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NEUTROSOPHIC $\mathcal{N}-$IDEALS ON SHEFFER STROKE BCK-ALGEBRAS

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Abstract. In this study, a neutrosophic $\mathcal{N}-$subalgebra and neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebras are defined. It is shown that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (ideal) of a Sheffer stroke BCK-algebra is a subalgebra (ideal) of this algebra and vice versa. Then we present that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCK-algebra forms a complete distributive modular lattice and that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebra is the neutrosophic $\mathcal{N}-$subalgebra but the inverse does not usually hold. Also, relationships between neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCK-algebras and homomorphisms are analyzed. Finally, we determine special subsets of a Sheffer stroke BCK-algebra by means of $\mathcal{N}-$functions on this algebraic structure and examine the cases in which these subsets are its ideals.

Key words and Phrases: Sheffer stroke BCK-algebra, subalgebra, ideal, neutrosophic $\mathcal{N}-$ subalgebra, neutrosophic $\mathcal{N}-$ideal

1. INTRODUCTION

Sheffer stroke (or Sheffer operation) was introduced by H. M. Sheffer and is one of the two operators that can be used by itself, without any other logical operators to build a logical formal system [15]. Since it provides new, basic and easily applicable axiom systems for many algebraic structures, this operation has many applications in algebraic structures such as orthoimplication algebras [1], ortholattices [3], Boolean algebras [9], the fuzzy implivative ideals of heffer stroke BG-algebras [13]. Moreover, BCK-algebras were introduced by Imai and Iséki [4].

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These algebras are derived from two different motivations which one of these motivations is based on set theory and another is based on classical and non-classical propositional calculi. BCK-algebras have been applied to many mathematical areas such as group theory, functional analysis, probability theory and topology. Recently, some types of BCK-algebras with Sheffer stroke are defined and relationships between other Sheffer stroke algebras and these algebraic structures are examined ([12], [14]).

On the other side, the fuzzy sets introduced by Zadeh [19] is defined as a generalization of ordinary sets and has the truth (t) (membership) function and positive meaning of information. This causes that scientists have studied to find negative meaning of information. Thus, Atanassov introduced the intuitionistic fuzzy sets [2] as a generalization of fuzzy sets and this notion has truth (t) (membership) and the falsehood (f) (nonmembership) functions. Then the neutrosophic sets are introduced by Smarandache as a generalization of the intuitionistic fuzzy sets and these sets have the indeterminacy/neutrality (i) function with membership and nonmembership functions [16]-[17]. These sets are used in the algebraic structures such as BCK/BCI-algebras, strong Sheffer stroke non-associative MV-algebras, Sheffer Stroke Hilbert algebras and Sheffer stroke BL-algebras ([5]-[8], [10]-[11], [18]).

Notions of Sheffer stroke BCK-algebras, neutrosophic $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures are presented. Then we define neutrosophic $\mathcal{N}-$subalgebra and a neutrosophic $\mathcal{N}-$ideal on Sheffer stroke BCK-algebras and give some properties. It is proved that the level set of a neutrosophic $\mathcal{N}-$subalgebra (ideal) of a Sheffer stroke BCK-algebra is its subalgebra (ideal) and vice versa. Also, it is shown that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCK-algebra forms a complete distributive modular lattice, and that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebra is its neutrosophic $\mathcal{N}-$subalgebra. Besides, some subsets of a Sheffer stroke BCK-algebra are introduced by means of the $\mathcal{N}-$functions T_N, I_N and F_N and its any elements x_t, x_i, x_f. Indeed, it is propounded that these subsets are ideals of this algebra if its neutrosophic $\mathcal{N}-$structure is the neutrosophic $\mathcal{N}-$ideal.

2. PRELIMINARIES

In this section, basic definitions and notions about Sheffer stroke BCK-algebras and neutrosophic $\mathcal{N}-$structures.

Definition 2.1. [3] *Let $\mathcal{A} = \langle A, |\rangle$ be a groupoid. The operation $|$ on A is said to be a Sheffer operation (or Sheffer stroke) if it satisfies the following conditions for all $x, y, z \in A$:*
$(S1)$ $x|y = y|x$,
$(S2)$ $(x|x)|(x|y) = x$,
$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2.2. [14] *A Sheffer stroke BCK-algebra is an algebra* $(A, |, 0)$ *of type* $(2, 0)$ *such that* 0 *is the constant in* A, $|$ *is Sheffer stroke and the following axioms are satisfied:*

$(sBCK-1)$ $((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|(x|(z|z))))|$ $(z|(y|y)) = 0|0$,

$(sBCK-2)$ $(x|(y|y))|(x|(y|y)) = 0$ *and* $(y|(x|x))|(y|(x|x)) = 0$ *imply* $x = y$,

for all $x, y, z \in A$.

Lemma 2.3. [14] *Let* $(A, |, 0)$ *be a Sheffer stroke BCK-algebra. Then the following properties hold for all* $x, y, z \in A$*:*

(1) $(x|(x|x))|(x|x) = x$,
(2) $(x|(x|x))|(x|(x|x)) = 0$,
(3) $x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 0|0$,
(4) $(0|0)|(x|x) = x$,
(5) $x|0 = 0|0$,
(6) $(x|(0|0))|(x|(0|0)) = x$,
(7) $(0|(x|x))|(0|(x|x)) = 0$,
(8) $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z)))$,
(9) $(x|((y|(z|z))|(y|(z|z))))|((y|((x|(z|z))|(x|(z|z))))|(y|((x|(z|z))|(x|(z|z))))) = 0|0$,
(10) $((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = 0|0$.

Lemma 2.4. [14] *Let* $(A, |, 0)$ *be a Sheffer stroke BCK-algebra. A binary relation* $\leq$ *is defined on* A *as follows:*

$$x \leq y \ if \ and \ only \ if \ (x|(y|y))|(x|(y|y)) = 0.$$

Then the binary relation $\leq$ *is a partial order on* A *such that* $0 \leq x$*, for each* $x \in A$*. Morever, we have* $y \leq x|(y|y)$*, and* $x \leq z$ *implies* $(x|(y|y))|(x|(y|y)) \leq (z|(y|y))j|(z|(y|y))$*, for all* $x, y, z \in A$*. Also,* $1 = 0|0$ *is the greatest element and* $0 = 1|1$ *is the least element of* A*.*

Lemma 2.5. [14] *Let* $(A, |, 0)$ *be a Sheffer stroke BCK-algebra. Then Then the following features are hold for all* $x, y, z \in A$*:*

(i) $x \leq z$ *implies* $(y|(z|z))|(y|(z|z)) \leq (y|(x|x))|(y|(x|x))$,
(ii) $((x|(y|y))|(x|(y|y)))|(z|z) = ((x|(z|z))|(x|(z|z)))|(y|y)$,
(iii) $(x|(y|y))|(x|(y|y)) \leq z \Leftrightarrow (x|(z|z))|(x|(z|z)) \leq y$,
(iv) $(x|(y|y))|(x|(y|y)) \leq x$,
(v) $x \leq y|(x|x)$,
(vi) $x \leq (x|(y|y))|(y|y)$,
(vii) *if* $x \leq y$*, then* $z|(x|x) \leq z|(y|y)$,

Definition 2.6. [5] $\mathcal{F}(A, [-1, 0])$ *denotes the collection of functions from a set* A *to* $[-1, 0]$ *and a element of* $\mathcal{F}(A, [-1, 0])$ *is called a negative-valued function from* A *to* $[-1, 0]$ *(briefly,* $\mathcal{N}-$*function on* A*). An* $\mathcal{N}-$*structure refers to an ordered pair* (A, f) *of* A *and* $\mathcal{N}-$*function* f *on* A*.*

Definition 2.7. [8] *A neutrosophic $\mathcal{N}-$structure over a nonempty universe A is defined by*

$$A_N := \frac{A}{(T_N, I_N, F_N)} = \{\frac{A}{(T_N(x), I_N(x), F_N(x))} : x \in A\}$$

where T_N, I_N and F_N are $\mathcal{N}-$function on A, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure A_N over X satisfies the condition

$$(\forall x \in A)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0).$$

3. NEUTROSOPHIC $\mathcal{N}-$STRUCTURES

In this section, neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCK-algebras are presented. Unless otherwise specified, A denotes a Sheffer stroke BCK-algebra.

Definition 3.1. *A neutrosophic $\mathcal{N}-$subalgebra A_N of a Sheffer stroke BCK-algebra A is a neutrosophic $\mathcal{N}-$structure on A satisfying the condition*

$$\begin{array}{c} T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\}, \\ \min\{I_N(x), I_N(y)\} \leq I_N((x|(y|y))|(x|(y|y))) \\ \textit{and} \\ \min\{F_N(x), F_N(y)\} \leq F_N((x|(y|y))|(x|(y|y))), \end{array} \tag{1}$$

for all $x, y \in A$.

Example 3.2. *Consider the Sheffer stroke BCK-algebra A where $A = \{0, x, y, 1\}$ and Sheffer stroke $|$ on A has the Cayley table* [14] *in Table 1:*

TABLE 1. Cayley table of Sheffer stroke $|$ on A

$\vert$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \left\{\frac{u}{(-1, -0.2, -0.1)} : \ u = 0, 1\right\} \cup \left\{\frac{u}{(-0.2, -1, -1)} : \ u = x, y\right\}$$

on A is a neutrosophic $\mathcal{N}-$subalgebra of A.

Definition 3.3. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A and u, v, w be any elements of $[-1, 0]$ such that $-3 \leq u+v+w \leq 0$. For the sets*

$$T_N^u := \{x \in A : T_N(x) \leq u\},$$
$$I_N^v := \{x \in A : v \leq I_N(x)\}$$

and

$$F_N^w := \{x \in A : w \leq F_N(x)\},$$

the set

$$A_N(u, v, w) := \{x \in A : T_N(x) \leq u, v \leq I_N(x) \text{ and } w \leq F_N(x)\}$$

is called the $(u, v, w)-$level set of A_N. Also, $A_N(u, v, w) = T_N^u \cap I_N^v \cap F_N^w$.

Definition 3.4. [12] *Let A be a Sheffer stroke BCK-algebra. Then a nonempty subset B of A is called a subalgebra of A if $(x|(y|y))|(x|(y|y)) \in B$, for all $x, y \in B$.*

Example 3.5. *Consider the Sheffer stroke BCK-algebra A in Example 3.2. Then a subset $\{0, 1\}$ of A is a subalgebra of A.*

Theorem 3.6. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A and u, v, w be any elements of $[-1, 0]$ with $-3 \leq u+v+w \leq 0$. If A_N is a neutrosophic $\mathcal{N}-$subalgebra of A, then the nonempty level set $A_N(u, v, w)$ of A_N is a subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A and x, y be any elements of $A_N(u, v, w)$, for $u, v, w \in [-1, 0]$ with $-3 \leq u+v+w \leq 0$. Then $T_N(x), T_N(y) \leq u; v \leq I_N(x), I_N(y)$ and $w \leq F_N(x), F_N(y)$. Since

$$T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\} \leq u,$$
$$v \leq \min\{I_N(x), I_N(y)\} \leq I_N((x|(y|y))|(x|(y|y)))$$

and

$$w \leq \min\{F_N(x), F_N(y)\} \leq F_N((x|(y|y))|(x|(y|y))),$$

for all $x, y \in A$, it is obtained that $(x|(y|y))|(x|(y|y)) \in T_N^u, I_N^v, F_N^w$. Then

$$(x|(y|y))|(x|(y|y)) \in T_N^u \cap I_N^v \cap F_N^w = A_N(u, v, w),$$

and so, $A_N(u, v, w)$ is a subalgebra of A. □

Theorem 3.7. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A and T_N^u, I_N^v and F_N^w be subalgebras of A, for all $u, v, w \in [-1, 0]$ with $-3 \leq u+v+w \leq 0$. Then A_N is a neutrosophic $\mathcal{N}-$subalgebra of A_N.*

Proof. Let T_N^u, I_N^v and F_N^w be subalgebras of A, for all $u, v, w \in [-1, 0]$ with $-3 \leq u+v+w \leq 0$. Suppose that

$$u_1 = \max\{T_N(x), T_N(y)\} < T_N((x|(y|y))|(x|(y|y))) = u_2,$$
$$v_1 = I_N((x|(y|y))|(x|(y|y))) < \min\{I_N(x), I_N(y)\} = v_2$$

and

$$w_1 = F_N((x|(y|y))|(x|(y|y))) < \min\{F_N(x), F_N(y)\} = w_2,$$

for some $x, y \in A$. If $u = \frac{1}{2}(u_1 + u_2), v = \frac{1}{2}(v_1 + v_2) and w = \frac{1}{2}(w_1 + w_2)$ are elements of $[-1, 0)$, then $u_1 < u < u_2$, $v_1 < v < v_2$ and $w_1 < w < w_2$. Thus, $x, y \in T_N^u, I_N^v, F_N^w$ but $(x|(y|y))|(x|(y|y)) \notin T_N^u, I_N^v, F_N^w$ which is a contradiction. So,

$$T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\},$$
$$\min\{I_N(x), I_N(y)\} \leq I_N((x|(y|y))|(x|(y|y)))$$

and

$$\min\{F_N(x), F_N(y)\} \leq F_N((x|(y|y))|(x|(y|y))),$$

for all $x, y \in A$. Hence, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

Theorem 3.8. *Let $\{A_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$ subalgebras of a Sheffer stroke BCK-algebra A. Then $\{A_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive modular lattice.*

Proof. Let α be a nonempty subset of $\{A_{N_i} : i \in \mathbb{N}\}$. Since every A_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of A, for all $i \in \mathbb{N}$, it satisfies the condition (1), for all $x, y \in A$, and so, $\bigcap \alpha$ satisfies the condition (1). Then $\bigcap \alpha$ is a neutrosophic $\mathcal{N}-$subalgebra of A. Let β be a family of all neutrosophic $\mathcal{N}-$subalgebras of A containing $\bigcup\{A_{N_i} : i \in \mathbb{N}\}$. Hence, $\bigcap \beta$ is a neutrosophic $\mathcal{N}-$subalgebra of A. If $\bigwedge_{i\in\mathbb{N}} A_{N_i} = \bigcap_{i\in\mathbb{N}} A_{N_i}$ and $\bigvee_{i\in\mathbb{N}} A_{N_i} = \bigcap B$, then $(\{A_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Also, this lattice is distibutive by the definitions of $\bigvee$ and $\bigwedge$. Since every distributive lattice is modular, then this lattice is modular. □

Lemma 3.9. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCK-algebra A. Then*

$$T_N(0) \leq T_N(x), I_N(x) \leq I_N(0) \text{ and } F_N(x) \leq F_N(0), \tag{2}$$

for all $x \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A. Then it follows from Lemma 2.3 (2) that

$$T_N(0) = T_N((x|(x|x))|(x|(x|x))) \leq \max\{T_N(x), T_N(x)\} = T_N(x),$$
$$I_N(x) = \min\{I_N(x), I_N(x)\} \leq I_N((x|(x|x))|(x|(x|x))) = I_N(0)$$

and

$$F_N(x) = \min\{F_N(x), F_N(x)\} \leq F_N((x|(x|x))|(x|(x|x))) = F_N(0),$$

for all $x \in A$. □

The inverse of Lemma 3.9 does not usually hold.

Example 3.10. *Consider the Sheffer stroke BCK-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure*

$$A_N = \left\{ \frac{y}{(-0.3, -0.7, -0.6)} : \right\} \cup \left\{ \frac{u}{(-0.91, 0, 0)} : \ u \in A - \{y\} \right\}$$

on A satisfies the condition (2) but it is not a neutrosophic $\mathcal{N}-$subalgebra of A since $I_N((1|(x|x))|(1|(x|x))) = I_N(y) = -0.7 < 0 = \min\{I_N(1), I_N(x)\}$.

Lemma 3.11. *A neutrosophic $\mathcal{N}-$subalgebra A_N of a Sheffer stroke BCK-algebra A satisfies*

$$T_N((x|(y|y))|(x|(y|y))) \leq T_N(y),$$
$$I_N(y) \leq I_N((x|(y|y))|(x|(y|y)))$$

and

$$F_N(y) \leq F_N((x|(y|y))|(x|(y|y))),$$

for all $x, y \in A$ if and only if T_N, I_N and F_N are constant.

Proof. Let A_N be a a neutrosophic $\mathcal{N}-$subalgebra of A satisfying

$$T_N((x|(y|y))|(x|(y|y))) \leq T_N(y),$$
$$I_N(y) \leq I_N((x|(y|y))|(x|(y|y)))$$

and

$$F_N(y) \leq F_N((x|(y|y))|(x|(y|y))),$$

for any $x, y \in A$. Since $T_N(x) = T_N((x|(0|0))|(x|(0|0))) \leq T_N(0)$, $I_N(0) \leq I_N((x|(0|0))|(x|(0|0))) = I_N(x)$ and $F_N(0) \leq F_N((x|(0|0))|(x|(0|0))) = F_N(x)$ from Lemma 2.3 (6), it follows from Lemma 3.9 that $T_N(x) = T_N(0)$, $I_N(x) = I_N(0)$ and $F_N(x) = F_N(0)$, for all $x \in A$. Thus, T_N, I_N and F_N are constant. Conversely, it is obvious since T_N, I_N and F_N are constant. □

Definition 3.12. *A neutrosophic $\mathcal{N}-$structure A_N on a Sheffer stroke BCK-algebra A is called a neutrosophic $\mathcal{N}-$ideal of A if*

$$\begin{array}{c} T_N(0) \leq T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\}, \\ \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \leq I_N(x) \leq I_N(0) \\ \text{and} \\ \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq F_N(x) \leq F_N(0), \end{array} \tag{3}$$

for all $x, y \in A$.

Example 3.13. *Consider the Sheffer stroke BCK-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure*

$$A_N = \left\{ \frac{u}{(-0.87, -0.23, -0.12)} : \ u = 0, y \right\} \cup \left\{ \frac{u}{(-0.34, -0.41, -0.56)} : \ u = x, 1 \right\}$$

on A is a neutrosophic $\mathcal{N}-$ideal of A.

Lemma 3.14. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A if and only if*

(1) $x \leq y$ implies $T_N(x) \leq T_N(y)$, $I_N(y) \leq I_N(x)$ and $F_N(y) \leq F_N(x)$,
(2) $T_N((x|x)|(y|y)) \leq \max\{T_N(x), T_N(y)\}$,
$\min\{I_N(x), I_N(y)\} \leq I_N((x|x)|(y|y))$ and
$\min\{F_N(x), F_N(y)\} \leq F_N((x|x)|(y|y))$,

for all $x, y \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A.

(1) Assume that $x \leq y$. Then $(x|(y|y))|(x|(y|y)) = 0$. Hence, it is obtained from Lemma 3.9 that

$$\begin{aligned} T_N(x) &\leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \\ &= \max\{T_N(0), T_N(y)\} \\ &= T_N(y), \end{aligned}$$

$$\begin{aligned} I_N(y) &= \min\{I_N(0), I_N(y)\} \\ &= \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \\ &\leq I_N(x) \end{aligned}$$

and

$$\begin{aligned} F_N(y) &= \min\{F_N(0), F_N(y)\} \\ &= \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \\ &\leq F_N(x), \end{aligned}$$

for all $x, y \in A$.

(2) Since
$(((((x|x)|(y|y))|(y|y))|(((x|x)|(y|y))|(y|y)))|(x|x))|$
$(((((x|x)|(y|y))|(y|y))|(((x|x)|(y|y))|(y|y)))|(x|x))$
$= (((x|x)|(y|y))|(((x|x)|(y|y))|((x|x)|(y|y))))|$
$(((x|x)|(y|y))|(((x|x)|(y|y))|((x|x)|(y|y))))$
$= 0$
from (S1), (S2), Lemma 2.3 (2) and (3), it follows from Lemma 2.4 that $((x|x)|(y|y))|(y|y) \leq x$, for all $x, y \in A$. Thus, we get from (1) that

$$\begin{aligned} T_N((x|x)|(y|y)) &\leq \max\{T_N((((x|x)|(y|y))|(y|y))| \\ &\quad (((x|x)|(y|y))|(y|y))), T_N(y)\} \\ &\leq \max\{T_N(x), T_N(y)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N(x), I_N(y)\} &\leq \min\{I_N((((x|x)|(y|y))|(y|y))| \\ &\quad (((x|x)|(y|y))|(y|y))), I_N(y)\} \\ &\leq I_N((x|x)|(y|y)) \end{aligned}$$

and

$$\begin{aligned} \min\{F_N(x), F_N(y)\} &\leq \min\{F_N((((x|x)|(y|y))|(y|y))| \\ &\quad (((x|x)|(y|y))|(y|y))), F_N(y)\} \\ &\leq F_N((x|x)|(y|y)), \end{aligned}$$

for all $x, y \in A$.

Conversely, let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the properties (1) and (2). Since 0 is the least element of A, we have from (1) that

$T_N(0) \leq T_N(x)$, $I_N(x) \leq I_N(0)$ and $F_N(x) \leq F_N(0)$, for all $x \in A$. Since $x \leq (x|(y|y))|(y|y)$ from Lemma 2.4, we obtain from (1), (2) and (S2) that

$$\begin{aligned} T_N(x) &\leq T_N((x|(y|y))|(y|y)) \\ &= T_N((((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))|(y|y)) \\ &\leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} &\leq I_N((((x|(y|y))|(x|(y|y)))| \\ &\quad ((x|(y|y))|(x|(y|y))))|(y|y)) \\ &= I_N((x|(y|y))|(y|y)) \\ &\leq I_N(x) \end{aligned}$$

and

$$\begin{aligned} \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} &\leq F_N((((x|(y|y))|(x|(y|y)))| \\ &\quad ((x|(y|y))|(x|(y|y))))|(y|y)) \\ &= F_N((x|(y|y))|(y|y)) \\ &\leq F_N(x), \end{aligned}$$

for all $x, y \in A$. Thereby, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Lemma 3.15. *Let A_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebra A. Then*

(1) $T_N(y) \leq T_N(x|(y|y))$, $I_N(x|(y|y)) \leq I_N(y)$ and $F_N(x|(y|y)) \leq F_N(y)$,
(2) $T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\}$,
$\min\{I_N(x), I_N(y)\} \leq I_N((x|(y|y))|(x|(y|y)))$ and
$\min\{F_N(x), F_N(y)\} \leq F_N((x|(y|y))|(x|(y|y)))$,
(3) $T_N(x) \leq T_N((x|(y|y))|(y|y))$, $I_N((x|(y|y))|(y|y)) \leq I_N(x)$ and
$F_N((x|(y|y))|(y|y)) \leq F_N(x)$,

for all $x, y \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Then

(1) Since $y \leq x|(y|y)$ from Lemma 2.5 (v), it follows from Lemma 3.14 (1) that $T_N(y) \leq T_N(x|(y|y))$, $I_N(x|(y|y)) \leq I_N(y)$ and $F_N(x|(y|y)) \leq F_N(y)$, for all $x, y \in A$.
(2) Since $(x|(y|y))|(x|(y|y)) \leq x$ from Lemma 2.5 (iv), it is obtained from Lemma 3.14 (1) that

$$T_N((x|(y|y))|(x|(y|y))) \leq T_N(x) \leq \max\{T_N(x), T_N(y)\},$$

$$\min\{I_N(x), I_N(y)\} \leq I_N(x) \leq I_N((x|(y|y))|(x|(y|y)))$$

and

$$\min\{F_N(x), F_N(y)\} \leq F_N(x) \leq F_N((x|(y|y))|(x|(y|y))),$$

for all $x, y \in A$.

(3) Since $x \leq (x|(y|y))|(y|y)$ from Lemma 2.5 (vi), we have from Lemma 3.14 (i) that $T_N(x) \leq T_N((x|(y|y))|(y|y))$, $I_N((x|(y|y))|(y|y)) \leq I_N(x)$ and $F_N((x|(y|y))|(y|y)) \leq F_N(x)$, for all $x, y \in A$.

□

Theorem 3.16. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A if and only if*

$$\begin{array}{l}(y|(z|z))|(y|(z|z)) \leq x \ \textit{ implies } \ T_N(y) \leq \max\{T_N(x), T_N(z)\}, \\ \min\{I_N(x), I_N(z)\} \leq I_N(y) \ \textit{ and } \ \min\{F_N(x), F_N(z)\} \leq F_N(y),\end{array} \tag{4}$$

for all $x, y, z \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A and $(y|(z|z))|(y|(z|z)) \leq x$. Then it follows from Lemma 3.14 (1) that

$$T_N(y) \leq \max\{T_N((y|(z|z))|(y|(z|z))), T_N(z)\} \leq \max\{T_N(x), T_N(z)\},$$

$$\min\{I_N(x), I_N(z)\} \leq \min\{I_N((y|(z|z))|(y|(z|z))), I_N(z)\} \leq I_N(y)$$

and

$$\min\{F_N(x), F_N(z)\} \leq \min\{F_N((y|(z|z))|(y|(z|z))), F_N(z)\} \leq F_N(y),$$

for all $x, y, z \in A$.

Conversely, let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the condition (4). Since $(0|(x|x))|(0|(x|x)) = 0 \leq x$ from Lemma 2.3 (7) and Lemma 2.4, it is obtained from the condition (4) that $T_N(0) \leq \max\{T_N(x), T_N(x)\} = T_N(x)$, $I_N(x) = \min\{I_N(x), I_N(x)\} \leq I_N(0)$ and $F_N(x) = \min\{F_N(x), F_N(x)\} \leq F_N(0)$, for all $x \in A$. Since $(x|(y|y))|(x|(y|y)) \leq (x|(y|y))|(x|(y|y))$, for all $x, y \in A$, it follows from the condition (4) that

$$T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$\min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \leq I_N(x)$$

and

$$\min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq F_N(x),$$

for all $x, y \in A$. Hence, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Definition 3.17. [12] *A nonempty subset I of a Sheffer stroke BCK-algebra A is called an ideal of A if it satisfies*
(I1) $0 \in I$,
(I2) $(x|(y|y))|(x|(y|y)) \in I$ *and* $y \in I$ *implies* $x \in I$, *for all* $x, y \in A$.

Example 3.18. *Consider the Sheffer stroke BCK-algebra A in Example 3.2. Then subsets A itself, $\{0, p\}$, $\{0, q\}$ and $\{0\}$ of A are ideals of A.*

Theorem 3.19. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A and u, v, w be any elements of $[-1, 0]$ with $-3 \leq u + v + w \leq 0$. If A_N is a neutrosophic $\mathcal{N}-$ideal of A, then the nonempty (u, v, w)-level set $A_N(u, v, w)$ of A_N is an ideal of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A and $A_N(u,v,w)$ be a nonempty subset of A, for $u,v,w \in [-1,0]$ with $-3 \leq u+v+w \leq 0$. Since $T_N(0) \leq T_N(x) \leq u, v \leq I_N(x) \leq I_N(0)$ and $w \leq F_N(x) \leq F_N(0)$, for all $x \in A$, it follows that $0 \in T_N(u,v,w)$. Let $(x|(y|y))|(x|(y|y)) \in A_N(u,v,w)$ and $y \in A_N(u,v,w)$. Since $T_N((x|(y|y))|(x|(y|y))) \leq u, T_N(y) \leq u$; $v \leq I_N((x|(y|y))|(x|(y|y))), v \leq I_N(y)$; $w \leq F_N((x|(y|y))|(x|(y|y)))$ and $w \leq F_N(y)$, it is obtained that

$$T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq u,$$

$$v \leq \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \leq I_N(x)$$

and

$$w \leq \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq F_N(x),$$

for all $x,y \in A$. Thus, $x \in A_N(u,v,w)$. Hence, $A_N(u,v,w)$ is an ideal of A. □

Theorem 3.20. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCK-algebra A and T_N^u, I_N^v, F_N^w be ideals of A, for all $u,v,w \in [-1,0]$ with $-3 \leq u+v+w \leq 0$. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$structure on A and T_N^u, I_N^v, F_N^w be ideals of A, for all $u,v,w \in [-1,0]$ with $-3 \leq u+v+w \leq 0$. Suppose that $T_N(x) < T_N(0)$, $I_N(0) < I_N(x)$ and $F_N(0) < F_N(x)$, for some $x \in A$. If $u = \frac{1}{2}(T_N(0)+T_N(x))$, $v = \frac{1}{2}(I_N(0)+I_N(x))$ and $w = \frac{1}{2}(F_N(0)+F_N(x))$ are elements of $[-1,0)$, then $T_N(x) < u < T_N(0)$, $I_N(0) < v < I_N(x)$ and $F_N(0) < w < F_N(x)$, and so, $0 \notin T_N^u, I_N^v, F_N^w$ which is a contradiction with (I1). Thus, $T_N(0) \leq T_N(x)$, $I_N(x) \leq I_N(0)$ and $F_N(x) \leq F_N(0)$, for all $x \in A$. Assume that

$$u_1 = \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} < T_N(x) = u_2,$$

$$v_1 = I_N(x) < \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} = v_2,$$

and

$$w_1 = F_N(x) < \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} = w_2.$$

If $u^0 = \frac{1}{2}(u_1+u_2)$, $v^0 = \frac{1}{2}(v_1+v_2)$ and $w^0 = \frac{1}{2}(w_1+w_2)$ are elements of $[-1,0)$, then $u_1 < u^0 < u_2$, $v_1 < v^0 < v_2$ and $w_1 < w^0 < w_2$. So, $(x|(y|y))|(x|(y|y)) \in T_N^{u^0}, I_N^{v^0}, F_N^{w^0}$ and $y \in T_N^{u^0}, I_N^{v^0}, F_N^{w^0}$ but $a \notin T_N^{\alpha^*}, I_N^{\beta^*}, F_N^{\gamma^*}$, which is a contradiction with (I2). Hence,

$$T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$\min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \leq I_N(x)$$

and

$$\min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq I_N(x),$$

for all $x,y \in A$. HTherefore, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Definition 3.21. *Let $(A,|_A,0_A)$ and $(B,|_B,0_B)$ be Sheffer stroke BCK-algebras. Then a mapping $\rho : A \longrightarrow B$ is called a homomorphism if $\rho(x|_A y) = \rho(x)|_B\rho(y)$, for all $x,y \in A$ and $f(0_A) = 0_B$.*

Theorem 3.22. *Let $(A,|_A,0_A)$ and $(B,|_B,0_B)$ be Sheffer stroke BCK-algebras, $\rho : A \longrightarrow B$ be a surjective homomorphism and $B_N = \dfrac{B}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$structure on B. Then B_N is a neutrosophic $\mathcal{N}-$ideal of B if and only if $B_N^\rho = \dfrac{A}{(T_N^\rho, I_N^\rho, F_N^\rho)}$ is a neutrosophic $\mathcal{N}-$ideal of A where the $\mathcal{N}-$functions $T_N^\rho, I_N^\rho, F_N^\rho : A \longrightarrow [-1,0]$ on A are defined by $T_N^\rho(x) = T_N(\rho(x)), I_N^\rho(x) = I_N(\rho(x))$ and $F_N^\rho(x) = F_N(\rho(x))$, for all $x \in A$, respectively.*

Proof. Let $(A,|,0)$ and $(B,|,0)$ be Sheffer stroke BCK-algebras, $\rho : A \longrightarrow B$ be a surjective homomorphism and $B_N = \dfrac{B}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$ideal of B. Then $T_N^\rho(0_A) = T_N(\rho(0_A)) = T_N(0_B)) \leq T_N(y) = T_N(\rho(x)) = T_N^\rho(x)$, $I_N^\rho(x) = I_N(\rho(x)) = I_N(y) \leq I_N(0_B) = I_N(\rho(0_A)) = I_N^\rho(0_A)$ and $F_N^\rho(x) = F_N(\rho(x)) = F_N(y) \leq F_N(0_B) = F_N(\rho(0_A)) = F_N^\rho(0_A)$, for all $a \in A$. Also,

$$\begin{aligned} T_N^\rho(x_1) &= T_N(\rho(x_1)) \\ &\leq \max\{T_N((\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))|_B \\ &\quad (\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))), T_N(\rho(x_2))\} \\ &= \max\{T_N(\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))), T_N(\rho(x_2))\} \\ &= \max\{T_N^\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), T_N^f(x_2)\}, \end{aligned}$$

$$\begin{aligned} &\min\{I_N^\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), I_N^\rho(x_2)\} \\ &= \min\{I_N(\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))), I_N(\rho(x_2))\} \\ &= \min\{I_N((\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))|_B \\ &\quad (\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))), I_N(\rho(x_2))\} \\ &\leq I_N(\rho(x_1)) \\ &= I_N^\rho(x_1) \end{aligned}$$

and

$$\begin{aligned} &\min\{F_N^\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), F_N^\rho(x_2)\} \\ &= \min\{F_N(\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))), F_N(\rho(x_2))\} \\ &= \min\{F_N((\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))|_B \\ &\quad (\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))), F_N(\rho(x_2))\} \\ &\leq F_N(\rho(x_1)) \\ &= F_N^\rho(x_1) \end{aligned}$$

for all $x_1, x_2 \in A$. So, $B_N^\rho = \dfrac{A}{(T_N^\rho, I_N^\rho, F_N^\rho)}$ is a neutrosophic $\mathcal{N}-$ideal of A.

Conversely, let B_N^ρ be a neutrosophic $\mathcal{N}-$ideal of A. Hence, $T_N(0_B) = T_N(\rho(0_A)) = T_N^\rho(0_A) \leq T_N^\rho(x) = T_N(\rho(x)) = T_N(y)$, $I_N(y) = I_N(\rho(x)) = I_N^\rho(x) \leq I_N^\rho(0_A) = I_N(\rho(0_A)) = I_N(0_B)$ and $F_N(y) = F_N(\rho(x)) = F_N^\rho(x) \leq F_N^\rho(0_A) =$

$F_N(\rho(0_A)) = F_N(0_B)$, for all $x \in B$. Moreover,

$$\begin{aligned} T_N(y_1) &= T_N(\rho(x_1)) \\ &= T_N^\rho(x_1) \\ &\leq \max\{T_N^\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), T_N^\rho(x_2)\} \\ &= \max\{T_N(\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))), T_N(\rho(x_2))\} \\ &= \max\{T_N((\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))|_B \\ &\quad (\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))), T_N(\rho(x_2))\} \\ &= \max\{T_N((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))), T_N(y_2)\}, \end{aligned}$$

$$\begin{aligned} &\min\{I_N((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))), I_N(y_2)\} \\ &= \min\{I_N((\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))|_B \\ &\quad (\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))), I_N(\rho(x_2))\} \\ &= \min\{I_N(\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))), I_N(\rho(x_2))\} \\ &= \min\{I_N^\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), I_N^\rho(x_2)\} \\ &\leq I_N^\rho(x_1) \\ &= I_N(\rho(x_1)) \\ &= I_N(y_1) \end{aligned}$$

and

$$\begin{aligned} &\min\{F_N((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))), F_N(y_2)\} \\ &= \min\{F_N((\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))|_B \\ &\quad (\rho(x_1)|_B(\rho(x_2)|_B\rho(x_2)))), F_N(\rho(x_2))\} \\ &= \min\{IF_N(\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))), F_N(\rho(x_2))\} \\ &= \min\{F_N^\rho((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), F_N^\rho(x_2)\} \\ &\leq F_N^\rho(x_1) \\ &= F_N(\rho(x_1)) \\ &= F_N(y_1), \end{aligned}$$

for all $y_1, y_2 \in B$. Thus, $B_N = \dfrac{B}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}-$ideal of B. □

Theorem 3.23. *Every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebra A is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Since $(x|(y|y))|(x|(y|y)) \leq x$ from Lemma 2.5 (iv), it is obtained from Lemma 3.14 (1) that

$$T_N((x|(y|y))|(x|(y|y))) \leq T_N(x) \leq \max\{T_N(x), T_N(y)\},$$

$$\min\{I_N(x), I_N(y)\} \leq I_N(x) \leq I_N((x|(y|y))|(x|(y|y)))$$

and

$$\min\{F_N(x), F_N(y)\} \leq F_N(x) \leq F_N((x|(y|y))|(x|(y|y))),$$

for all $x, y \in A$. Thus, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

The inverse of Theorem 3.23 is generally not true.

Example 3.24. *In Example 3.2, the neutrosophic $\mathcal{N}-$subalgebra of A is not a neutrosophic $\mathcal{N}-$ideal of A since* $\max\{T_N((x|(1|1))|(x|(1|1))), T_N(1)\} = -1 < -0.2 = T_N(x)$.

Lemma 3.25. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCK-algebra A satisfying*

$$\begin{aligned} & T_N(x|(y|y)) \leq \max\{T_N((x|((y|(z|z))|(y|(z|z))))|(x \\ & \qquad |((y|(z|z))|(y|(z|z))))), T_N(x|(z|z))\} \\ & \min\{I_N((x|((y|(z|z))|(y|(z|z))))|(x|((y|(z| \\ & z))|(y|(z|z))))), I_N(x|(z|z))\} \leq I_N(x|(y|y)) \\ & \qquad \text{and} \\ & \min\{F_N((x|((y|(z|z))|(y|(z|z))))|(x|((y|(z| \\ & z))|(y|(z|z))))), F_N(x|(z|z))\} \leq F_N(x|(y|y)), \end{aligned} \tag{5}$$

for all $x, y, z \in A$. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of A satisfying the condition (5). Then we have from Lemma 3.9 that $T_N(0) \leq T_N(x), I_N(x) \leq I_N(0)$ and $F_N(x) \leq F_N(0)$, for all $x \in A$. By substituting $[x := 0|0]$, $[y := x]$ and $[z := y]$ in the condition (5), simultaneously, it is obtained from Lemma 2.3 (4) that

$$\begin{aligned} T_N(x) &= T_N((0|0)|(x|x)) \\ &\leq \max\{T_N(((0|0)|((x|(y|y))|(x|(y|y))))|((0| \\ &\quad 0)|((x|(y|y))|(x|(y|y))))), T_N((0|0)|(y|y))\} \\ &= \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\}, \end{aligned}$$

$$\begin{aligned} & \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \\ & = \min\{I_N(((0|0)|((x|(y|y))|(x|(y|y))))|((0| \\ & \quad 0)|((x|(y|y))|(x|(y|y))))), I_N((0|0)|(y|y))\} \\ & \leq I_N((0|0)|(x|x)) \\ & = I_N(x) \end{aligned}$$

and

$$\begin{aligned} & \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \\ & = \min\{F_N(((0|0)|((x|(y|y))|(x|(y|y))))|((0| \\ & \quad 0)|((x|(y|y))|(x|(y|y))))), F_N((0|0)|(y|y))\} \\ & \leq F_N((0|0)|(x|x)) \\ & = F_N(x), \end{aligned}$$

for all $x, y \in A$. Therefore, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Lemma 3.26. *Let A_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A. Then the subsets $A_{T_N} = \{x \in A : T_N(x) = T_N(0)\}$, $A_{I_N} = \{x \in A : I_N(x) = I_N(0)\}$ and $A_{F_N} = \{x \in A : F_N(x) = F_N(0)\}$ of A are ideals of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Then it is obvious that $0 \in A_{T_N}, A_{I_N}, A_{F_N}$. Assume that $(x|(y|y))|(x|(y|y)), y \in A_{T_N}, A_{I_N}, A_{F_N}$. Since

$$T_N(y) = T_N(0) = T_N((x|(y|y))|(x|(y|y))),$$

$$I_N(y) = I_N(0) = I_N((x|(y|y))|(x|(y|y)))$$

and

$$F_N(y) = F_N(0) = F_N((x|(y|y))|(x|(y|y))),$$

it is obtained that

$$T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} = \max\{T_N(0), T_N(0)\} = T_N(0),$$
$$I_N(0) = \min\{I_N(0), I_N(0)\} = \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \leq I_N(x)$$

and

$$F_N(0) = \min\{F_N(0), F_N(0)\} = \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq F_N(x).$$

Since $T_N(x) = T_N(0)$, $I_N(x) = I_N(0)$ and $F_N(x) = F_N(0)$, we get that $x \in A_{T_N}, A_{I_N}, A_{F_N}$. Thus, A_{T_N}, A_{I_N} and A_{F_N} are ideals of A. □

Definition 3.27. *Let A be a Sheffer stroke BCK-algebra. Define the subsets*

$$A_N^{x_t} := \{x \in A : T_N(x) \leq T_N(x_t)\},$$
$$A_N^{x_i} := \{x \in A : I_N(x_i) \leq I_N(x)\}$$

and

$$A_N^{x_f} := \{x \in A : F_N(x_f) \leq F_N(x)\}$$

of A, for all $x_t, x_i, x_f \in A$. Moreover, $x_t \in A_N^{x_t}, x_i \in A_N^{x_i}$ and $x_f \in A_N^{x_f}$.

Example 3.28. *Consider the Sheffer stroke BCK-algebra A in Example 3.2. Let*

$$T_N(u) = \begin{cases} -0.08, & \text{if } u = 0, 1 \\ -0.58, & \text{if } u = x \\ -0.15, & u = y, \end{cases} \qquad I_N(u) = \begin{cases} -0.29, & \text{if } u = 1 \\ -0.001, & \text{otherwise,} \end{cases}$$

$$F_N(u) = \begin{cases} -0.86, & \text{if } u = 0 \\ 0, & \text{otherwise,} \end{cases} \qquad x_t = y, x_i = 1 \text{ and } x_f = x.$$

Then

$$A_N^{x_t} = \{u \in A : T_N(u) \leq T_N(y)\} = \{x, y\},$$
$$A_N^{x_i} = \{u \in A : I_N(1) \leq I_N(u)\} = A$$

and

$$A_N^{x_f} = \{u \in A : F_N(x) \leq F_N(u)\} = \{x, y, 1\}.$$

Theorem 3.29. *Let x_t, x_i and x_f be any elements of a Sheffer stroke BCK-algebra A. If A_N is a neutrosophic $\mathcal{N}-$ideal of A, then $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are ideals of A.*

Proof. Let x_t, x_i and x_f be any elements of A and A_N be a neutrosophic $\mathcal{N}-$ideal of A. Since $T_N(0) \leq T_N(x_t)\}$, $I_N(x_i) \leq I_N(0)$ and $F_N(x_f) \leq F_N(0)$, for all $x_t, x_i, x_f \in A$, it follows that $0 \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$. Assume that $(x|(y|y))|(x|(y|y)), y \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$. Since

$$T_N((x|(y|y))|(x|(y|y))), T_N(y) \leq T_N(x_t),$$
$$I_N(x_i) \leq I_N((x|(y|y))|(x|(y|y))), I_N(y)$$

and

$$F_N(x_f) \leq F_N((x|(y|y))|(x|(y|y))), F_N(b),$$

it is obtained that

$$T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq T_N(x_t),$$
$$I_N(x_i) \leq \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \leq I_N(x)$$

and

$$F_N(x_f) \leq \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq F_N(x),$$

which means that $x \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$. Thereby, $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are ideals of A. $\square$

Example 3.30. *Consider the Sheffer stroke BCH-algebra A in Example 3.2. For a neutrosophic $\mathcal{N}-$ideal*

$$A_N = \{\frac{0}{(-0.74, -0.26, -0.67)}\} \cup \{\frac{u}{(-0.002, -0.301, -0.85)} : A - \{0\}\}$$

of A and $x_t = 1, x_i = y, x_f = 0 \in S$, the subsets

$$A_N^{x_t} = \{u \in A : T_N(u) \leq T_N(1)\} = A,$$

$$A_N^{x_i} = \{u \in A : I_N(y) \leq I_N(u)\} = A$$

and

$$A_N^{x_f} = \{u \in A : F_N(0) \leq F_N(u)\} = \{0\}$$

of A are ideals of A.

Theorem 3.31. *Let x_t, x_i and x_f be any elements of a Sheffer stroke BCK-algebra A and A_N be a neutrosophic $\mathcal{N}-$structure on A.*

(1) If $A_N^{ax_t}, A_N^{x_i}$ and $A_N^{x_f}$ are ideals of A, then

$$\max\{T_N((y|(z|z))|(y|(z|z))), T_N(z)\} \leq T_N(x) \Rightarrow T_N(y) \leq T_N(x),$$

$$I_N(x) \leq \min\{I_N((y|(z|z))|(y|(z|z))), I_N(z)\} \Rightarrow I_N(x) \leq I_N(y) \quad \textit{and} \quad (6)$$

$$F_N(x) \leq \min\{F_N((y|(z|z))|(y|(z|z))), F_N(z)\} \Rightarrow F_N(x) \leq F_N(y),$$

for all $x, y, z \in A$.

(2) If A_N satisfies the condition (6) and

$$T_N(0) \leq T_N(x), \quad I_N(x) \leq I_N(0) \quad \textit{and} \quad F_N(x) \leq F_N(0), \quad (7)$$

for all $x \in A$, then $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are ideals of A, for all $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$.

Proof. Let x_t, x_i and x_f be any elements of A and A_N be a neutrosophic $\mathcal{N}-$structure on A.

(1) Assume that $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are ideals of A and

$$\max\{T_N((y|(z|z))|(y|(z|z))), T_N(z)\} \leq T_N(x),$$

$$I_N(x) \leq \min\{I_N((y|(z|z))|(y|(z|z))), I_N(z)\}$$

and

$$F_N(x) \leq \min\{F_N((y|(z|z))|(y|(z|z))), F_N(z)\}.$$

Since $(y|(z|z))|(y|(z|z)), z \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$ where $x_t = x_i = x_f = x$, it follows that $y \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$ in which $x_t = x_i = x_f = x$. Hence, $T_N(y) \leq T_N(x)$, $I_N(x) \leq I_N(y)$ and $F_N(x) \leq F_N(y)$, for all $x, y, z \in A$.

(2) Let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (6) and (7), for any $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$. Then it is obtained from the condition (7) that $0 \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$. Assume that $(x|(y|y))|(x|(y|y)), y \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$. So,

$$T_N((x|(y|y))|(x|(y|y))), T_N(y) \leq T_N(x_t),$$

$$I_N(x_i) \leq I_N((x|(y|y))|(x|(y|y))), I_N(y)$$

and

$$F_N(x_f) \leq F_N((x|(y|y))|(x|(y|y))), F_N(y).$$

Since

$$\max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq T_N(x_t),$$

$$I_N(x_i) \leq \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(x_f) \leq \min\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

we get from the condition (6) that $T_N(x) \leq T_N(x_t)$, $I_N(x_i) \leq I_N(x)$ and $F_N(x_f) \leq F_N(x)$. Thus, $x \in A_N^{x_t}, A_N^{x_i}, A_N^{x_f}$. Thus, $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are ideals of A.

□

Example 3.32. *Consider the Sheffer stroke BCH-algebra A in Example 3.2. Let*

$$T_N(u) = \begin{cases} -0.43, & \text{if } u = 0, x \\ -0.004, & \text{otherwise,} \end{cases} \qquad I_N(u) = \begin{cases} -1, & \text{if } u = y, 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$F_N(u) = \begin{cases} 0, & \text{if } u = 0 \\ -0.923, & \text{otherwise,} \end{cases} \qquad \text{and } \ x_t = y, x_i = x \ x_f = 0 \in A.$$

Then the ideals

$$A_N^{x_t} = A, A_N^{x_i} = \{0, x\} \text{ and } A_N^{x_f} = \{0\}$$

of A satisfy the condition (6).

Let

$$A_N = \left\{\frac{0}{(-1, -0.3, -0.001)}\right\} \cup \left\{\frac{u}{(-0.003, -0.48, -1)} \ : \ A - \{0\}\right\}$$

be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (6) and (7). Then the subsets $A_N^{x_t} = \{0\}, A_N^{x_i} = A$ and $A_N^{x_f} = A$ of A are ideals of A, where $x_t = 0, x_i = x$ and $x_f = 1$.

4. CONCLUSION

In this paper, a neutrosophic $\mathcal{N}-$subalgebra (ideal) and a level-set of neutrosophic $\mathcal{N}-$ structures on Sheffer stroke BCK-algebras are defined, and it is given that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (ideal) of a Sheffer stroke BCK-algebra is a subalgebra (an ideal) of this algebraic structure and the inverse is always true. Infact, we prove that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCK-algebra forms a complete distributive modular lattice, and examine the cases which $\mathcal{N}-$functions are constant. Also, some properties of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebra are presented. Homomorphisms between Sheffer stroke BCK-algebras are introduced and neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCK-algebras are constructed by means of a surjective homomorphism. It is illustrated that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCK-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse does not mostly hold. Moreover, some subsets A_{T_N}, A_{I_N} and A_{F_N} of a Sheffer stroke BCK-algebra are its ideals for the neutrosophic $\mathcal{N}-$ideal defined by means of the $\mathcal{N}-$functions T_N, I_N and F_N. Finally, subsets $A_N^{x_t}$, $A_N^{x_i}$ and $A_N^{x_f}$ of a Sheffer stroke BCK-algebra are described for its any elements x_t, x_i, x_f, and it is stated that these subsets are ideals of this algebra if its neutrosophic $\mathcal{N}-$structure is the neutrosophic $\mathcal{N}-$ideal.

In future works, we want to study on various ideals and fuzzy structures on Sheffer stroke BCK-algebras.

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