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ON FREE PRODUCT OF N-COGROUPS

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Abstract. The structure of rings has been generalized into near-rings which are not as strong as the first one. The additive group in a near-ring is not necessary an abelian group and it is allowed to have only one sided distributive law. Moreover, if there exists an action from a near-ring N to a group Γ , then the group Γ is called an N-group. On the other hand, with a different axiom, an action from a near-ring into a group could obtain an N-cogroup. In this paper we apply the definition of free product of groups as an alternative way to build a product of N-cogroups. This product can be viewed as a functor and we prove that this functor is a left adjoint functor. Moreover using this functor one can obtain a category of F-algebras.

Key words: Near-rings, N-cogroups, free product, left adjoint functor, F-algebras.

Abstrak. Definisi ring dapat diperumum menjadi near-ring yang syaratnya tidak sekuat ring. Grup aditif dalam near-ring tidak harus komutatif dan diperbolehkan hanya mempunyai sifat distributif satu sisi saja. Selanjutnya, jika didefinisikan suatu aksi dari near-ring N ke suatu grup Γ, maka grup Γ tersebut disebut N-group. Di pihak lain, melalui aksioma yang berbeda, suatu aksi dari N ke Γ menghasilkan suatu struktur yang disebut N-cogroup. Dalam paper ini diterapkan pengertian hasil kali bebas (*free product*) pada N-cogroup sebagai cara membangun suatu produk atau hasil kali dua buah N-kogrup. Lebih jauh, hasil kali ini dapat dipandang sebagai fungtor yang adjoin kiri. Melalui fungtor adjoin kiri inilah diperoleh suatu F-aljabar.

Kata kunci: Near-ring, N-cogroup, hasil kali bebas, fungtor adjoin kiri, F-aljabar.

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1. Introduction

The existence of tensor product in module theory [7] and semimodules over commutative semiring [1] give a motivation to do the similar thing in near-rings and N-groups. A beginning work was presented by Mahmood [4] in which she anticipated the noncommutativity of the groups by splitting the tensor product of N-group and N-cogroup into two cases : left and right tensor product. Wijayanti [6] also defined the tensor product of N-groups due to make a dualization of near-rings and N-cogroups but still in commutative addition case.

To avoid the misused of additive notion in commutative groups case, we present something more general to obtain a product for groups in general case. In this paper we apply the definition of free product of groups (it is also as a coproduct of groups) as another alternative to build a product of N-cogroups. Furthermore we show that this product is a left adjoint functor, which is motivated by Wisbauer [9]. This product induces an F-algebra and together with its homomorphism form a category of F-algebras. We investigate then the homomorphisms in this category applying some results of Gumm [3].

In this section we recall the basic notions of near-rings and N-groups as we refer to Pilz [5] and Clay [2].

Definition 1.1. *A left near-ring N is a set with two binary operations $+$ and $\cdot$ such that:*

(i) $(N, +)$ *is a group (not necessary commutative).*
(ii) $(N, \cdot)$ *is a semigroup.*
(iii) *For any $n_1, n_2, n_3 \in N$, $n_1 \cdot (n_2 + n_3) = (n_1 \cdot n_2) + (n_1 \cdot n_3)$ (left distributive law).*

To make a simpler writing, we denote $n_1 \cdot n_2$ as $n_1 n_2$ for any $n_1, n_2 \in N$. A near-ring with the right distributive law is called a right near-ring. If a near-ring has both right and left distributive laws, we call it just a near-ring.

Definition 1.2. *Let N be a left near-ring, Γ a group with neutral element 0_Γ and $\mu : \Gamma \times N \to \Gamma$ where $(\gamma, n) := \gamma n$. Group (Γ, μ) is called a* right N-group *if for any $\gamma \in \Gamma, n_1, n_2 \in N$ the following axioms are satisfied :*

(i) $\gamma(n_1 + n_2) = \gamma n_1 + \gamma n_2$, *and*
(ii) $\gamma(n_1 n_2) = (\gamma n_1) n_2$.

In Clay [2] such Γ is also called *near-ring module* or *N-module.* Any left near-ring N is a right N-group.

Moreover we also recall the other structure constructed by an action from a near-ring to a group but with the different axioms.

Definition 1.3. *Let N be a left near-ring, Γ a group with neutral element 0_Γ and $\mu' : N \times \Gamma \to \Gamma$ where $(n, \gamma) := n\gamma$. Group (Γ, μ') is called a* left N-cogroup *if for any $\gamma_1, \gamma_2 \in \Gamma$, $n, m \in N$ the following axioms are satisfied :*

(i) $n(\gamma_1 + \gamma_2) = n\gamma_1 + n\gamma_2$, *and*

(ii) $(nm)\gamma = n(m\gamma)$.

In Clay [2] such Γ is also called *near-ring comodule* or *N-comodule.* Any left near-ring N is a left N-cogroup.

We recall the definition of morphisms between two near-rings, two N-groups and two N-cogroups.

Definition 1.4. (i) *Let N, N' be left near-rings. Then $h : N \to N'$ is called a* (near-ring) homomorphism *if for any $m, n \in N$ $h(m+n) = h(m) + h(n)$ and $h(mn) = h(m)h(n)$.*

(ii) *Let N be a left near-ring, Γ, Γ' right N-groups. Then $h : \Gamma \to \Gamma'$ is called an* N-homomorphism of N-group *if for any $\gamma, \gamma' \in \Gamma$ $h(\gamma + \gamma') = h(\gamma) + h(\gamma')$ and $h(\gamma n) = h(\gamma)n$.*

(iii) *Let N be a left near-ring, Λ, Λ' left N-cogroups. Then $h : \Lambda \to \Lambda'$ is called an* N-homomorphism of N-cogroup *if for any $\lambda, \lambda' \in \Gamma$ $h(\lambda + \lambda') = h(\lambda) + h(\lambda')$ and $h(n\lambda) = nh(\lambda)$.*

Following the situation in module theory, in which we can define a bimodule, in N-group structure we use the same method to yield a bigroup as

Definition 1.5. *Let N and M be two left near-rings and Γ a group. If*

(i) Γ *is a right N-group;*

(ii) Γ *is a left M-cogroup;*

(iii) $(m\gamma)n = m(\gamma n)$ *for all $n \in N$, $m \in M$ and $\gamma \in \Gamma$,*

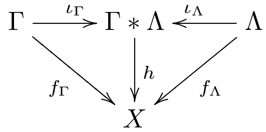
then Γ is called an (N, M)-bigroup.

Any left near-ring N is then a trivial example of (N, N)-bigroup.

2. **Free Product of Groups**

We give first some notions which have an important roles in the discussion. For any group Γ, e_Γ means the neutral element in Γ and the identity homomorphism denoted by $I_\Gamma : \Gamma \to \Gamma$.

Given two groups Γ and Λ. The *free product of* Γ *and* Λ is a group, denoted by $\Gamma * \Lambda$, with embedding homomorphisms $\iota_\Gamma : \Gamma \to \Gamma * \Lambda$ and $\iota_\Lambda : \Lambda \to \Gamma * \Lambda$ such that given any group X with homomorphism $f_\Gamma : \Gamma \to X$ and $f_\Lambda : \Lambda \to X$ there is a unique homomorphism $h : \Gamma * \Lambda \to X$, such that $h \circ \iota_\Gamma = f_\Gamma$ and $h \circ \iota_\Lambda = f_\Lambda$, which is described by the following commutative diagram :



An arbitrary element of $\Gamma * \Lambda$ looks something like $\gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_k\lambda_k$ for some $k \in \mathbb{N}$. It could start with an element of Λ or end with an element of Γ, but important is that the entries from the two groups alternate. The operation in $\Gamma * \Lambda$ is the composition by just sticking sequences together like for a free group.

By the embedding homomorphism ι_Γ means $\iota_\Gamma(\gamma) = \gamma e_\Lambda$ and ι_Λ is analog. For any group X and homomorphism $f_\Gamma : \Gamma \to X$ and $f_\Lambda : \Lambda \to X$, we can send $\gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_k\lambda_k$ to $f_\Gamma(\gamma_1)f_\Lambda(\lambda_1)f_\Gamma(\gamma_2)f_\Lambda(\lambda_2)\dots f_\Gamma(\gamma_k)f_\Lambda(\lambda_k)$, that is the $h := f_\Gamma * f_\Lambda$.

Now we apply the free product of groups to an N-group and an N-cogroup as follow. Let N be a left near-ring, Γ an (N,N)-bigroup and Λ a left N-cogroup. As groups we obtain a group of free product $\Gamma * \Lambda$ and moreover it yields also a left N-cogroup.

Proposition 2.1. *Let N be a left near-ring, Γ an N-group and Λ a group. Then $\Gamma * \Lambda$ is a left N-cogroup.*

Proof : We define first the following action :

$$\begin{array}{rcl} N \times (\Gamma * \Lambda) & \to & \Gamma * \Lambda \\ (n, \gamma_1\lambda_1 \dots \gamma_k\lambda_k) & \mapsto & n\gamma_1\lambda_1 \dots n\gamma_k\lambda_k, \end{array}$$

and show that the axioms of a left N-cogroup are satisfied. Let $\gamma_1\lambda_1 \dots \gamma_k\lambda_k$, $\gamma_1'\lambda_1' \dots \gamma_k'\lambda_k' \in \Gamma * \Lambda$ and $n, m \in N$, then

$$\begin{array}{rcl} n(\gamma_1\lambda_1 \dots \gamma_k\lambda_k\gamma_1'\lambda_1' \dots \gamma_k'\lambda_k') & = & n\gamma_1\lambda_1 \dots n\gamma_k\lambda_k n\gamma_1'\lambda_1' \dots n\gamma_k'\lambda_k' \\ & = & (n\gamma_1\lambda_1 \dots n\gamma_k\lambda_k)(n\gamma_1'\lambda_1' \dots n\gamma_k'\lambda_k') \\ & = & n(\gamma_1\lambda_1 \dots \gamma_k\lambda_k)n(\gamma_1'\lambda_1' \dots \gamma_k'\lambda_k'); \\ (nm)\gamma_1\lambda_1 \dots \gamma_k\lambda_k & = & nm\gamma_1\lambda_1 \dots nm\gamma_k\lambda_k \\ & = & n(m\gamma_1\lambda_1 \dots m\gamma_k\lambda_k) \\ & = & n(m(\gamma_1\lambda_1 \dots \gamma_k\lambda_k)). \quad \square \end{array}$$

An immediate consequence of Proposition (2.1) is if Γ' is an N-group we can construct the following free product $\Gamma * (\Gamma' * \Lambda)$. But we shall recognize that $\Gamma * (\Gamma' * \Lambda)$ is not always the same as $(\Gamma * \Gamma') * \Lambda$, since $(\Gamma * \Gamma')$ is not necessarily a right N-group such that $(\Gamma * \Gamma') * \Lambda$ can not be obtained. For any $\Gamma_1, \Gamma_1, \dots, \Gamma_n$ N-groups, denote

$$\Gamma_1 * \Gamma_2 * \dots * \Gamma_n := \Gamma_1 * (\Gamma_2 * (\cdots (\Gamma_{n-1} * \Gamma_n))).$$

If we keep Γ fixed and replace the left N-cogroup Λ with any left N-cogroup, we obtain a functor from category of groups GRP into category of left N-cogroup N-COGRP. The functor is denoted by $\Gamma * - : \text{GRP} \to N - \text{COGRP}$, where for any $\Lambda \in \text{Obj(GRP)}$, $\Gamma * - : \Gamma \mapsto \Gamma * \Lambda$ and for any $g \in \text{Mor(GRP)}$, $\Gamma * - : g \mapsto g * \Lambda$.

Suppose Λ and Δ are groups and $g : \Lambda \to \Delta$ is a group homomorphism. Applying the functor $\Gamma * -$ we have the following diagram :

$$\begin{CD} \Lambda @>{g}>> \Delta \\ @V{\Gamma*-}VV @VV{\Gamma*-}V \\ \Gamma * \Lambda @>{\Gamma*g}>> \Gamma * \Delta \end{CD}$$

where the first row is the situation in category GRP, meanwhile the second row is the situation in category N-COGRP.

We define

$$(\Gamma * g)(\gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_k\lambda_k) := \gamma_1 g(\lambda_1)\gamma_2 g(\lambda_2) \dots \gamma_k g(\lambda_k).$$

Now we show that $\Gamma * -$ is a covariant functor. Suppose a composition of morphisms in GRP :

$$\begin{CD} \Lambda @>{g}>> \Delta @>{g'}>> \Delta' \\ @V{\Gamma*-}VV @VV{\Gamma*-}V @VV{\Gamma*-}V \\ \Gamma * \Lambda @>{\Gamma*g}>> \Gamma * \Delta @>{\Gamma*g'}>> \Gamma * \Delta' \end{CD}$$

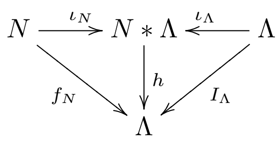
and the following property

$$\begin{aligned} (\Gamma * (g' \circ g))(\gamma_1\lambda_1 \dots \gamma_k\lambda_k) &= (I_\Gamma * (g' \circ g))(\gamma_1\lambda_1 \dots \gamma_k\lambda_k) \\ &= \gamma_1(g' \circ g)(\lambda_1) \dots \gamma_k(g' \circ g)(\lambda_k) \\ &= \gamma_1 g'(g(\lambda_1)) \dots \gamma_k g'(g(\lambda_k)) \\ &= (I_\Gamma * g')(\gamma_1 g(\lambda_1) \dots \gamma_k g(\lambda_k)) \\ &= (I_\Gamma * g') \circ (I_\Gamma * g)(\gamma_1\lambda_1 \dots \gamma_k\lambda_k). \end{aligned}$$

Obviously one can also prove that $\Gamma * I_\Lambda = I_{\Gamma * \Lambda}$.

Proposition 2.2. *Let N be a left near-ring and Λ a left N-cogroup. Then there is a homomorphism of N-cogroup $h : N * \Lambda \to \Lambda$.*

Proof : From the definition of free product, for the identity homomorphism I_Λ : $\Lambda \to \Lambda$ and any $f_N : N \to \Lambda$ we have the following diagram:



where ι_N and ι_Λ are the embedding homomorphisms. According to the definition of free product, the existence of h can be taken by $h := f_N * I_\Lambda$, a group homomorphism, and satisfies $I_\Lambda = h \circ \iota_\Lambda$ and $f_N = h \circ \iota_N$. We show that this h is an

N-cogroup homomorphism. Take any $n_1\lambda_1 \dots n_k\lambda_k$, $n'_1\lambda'_1 \dots n'_k\lambda'_k \in N * \Lambda$ and $m \in N$,

$$\begin{aligned} h(n_1\lambda_1 \dots n_k\lambda_k)(n'_1\lambda'_1 \dots n'_k\lambda'_k) &= (f_N * \iota_\Lambda)(n_1\lambda_1 \dots n_k\lambda_k n'_1\lambda'_1 \dots n'_k\lambda'_k) \\ &= f_N(n_1)\lambda_1 \dots f(n_k)\lambda_k f_N(n'_1)\lambda'_1 \dots f(n'_k)\lambda'_k \\ &= (f_N(n_1)\lambda_1 \dots f(n_k)\lambda_k)(f_N(n'_1)\lambda'_1 \dots f(n'_k)\lambda'_k) \\ &= (f_N * \iota_\Lambda)(n_1\lambda_1 \dots n_k\lambda_k)(f_N * \iota_\Lambda)(n'_1\lambda'_1 \dots n'_k\lambda'_k) \\ &= h(n_1\lambda_1 \dots n_k\lambda_k)h(n'_1\lambda'_1 \dots n'_k\lambda'_k); \\ h(m(n_1\lambda_1 \dots n_k\lambda_k)) &= h(mn_1\lambda_1 \dots mn_k\lambda_k) \\ &= (f_N * \iota_\Lambda)(mn_1\lambda_1 \dots mn_k\lambda_k) \\ &= f_N(mn_1)\lambda_1 \dots f_N(mn_k)\lambda_k \\ &= mf_N(n_1)\lambda_1 \dots mf_N(n_k)\lambda_k) \\ &= m(f_N * \iota_\Lambda)(n_1\lambda_1 \dots n_k\lambda_k) \\ &= mh(n_1\lambda_1 \dots n_k\lambda_k). \quad \square \end{aligned}$$

We denote the set of N-cogroup homomorphisms from N-cogroup Γ to Λ as $\mathrm{Hom}(\Gamma, \Lambda)$.

Proposition 2.3. *Let N be a left near-ring, Γ an (N, N)-bigroup, Λ a left N-cogroup and $\Gamma * \Lambda$ free product of Γ and Λ. For any left N-cogroup X denote*

$$\mathrm{Hom}(\Gamma, X) \times \mathrm{Hom}(\Lambda, X) = \{(f, g) \mid f \in \mathrm{Hom}(\Gamma, X), g \in \mathrm{Hom}(\Lambda, X)\}.$$

There is a bijection

$$\mathrm{Hom}(\Gamma * \Lambda, X) \xrightarrow{\simeq} \mathrm{Hom}(\Gamma, X) \times \mathrm{Hom}(\Lambda, X).$$

Proof : Define the map $\alpha : \mathrm{Hom}(\Gamma * \Lambda, X) \to \mathrm{Hom}(\Gamma, X) \times \mathrm{Hom}(\Lambda, X)$ by $\alpha(h) := (h \circ \iota_\Gamma, h \circ \iota_\Lambda)$ for any $h \in \mathrm{Hom}(\Gamma * \Lambda, X)$. By definition of free product of Γ and Λ, this map is well defined.

Now take any $h, h' \in \mathrm{Hom}(\Gamma * \Lambda, X)$ where $\alpha(h) = \alpha(h')$. It means $(h \circ \iota_\Gamma, h \circ \iota_\Lambda) = (h' \circ \iota_\Gamma, h' \circ \iota_\Lambda)$. Then $h \circ \iota_\Gamma = h' \circ \iota_\Gamma$ and $h \circ \iota_\Lambda = h' \circ \iota_\Lambda$. For any $\gamma \in \Gamma$ we have

$$\begin{aligned} (h \circ \iota_\Gamma)(\gamma) &= (h' \circ \iota_\Gamma)(\gamma) \\ h(\gamma e_\Lambda) &= h'(\gamma e_\Lambda). \end{aligned}$$

By analog we obtain $h(e_\Gamma \lambda) = h'(e_\Gamma \lambda)$. For any $\gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_k\lambda_k \in \Gamma * \lambda$ we consider that

$$\gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_k\lambda_k = \gamma_1 e_\Lambda e_\Gamma \lambda_1 \gamma_2 e_\Lambda e_\Gamma \lambda_2 \dots \gamma_k e_\Lambda e_\Gamma \lambda_k.$$

Hence

$$\begin{aligned} h(\gamma_1\lambda_1\gamma_2\lambda_2\dots\gamma_k\lambda_k) &= h(\gamma_1\lambda_1)h(\gamma_2\lambda_2)\dots h(\gamma_k\lambda_k) \\ &= h(\gamma_1 e_\Lambda e_\Gamma\lambda_1)h(\gamma_2 e_\Lambda e_\Gamma\lambda_2)\dots h(\gamma_k e_\Lambda e_\Gamma\lambda_k) \\ &= h(\gamma_1 e_\Lambda)h(e_\Gamma\lambda_1)h(\gamma_2 e_\Lambda)h(e_\Gamma\lambda_2)\dots h(\gamma_k e_\Lambda)h(e_\Gamma\lambda_k) \\ &= h'(\gamma_1 e_\Lambda)h'(e_\Gamma\lambda_1)h'(\gamma_2 e_\Lambda)h'(e_\Gamma\lambda_2)\dots h'(\gamma_k e_\Lambda)h'(e_\Gamma\lambda_k) \\ &= h'(\gamma_1\lambda_1\gamma_2\lambda_2\dots\gamma_k\lambda_k) \end{aligned}$$

or $h = h'$. Thus α is injective.

Now for any $(f, g) \in \mathrm{Hom}(\Gamma, X) \times \mathrm{Hom}(\Lambda, X)$, there is $h := f * g \in \mathrm{Hom}(\Gamma * \Lambda, X)$ such that $\alpha(h) = (\iota_\Gamma \circ h, \iota_\Lambda \circ h) = (f, g)$. Thus α is surjective. □

Corollary 2.4. *Let N be a left near-ring and $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ (N,N)-bigroups. For any left N-cogroup X,*

$$\mathrm{Hom}(\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n, X) \xrightarrow{\simeq} \mathrm{Hom}(\Gamma_1, X) \times \mathrm{Hom}(\Gamma_2, X) \times \cdots \times \mathrm{Hom}(\Gamma_n, X).$$

Proof : It is a consequence of Proposition (2.3). □

Corollary (2.4) briefly can be written by

$$\mathrm{Hom}(\coprod_i \Gamma_i, X) \xrightarrow{\simeq} \prod_i \mathrm{Hom}(\Gamma_i, X).$$

3. Adjoint Functor of $\Gamma * -$

Now we investigate the right adjoint functors of functor $\Gamma * - : \mathrm{GRP} \to \mathrm{N-COGRP}$. From the definition of free product itself we find that forgetful functor $U(-) : \mathrm{N-COGRP} \to \mathrm{GRP}$ is one of the right adjoint functor of $\Gamma * -$.

Proposition 3.1. *Let N be a left near-ring and Γ an (N,N)-bigroup. For any $\Lambda \in Obj(\mathrm{GRP})$ and $\Lambda' \in Obj(\mathrm{N-COGRP})$ there is an injective mapping*

$$\varphi_1 : \mathrm{Mor}_{\mathrm{COGRP}}(\Gamma * \Lambda, \Lambda') \longrightarrow \mathrm{Mor}_{\mathrm{GRP}}(\Lambda, \Lambda'),$$

which has a right inverse.

Proof : Recall the definition of free product. Let

$$\varphi_1 : \mathrm{Mor}_{\mathrm{COGRP}}(\Gamma * \Lambda, \Lambda') \to \mathrm{Mor}_{\mathrm{GRP}}(\Lambda, \Lambda'),$$

where for any $h \in \mathrm{Mor}_{\mathrm{COGRP}}(\Gamma * \Lambda, \Lambda')$ we define $\varphi_1(h) := h \circ \iota_\Lambda$, where $\iota_\Lambda : \Lambda \to \Gamma * \Lambda$ an embedding homomorphism. Moreover we also have

$$\varphi_2 : \mathrm{Mor}_{\mathrm{GRP}}(\Lambda, \Lambda') \to \mathrm{Mor}_{\mathrm{COGRP}}(\Gamma * \Lambda, \Lambda').$$

For any $g : \Gamma \to \Lambda'$ and $f : \Lambda \to \Lambda'$, there exists a unique $h = g * f$ such that $h \circ \iota_\Lambda = f$. Thus for any $f \in \mathrm{Mor}_{\mathrm{GRP}}(\Lambda, \Lambda')$ we define the map $\varphi_2(f) := h$ such that $h \circ \iota_\Lambda = f$. Note that this h exists by the definition of free product of Γ and Λ.

We now prove that φ_2 is a right inverse of φ_1. For any $f:\Lambda\to\Lambda'$,

$$\begin{array}{rcl}\varphi_1\circ\varphi_2(f) & = & \varphi_1(h), (\text{ where } h\circ\iota_\Lambda = f)\\ & = & h\circ\iota_\Lambda\\ & = & f.\end{array}$$

Now assume $\varphi_2\circ\varphi_1$ is an identity mapping and $g_1 * f \neq g_2 * f$. We have

$$\begin{array}{rcl}g_1 * f & = & (\varphi_2\circ\varphi_1)(g_1 * f)\\ & = & \varphi_2((g_1 * f)\circ\iota_\Lambda)\\ & = & \varphi_2(f)\\ & = & h, \text{ where } h\circ\iota_\Lambda = f.\end{array}$$

The h in the last row could be $g_1 * f$ or $g_1 * f$. Thus φ_2 is not a left inverse of φ_1. □

The other right adjoint functor of $\Gamma * -$ is $\mathrm{Mor}_{\mathrm{GRP}}(\Gamma, -)$ as we can see in the following proposition.

Proposition 3.2. *Let N be a left near-ring, Γ an (N,N)-bigroup and Λ a left N-cogroup. For any $\Lambda\in Obj(\mathrm{GRP})$ and $\Lambda'\in Obj(\mathrm{N-COGRP})$ there is a bijection*

$$\mathrm{Mor}_{\mathrm{COGRP}}(\Gamma*\Lambda,\Lambda')\xrightarrow{\simeq}\mathrm{Map}(\Lambda,\mathrm{Mor}_{\mathrm{GRP}}(\Gamma,\Lambda')).$$

Proof : Let

$$\begin{array}{rcl}\psi_1:\mathrm{Mor}_{\mathrm{COGRP}}(\Gamma*\Lambda,\Lambda') & \to & \mathrm{Map}(\Lambda,\mathrm{Mor}_{\mathrm{GRP}}(\Gamma,\Lambda'))\\ h & \mapsto & [\lambda\mapsto h(-\lambda)]\end{array}$$

for all $h\in\mathrm{Mor}_{\mathrm{COGRP}}(\Gamma*\Lambda,\Lambda')$ and

$$\begin{array}{rcl}\psi_2:\mathrm{Map}(\Lambda,\mathrm{Mor}_{\mathrm{GRP}}(\Gamma,\Lambda')) & \to & \mathrm{Mor}_{\mathrm{COGRP}}(\Gamma*\Lambda,\Lambda')\\ h' & \mapsto & [\gamma_1\lambda_1\dots\gamma_k\lambda_k\mapsto h'(\lambda_1)(\gamma_1)\dots h'(\lambda_k)(\gamma_k)]\end{array}$$

for all $h'\in\mathrm{Map}(\Lambda,\mathrm{Mor}_{\mathrm{GRP}}(\Gamma,\Lambda'))$.

We show that both ψ_1 and ψ_2 are inverse each other. Take any $h\in\mathrm{Mor}_{\mathrm{COGRP}}(\Gamma*\Lambda,\Lambda')$, it yields $(\psi_2\circ\psi_1)(h)=\psi_2(\psi_1(h))=h$ since

$$\psi_2:\psi_1(h)\mapsto[\gamma_1\lambda_1\dots\gamma_k\lambda_k\mapsto\psi_1(h)(\lambda_1)(\gamma_1)\dots\psi_1(h)(\lambda_k)(\gamma_k)]$$

where $\psi_1(h)(\lambda)(\gamma)=h(-\lambda)(\gamma)=h(\gamma\lambda)$.

Now take any $h'\in\mathrm{Map}(\Lambda,\mathrm{Mor}_{\mathrm{GRP}}(\Gamma,\Lambda'))$, it yields $(\psi_1\circ\psi_2)(h')=\psi_1(\psi_2(h'))=h'$ since

$$\psi_1:\psi_2(h')\mapsto[\lambda\mapsto\psi_2(h')(-\lambda)]$$

where $\psi_2(h')(\gamma_1\lambda_1\dots\gamma_k\lambda_k)=h'(\lambda_1)(\gamma_1)\dots h'(\lambda_k)(\gamma_k)$ for all $\gamma\in\Gamma$. □

For special cases one obtains the following corollaries.

Corollary 3.3. *Let N be a left near-ring and Γ an (N,N)-bigroup. Then*

$$\mathrm{Mor}_{\mathrm{COGRP}}(\Gamma*N,\Gamma)\xrightarrow{\simeq}\mathrm{Map}(N,\mathrm{End}_{\mathrm{GRP}}(\Gamma,\Gamma)).$$

Proof : From Proposition (3.2) if we take $\Lambda = N$ and $\Lambda' = \Gamma$, then we have

$$\mathrm{Mor}_{\mathrm{COGRP}}(\Gamma * N, \Gamma) \xrightarrow{\simeq} \mathrm{Map}(N, \mathrm{Mor}_{\mathrm{GRP}}(\Gamma, \Gamma)) \simeq \mathrm{Map}(N, \mathrm{End}_{\mathrm{GRP}}(\Gamma, \Gamma)). \quad \square$$

Corollary 3.4. *Let N be a left near-ring and Γ an (N, N)-bigroup. Then*

$$\mathrm{Mor}_{\mathrm{COGRP}}(N * \Gamma, \Gamma) \xrightarrow{\simeq} \mathrm{Map}(\Gamma, \mathrm{Mor}_{\mathrm{GRP}}(N, \Gamma)).$$

Proof : From Proposition (3.2) if we take $\Lambda = \Gamma$, $\Gamma = N$ and $\Lambda' = \Gamma$, then we have

$$\mathrm{Mor}_{\mathrm{COGRP}}(N * \Gamma, \Gamma) \xrightarrow{\simeq} \mathrm{Map}(\Gamma, \mathrm{Mor}_{\mathrm{GRP}}(N, \Gamma)). \quad \square$$

4. Category of $(\Gamma * -)$-algebras

To observe further the properties of free product of N-group and N-cogroup we introduce now the notions of $(\Gamma * -)$-algebra. Consider the definition of F-algebra from Gumm [3] and its properties to get more ideas of these functorial notions.

Let Γ be a left N-cogroup. We define the functor $(\Gamma * -) : \mathrm{N-COGRP} \to \mathrm{N-COGRP}$ as $(\Gamma * -) : \Lambda \mapsto \Gamma * \Lambda$. Moreover, for any left N-cogroup Λ and if there exists an N-cogroup homomorphism $f_\Gamma : \Gamma \to \Lambda$, then we obtain an N-cogroup homomorphism $h : (\Gamma * \Lambda) \to \Lambda$ by the definition of free product. This h is defined by $h := f_\Gamma * I_\Lambda$ such that $h \circ \iota_\Gamma = f_\Gamma$ and $h \circ \iota_\Lambda = I_\Lambda$. Thus we have $h : \Gamma * \Lambda \to \Lambda$ and call the pair (Λ, h) as a $(\Gamma * -)$-algebra.

Definition 4.1. Let (Λ, h) and (Λ', h') be $(\Gamma * -)$-algebras. An N-cogroup homomorphism $g : \Lambda \to \Lambda'$ is called *homomorphism* from Λ to Λ' if the following diagram is commutative, i.e.

$$\begin{array}{ccc} \Lambda & \xrightarrow{g} & \Lambda' \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle h'} \\ \Gamma * \Lambda & \xrightarrow{\Gamma * g} & \Gamma * \Lambda', \end{array}$$

where $\Gamma * g := I_\Gamma * g$.

We show now that the class of $(\Gamma * -)$-algebras with their homomorphisms form a category.

Proposition 4.2. *Let (Λ, h), (Λ', h'), (Λ'', h'') be $(\Gamma * -)$-algebras. Then*

(i) *The identity mapping $I_\Lambda : \Lambda \to \Lambda$ is a homomorphism;*

(ii) *If $t : \Lambda \to \Lambda'$ and $l : \Lambda' \to \Lambda''$ are homomorphisms, then the composition $l \circ t : \Lambda \to \Lambda''$ is also a homomorphism.*

Proof : (i) Given the following diagram :

$$\begin{array}{ccc} \Lambda & \xrightarrow{I_\Lambda} & \Lambda \\ \uparrow h & & \uparrow h \\ \Gamma * \Lambda & \xrightarrow{I_\Gamma * I_\Lambda} & \Gamma * \Lambda. \end{array}$$

For any $\gamma_1\lambda_1 \dots \gamma_k\lambda_k$ holds

$$\begin{aligned} (h \circ (I_\Gamma * I_\Lambda))(\gamma_1\lambda_1 \dots \gamma_k\lambda_k) &= h(\gamma_1\lambda_1 \dots \gamma_k\lambda_k) \\ &= (f_\Gamma * I_\Lambda)(\gamma_1\lambda_1 \dots \gamma_k\lambda_k) \\ &= f_\Gamma(\gamma_1)\lambda_1 \dots f_\Gamma(\gamma_k)\lambda_k \\ &= I_\Lambda(f_\Gamma(\gamma_1)\lambda_1 \dots f_\Gamma(\gamma_k)\lambda_k) \\ &= (I_\Lambda \circ (f_\Gamma * I_\Lambda))(\gamma_1\lambda_1 \dots \gamma_k\lambda_k) \\ &= (I_\Lambda \circ h)(\gamma_1\lambda_1 \dots \gamma_k\lambda_k). \end{aligned}$$

(ii) Given the following diagram :

$$\begin{array}{ccccc} \Lambda & \xrightarrow{t} & \Lambda' & \xrightarrow{l} & \Lambda' \\ \uparrow h & & \uparrow h' & & \uparrow h'' \\ \Gamma * \Lambda & \xrightarrow{I_\Gamma * t} & \Gamma * \Lambda' & \xrightarrow{I_\Gamma * l} & \Gamma * \Lambda'' \end{array}$$

$$h'' \circ (f_\Gamma * l) \circ (f_\Gamma * t) = (l \circ h') \circ (f_\Gamma * t) = l \circ t \circ h. \quad \square$$

We denote the category of $(\Gamma * -)$-algebra as $\mathrm{COGRP}^{(\Gamma * -)}$. Furthermore, in $\mathrm{COGRP}^{(\Gamma * -)}$ a bijective homomorphism and an isomorphism are coincide.

Now we give some direct consequences of the category of $(\Gamma * -)$-algebras.

Proposition 4.3. *Any bijective homomorphism in* $\mathrm{COGRP}^{(\Gamma * -)}$ *is an isomorphism and conversely.*

Proposition 4.4. *Let* (Λ, h), (Λ', h'), (Λ'', h'') *be* $(\Gamma * -)$*-algebras. Let* $t : \Lambda \to \Lambda'$ *and* $l : \Lambda' \to \Lambda''$ *be mappings such that* $g := l \circ t : \Lambda \to \Lambda'$ *is a homomorphism. Then*

(i) *if* t *is a surjective homomorphism, then* l *is a homomorphism;*

(ii) *if* l *is an injective homomorphism, then* t *is a homomorphism.*

Corollary 4.5. *Let* (Λ, h), (Λ', h'), (Λ'', h'') *be* $(\Gamma * -)$*-algebras. Let* $t : \Lambda \to \Lambda'$ *and* $l : \Lambda \to \Lambda''$ *be homomorphisms. If* t *is surjective, then there is a homomorphism* $g : \Lambda' \to \Lambda''$ *with* $g \circ t = l$ *if and only if* $\mathrm{Ker}\ t \subseteq \mathrm{Ker}\ l$.

5. Concluding Remarks

From the discussion above we conclude some important remarks. Free product of N-group and N-cogroup might be an alternative way to obtain a structure such as a tensor product in category of groups in which the involved groups are not necessary commutative. Since these free product is not associative, in general case it is not a monad. For any (N, N)-bigroup Γ, the free product $(\Gamma * -)$ is a left adjoint functor and it yields a more general structure, that is $(\Gamma * -)$-algebra. Together with their homomorphisms, the class of $(\Gamma * -)$-algebras form a category.

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