. When interchanging the labels, continue up to, $1 \leq i \leq \frac{n}{2}$. Since n is even we get balanced index set as $\{0, 2, 4, ..., n\}$.

Case 3) When m is even and n is odd, the total number of vertices $n(m+1)$ is odd. Therefore, to satisfy friendly labeling, $v_{f_s}(1) = \frac{n(m+1)-1}{2}$ and $v_{f_s}(0) = \frac{n(m+1)-1}{2}$. The vertex labeling is done as follows,

All the end vertices are labeled by 1 and first vertex in each edge is labeled by 0. Label $\frac{m-2}{2}$ middle vertices of each edge by 0 and $\frac{m-2}{2}$ middle vertices of each edge by 1. There will be one vertex in each edge is remaining. Label $\frac{n+1}{2}$ vertices by 1 and $\frac{n-1}{2}$ vertices by 0, in the remaining n vertices. Thus $\frac{n+1}{2}$ edges have $v_{f_s}(1) > v_{f_s}(0)$ and $\frac{n-1}{2}$ edges have $v_{f_s}(1) = v_{f_s}(0)$. Therefore highest balanced index is $\frac{n+1}{2}$. To get balanced index set use following steps.

Interchange the label of the vertex which has label 0 by 1 in E_i with the label of vertex which has label 0 in $E_{\frac{n+1}{2}+i}$, $1 \leq i \leq \frac{n-1}{2}$. At each interchange, number of edges labeled by 1 increases by 1. Since n is odd, the balanced index set is $\{1, 2, ..., \frac{n+1}{2}\}$.

Case 4) When m is even and n is even, $n(m+1)$ is even. Then for friendly labeling $v_{f_s}(1) = v_{f_s}(0) = \frac{n(m+1)}{2}$. To label the vertices of $C^c_{n,m}$, the labeling technique can be done as in Case 3. The highest edge balanced index is $\frac{n}{2}$. Using Case 3, the balanced index set is $\{0, 2, 4, ..., \frac{n}{2}\}$.

Therefore,

$$BI(C^c_{n,m}) = \begin{cases} \{1,3,5,...,n\}, & when\ m\ and\ n\ are\ odd; \\ \{0,2,4,...,n\}, & when\ m\ is\ odd\ and\ n\ is\ even; \\ \{1,2,3,...,\frac{n+1}{2}\}, & when\ m\ is\ even\ and\ n\ is\ odd; \\ \{0,2,4,...,\frac{n}{2}\}, & when\ m\ and\ n\ are\ even. \end{cases}$$

□

4. Significance in Network Optimization

The concepts and results presented in this paper have significant applications in the field of telecommunication networks. Specifically, the balanced index set and friendly labeling of graphs and semigraphs can be utilized to optimize network design, enhance routing efficiency, and improve fault tolerance in communication systems. By ensuring a balanced distribution of labels and edges, network designers can achieve more robust and efficient network structures.

Network Design Optimization: The balanced index set provides a framework for designing network topologies that minimize the difference between different types of connections. This can lead to more uniform and stable networks, which are less prone to congestion and failure.

Routing Efficiency: In telecommunication networks, efficient routing algorithms are crucial for minimizing latency and maximizing throughput. The friendly labeling and balanced index set can be applied to develop routing protocols that evenly distribute traffic, reducing the likelihood of bottlenecks and improving overall network performance.

Fault Tolerance and Reliability: A balanced network, as defined by the balanced index set, can enhance the fault tolerance of a telecommunication system. By maintaining a balanced structure, the network can better withstand node or edge failures, ensuring continued communication and reducing the risk of network partitioning.

Load Balancing: The concepts of friendly labeling and partial edge labeling can be used to achieve load balancing in telecommunication networks. By evenly distributing the load across different nodes and edges, the network can prevent overloads and ensure more efficient utilization of resources.

Scalability: As telecommunication networks grow in size and complexity, maintaining balance becomes increasingly important. The methods described in this paper provide a scalable approach to network design and management, allowing for the seamless integration of new nodes and connections while preserving network balance.

By applying the balanced index set and friendly labeling to telecommunication networks, researchers and engineers can develop more efficient, reliable, and scalable communication systems. The theoretical advancements presented in this paper pave the way for practical implementations that can significantly improve the performance and robustness of modern telecommunication networks.

Cite or refer to your theorem using: Theorem 2.1 (for example).

5. Conclusion

Labeled graph is the topic of current interest for many researchers as it has diversified applications. We discuss here balanced labeling and balance index set of $T(P_n)$, $T(W_n)$, $T(K_{m,n})$, $T(S_n)$ are obtained and balance index set of semigraph is introduced. Also, balance index set of semigraphs $C^c_{n,m}$ and $K^c_{n,m}$ are obtained. The derived labeling pattern is demonstrated by means of elegant illustrations which provides better under-standing of the derived results.

REFERENCES

[1] Kong, M. C., Lee, S. M., Seah, E., and Tang, A. S., A complete characterization of balanced graphs, *Journal of Combinatorial Mathematics and combinatorial computing*, **66** (2008), 125-136.

[2] Kwong, H., On balance index sets of rooted trees, *Ars. Combinatorica*, **91** (2009), 373-382.

[3] Alhevaz, A., Darkooti, M., Rahbani, H., Shang, Y., Strong equality of perfect roman and weak roman domination in trees, *Mathematics*, **77(997)**, (2019), 1-13.

[4] Kim, R. Y., Lee, S. M., and Ng, H. K., On balancedness of some graph construction, *Journal of Combinatorial Mathematics and combinatorial computing*, **66**(2008), 3-16.

[5] Tan, S. K., Liu, A., and Lee, S. M., On balanced graphs, *Congresses Numerantium*, **82** (1992), 59-64.

[6] Kwong, H., Lee, S. M., and Sarvate, On balance index sets of one point union of graphs, *Journal of Combinatorial mathematics and Combinatorial Computing*, **66** (2008), 113-127.

[7] Ng, H. K., Lee, S. M., and Tong, S. M., On balance index of the chain-sum graphs of cycle, *Utilitas Mathematica*,**77**, (2008), 113-123.

[8] Lee, S. M., Wang, W. C., and Wen, Y. H., On the balance index set of $(p, p + 1)$-graphs, *Journal of Combinatorial mathematics and Combinatorial Computing*, **62** (2007), 193-216.

[9] Kwong, H., and Shiu, W. C., An algebraic approach for finding balance index sets, *Australian Journal of Combinatorial* **45**,(2009), 139-155.

[10] Su, H. H., Lee, S. M., and Wang, H. C., On balance index set of trees of diameter four, *Journal of Combinatorial mathematics and Combinatorial Computing*, **78** (2011), 285-302.

[11] SampathKumar, E., Semigraphs and their applications, *Technical Report[DST/MS/022/94]*, India: Dept. of Science and Technology, 1999.