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ON QUASI BI-SLANT RIEMANNIAN MAPS FROM LORENTZIAN PARA SASAKIAN MANIFOLDS

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Abstract. We first introduce quasi bi-slant Riemannian maps and study such Riemannian maps from Lorentzian para Sasakian manifolds into Riemannian manifolds. We give necessary and sufficient conditions for the integrability of the distributions which are involved in the definition of the quasi bi-slant Riemannian map and investigate their leaves. We also obtain totally geodesic conditions for such maps. Moreover, we provide some examples for this notion.

Key words and Phrases: Riemannian map, Quasi bi-slant Riemannian map, Lorentzian para Sasakian manifolds.

1. Introduction

In differential geometry, there are so many important applications of Riemannian maps in Mathematics and Physics [22]. The properties of slant Riemannian maps became an interesting subject in differential geometry, both in complex geometry and contact geometry. Differentiable maps between Riemannian manifolds are important in differential geometry. There are certain types of differentiable maps between Riemannian manifolds whose existence influence the geometry of source manifolds and target manifolds. The geometric structures defined on both manifolds are compared by differentiable maps between Riemannian manifolds. Basic maps in this manner are isometric immersions and Riemannian submersions. Isometric immersions between Riemannian manifolds are characterized by their Jacobian matrices and the induced metric which is symmetric positive definite bilinear form. The theory of isometric immersions is an active research area and it plays an important role in the development of modern differential geometry. On the other

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hand, the Riemannian submersions are also useful to compare geometric structures. The theory of Riemannian submersions is also a very active research field for recent developments.

In 1966, the theory of Riemannian submersions was initiated by O' Neill [14] and in 1967, A. Gray [8] extended this theory. Later, this theory was widely studied by several geometers. Riemannian submersions are interesting and very important in several areas of Riemannian geometry. In particular, the Riemannian submersions have several important applications both in Mathematics and in Physics because of their applications in supergravity and superstring theories [1], Kaluza-Klein theory [3, 10], Yang-Mills theory [4] etc.

In 1992, the notion of Riemannian maps between Riemannian manifold was introduced by Fischer [6] as a generalization of isometric immersions and Riemannian submersions. Let $\mathcal{F} : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifold such that $0 < rank\mathcal{F} < \min\{m, n\}$, where $\dim M = m$ and $\dim N = n$. Let we denote the kernel space of $\mathcal{F}_*$ by $\ker \mathcal{F}_{*p}$ at $p \in M$. The tangent space of M at $p \in M$ is given by

$$\mathcal{T}_pM = \ker \mathcal{F}_{*p} \oplus \mathcal{H},$$

where $\mathcal{H}$ is orthogonal complementary space and $\mathcal{H} = (\ker \mathcal{F}_{*p})^\perp$ to $\ker \mathcal{F}_{*p}$ in $\mathcal{T}_pM$.

We denote the range of $\mathcal{F}_*$ by $range\mathcal{F}_{*p}$ at $p \in M$ and consider the orthogonal complementary space $(range\mathcal{F}_{*p})^\perp$ to $range\mathcal{F}_{*p}$ in the tangent space $\mathcal{T}_{\mathcal{F}(p)}N$ of N. As we know that $rank\mathcal{F} < \min\{m, n\}$, we always have $(range\mathcal{F}_{*p})^\perp \neq \{0\}$. Thus the tangent space $\mathcal{T}_{\mathcal{F}(p)}N$ of N has the following decomposition

$$\mathcal{T}_{\mathcal{F}(p)}N = (range\mathcal{F}_{*p}) \oplus (range\mathcal{F}_{*p})^\perp.$$

Now, a differentiable map $\mathcal{F} : (M, g_M) \to (N, g_N)$ is called Riemannian map at $p \in M$ if the horizontal restriction $\mathcal{F}^h_{*p} : (\ker \mathcal{F}_{*p})^\perp \to (range\mathcal{F}_{*p})$ is a linear isometry between the inner product spaces $((\ker \mathcal{F}_{*p})^\perp, g_M(p) \mid (\ker \mathcal{F}_{*p})^\perp)$ and $((range\mathcal{F}_{*p}), g_N(\mathcal{F}(p)) \mid (range\mathcal{F}_{*p}))$. Therefore, Fischer defined in [6] that a Riemannian map is a map which is isometric as it can be. It also follows that $\mathcal{F}_*$ satisfies the equation

$$g_N(\mathcal{F}_*U, \mathcal{F}_*V) = g_M(U, V), \tag{1}$$

for $U, V \in \mathcal{H}$.

It follows that isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker \mathcal{F}_* = 0$ and $(range\mathcal{F}_*)^\perp = 0$, respectively. Since the Riemannian map is subimmersion, the rank of the linear map $\mathcal{F}_{*p} : \mathcal{T}_pM \to \mathcal{T}_{\mathcal{F}(p)}N$ is constant for p in each connected component of M.

We note that a remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation $||\mathcal{F}_*||^2 = rank\mathcal{F}$, which is bridge between geometric optics and physical optics. Riemannian maps have been also studied in spacetime geometry under some regularity conditions [7].

The theory of Lorentzian submersions was introduced by Magid [13] and Falcitelli et al. [5]. In 2013, Gunduzalp and Sahin studied paracontact semi-Riemannian submersions [9]. Recently, S. Kumar et al. [11] defined and studied conformal semi-slant submersions from $LP-$Sasakian manifolds onto Riemannian manifolds and R. Prasad et al, introduced quasi bi-slant Lorentzian submersions from $LP-$Sasakian manifolds which generalizes hemi-slant, semi-slant and semi-invariant Riemannian submersions [18].

Inspired from the good and interesting results of above studies, we introduce the notion of quasi bi-slant Riemannian maps from $LP-$Sasakian manifolds into Riemannian manifolds as a generalization of bi-slant Riemannian maps, quasi hemi-slant Riemannian maps [16], slant Riemannian maps, semi-slant Riemannian maps ([2], [15], [12], [17]) and hemi-slant Riemannian maps.

In this research paper we tackle our work as follows: In the second section, we present several main informations related to quasi bi-slant Riemannian maps. In the third section, we give definition of $LP-$Sasakian manifolds and discuss certain interesting outcomes on quasi bi-slant submersions from $LP-$Sasakian manifolds into Riemannian manifolds. In the section 4 the geometry of leaves of distributions that are involved in the definition of considered maps is studied and in section 5, we give a necessary and sufficient condition for quasi bi-slant Riemannian maps to be totally geodesic. Finally, in the last section 6, we construct some non-trivial examples for considered maps.

2. Quasi bi-slant Riemannian maps

Let (M, g_M) and (N, g_N) be Riemannian manifolds and let $\mathcal{F} : (M, g_M) \to (N, g_N)$ be a smooth map. The differential map $\mathcal{F}_*$ of $\mathcal{F}$ can be viewed as a section of the bundle $Hom(TM, \mathcal{F}^{-1}TN) \to M$, where $\mathcal{F}^{-1}TN$ is the pullback bundle with fibers $(\mathcal{F}^{-1}TN)_p = T_{\mathcal{F}(p)}N$, $p \in M$. $Hom(TM, \mathcal{F}^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. In addition, the second fundamental form of $\mathcal{F}$ is given by

$$(\nabla \mathcal{F}_*)(U, V) = \nabla^{\mathcal{F}}_U \mathcal{F}_*(V) - \mathcal{F}_*(\nabla^M_U V), \tag{2}$$

for $U, V \in \Gamma(TM)$.

The second fundamental form is symmetric [1] and Sahin in [19] showed that the second fundamental form $(\nabla \mathcal{F}_*)(U, V)$, $\forall U, V \in \Gamma(TM)$, of a Riemannian map has no components in $range\mathcal{F}_*$.

Lemma 2.1. [21] *Let $\mathcal{F} : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then*

$$g_N((\nabla \mathcal{F}_*)(U, V), \mathcal{F}_*(W)) = 0, \forall U, V, W \in \Gamma((ker\mathcal{F}_*)^{\perp}).$$

As an outcome of Lemma 2.1, we get

$$(\nabla \mathcal{F}_*)(U, V) \in \Gamma((range\mathcal{F}_*)^{\perp}).$$

Let $\mathcal{F} : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds. The fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ defined by O'Nill's [14] for vector field E and F on M such that

$$\mathcal{A}_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F, \tag{3}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F, \tag{4}$$

where ∇ is the Levi-civita connection on g_M, $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections, respectively. On the other hand, from equations (3) and (4), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{5}$$

$$\nabla_X U = \mathcal{H}\nabla_X U + \mathcal{T}_X U, \tag{6}$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X, \tag{7}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{8}$$

for all $X, Y \in \Gamma(\ker \mathcal{F}_*)$ and $U, V \in \Gamma(\ker \mathcal{F}_*)^\perp$, where $\mathcal{V}\nabla_X Y = \widehat{\nabla}_X Y$. If U is basic, then $\mathcal{A}_U Y = \mathcal{H}\nabla_U Y$.

It is easily seen that for $p \in M$, $Y \in \mathcal{V}_p$ and $U \in \mathcal{H}_p$ the linear operators

$$\mathcal{T}_Y, \mathcal{A}_U : T_pM \to T_pM,$$

are skew-symmetric, that is

$$g_M(\mathcal{A}_U E, F) = -g_M(E, \mathcal{A}_U F) \text{ and } g_M(\mathcal{T}_Y E, F) = -g_M(E, \mathcal{T}_Y F), \tag{9}$$

for all $E, F \in T_pM$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}$ is the second fundamental form of the fibres of f. Since $\mathcal{T}_Y$ is skew-symmetric, we get $\mathcal{F}$ has totally geodesic fibres if and only if $\mathcal{T} = 0$.

Definition 2.2. *Let $(\mathcal{M}, \phi, \xi, \eta, g_{\mathcal{M}})$ be an almost contact metric manifold and (N, g_N) be a Riemannian manifold. A Riemannian map $\mathcal{F} : (\mathcal{M}, \phi, \xi, \eta, g_{\mathcal{M}}) \to (N, g_N)$ is called a quasi bi-slant Riemannian map if there exist four mutually orthogonal distributions $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2$ and $< \xi >$ such that*

$$\ker \mathcal{F}_* = \mathcal{D} \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >, \phi(\mathcal{D}) = \mathcal{D}, \phi(\mathcal{D}_1) \perp \mathcal{D}_2, \phi(\mathcal{D}_2) \perp \mathcal{D}_1.$$

The angle θ_1 between ϕX and the space $(\mathcal{D}_1)_p$ is constant and independent of the choice of the point $p \in \mathcal{M}$ and $X \in (\mathcal{D}_1)_p$ for any non-zero vector field $X \in (\mathcal{D}_1)_p$. Similarly, the angle θ_2 between ϕZ and the space $(\mathcal{D}_2)_q$ is constant and independent of the choice of the point $q \in \mathcal{M}$ and $Z \in (\mathcal{D}_2)_q$ for any non-zero vector field $Z \in (\mathcal{D}_2)_q$.

We give some examples of quasi bi-slant Riemannian maps.

Example 2.3. *Every quasi bi-slant submersion from an almost Hermitian manifold to a Riemannian manifold is a quasi bi-slant Riemannian map with $(range\mathcal{F}_*)^\perp = \{0\}$.*

Example 2.4. *Every quasi hemi-slant submersion from an almost Hermitian manifold to a Riemannian manifold is a quasi bi-slant Riemannian map with* $(range\mathcal{F}_*)^\perp = \{0\}$ *and* $\theta_2 = \frac{\pi}{2}$.

We say that quasi bi-slant Riemannian map $\mathcal{F} : (\mathcal{M}, \phi, \xi, \eta, g_\mathcal{M}) \to (N, g_N)$ is proper if $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Throughout in forthcoming sections of this paper, we take $\mathcal{F} : (\mathcal{M}, \phi, \xi, \eta, g_\mathcal{M}) \to (N, g_N)$ be a quasi bi-slant Riemannian map where $(\mathcal{M}, \phi, \xi, \eta, g_\mathcal{M})$ be an $LP-$Sasakian manifold and (N, g_N) be a Riemannian manifold i.e., from now on we will denote a quasi bi-slant Riemannian map from a Lorentzian para Sasakian manifold $(\mathcal{M}, \phi, \xi, \eta, g_\mathcal{M})$ onto a Riemannian manifold (N, g_N) by $\mathcal{F}$.

3. Quasi bi-slant Riemannian maps from Lorentzian para Sasakian manifolds

In this section we study the notion quasi bi-slant Riemannian maps from LP-Sasakian manifolds onto Riemannian manifolds.

Definition 3.1. *A* $(2m+1)$ *dimensional differentiable manifold* $\mathcal{M}$ *admitting a* $(1,1)$ *tensor field* ϕ, *a contravariant vector field* ξ, *a* $1-$*form* η *is called a Lorentzian para Sasakian manifold with Lorentzian metric* $g_\mathcal{M}$ *if they satisfy:*

$$\phi^2 = I + \eta \otimes \xi, \phi \circ \xi = 0, \eta \circ \xi = 0, \tag{10}$$
$$\eta(\xi) = -1, g_\mathcal{M}(X, \xi) = \eta(X), \tag{11}$$
$$g_\mathcal{M}(\phi X, \phi Y) = g_\mathcal{M}(X, Y) + \eta(X)\eta(Y), g_\mathcal{M}(\phi X, Y) = g_\mathcal{M}(X, \phi Y), \tag{12}$$
$$\nabla_X \xi = \phi X, \tag{13}$$
$$(\nabla_X \phi)Y = \eta(Y)X + g_\mathcal{M}(X, Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{14}$$

where ∇ *represents the operator of covariant differentiation with respect to the Lorentzian metric* $g_\mathcal{M}$. *In a Lorentzian para Sasakian manifold, it is clear that*

$$rank\ \phi = 2m. \tag{15}$$

Now, if we put

$$\Phi(X, Y) = \Phi(Y, X) = g_\mathcal{M}(X, \phi Y) = g_\mathcal{M}(\phi X, Y), \tag{16}$$

then the tensor field Φ *is symmetric* $(0,2)$ *tensor field, for any vector fields* X *and* Y.

Example 3.2. [11] *Let* $R^{2m+1} = \{(x_1, x_2, ..., x_m, y_1, y_2, \ldots, y_m, z) : x_i, y_i, z \in R, i = 1, 2, \ldots, m)\}$. *Consider* R^{2m+1} *with the following structure:*

$$\phi\left(\sum_{i=1}^{m}\left(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right)+Z\frac{\partial}{\partial z}\right)=-\sum_{i=1}^{m}Y_i\frac{\partial}{\partial x_i}-\sum_{i=1}^{m}X_i\frac{\partial}{\partial y_i}+\sum_{i=1}^{m}Y_iy_i\frac{\partial}{\partial z},$$

$$g_{R^{2m+1}} = -(\eta\otimes\eta)+\frac{1}{4}\sum_{i=1}^{m}(dx_i\otimes dx_i+dy_i\otimes dy_i),$$

$$\eta = -\frac{1}{2}\left(dz-\sum_{i=1}^{m}y_i dx_i\right),$$

$$\xi = 2\frac{\partial}{\partial z}.$$

Then, $(R^{2m+1}, \phi, \xi, \eta, g_{R^{2m+1}})$ *is a Lorentzian para-Sasakian manifold. The vector fields* $E_i = 2\frac{\partial}{\partial y_i}, E_{m+i} = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z})$ *and* ξ *form a* $\phi-$*basis for the contact metric structure.*

Let $\mathcal{F} : (\mathcal{M}, \phi, \xi, \eta, g_{\mathcal{M}}) \to (N, g_N)$ be a quasi bi-slant Riemannian maps. Then we have

$$T\mathcal{M} = \ker\mathcal{F}_* \oplus (\ker\mathcal{F}_*)^{\perp}. \tag{17}$$

Now, for any vector field $X \in \Gamma(\ker\mathcal{F}_*)$, we put

$$X = PX + QX + RX - \eta(X)\xi, \tag{18}$$

where P, Q and R are projection morphisms of $\ker\mathcal{F}_*$ onto $\mathcal{D}, \mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

For any vector field $X \in \Gamma(\ker\mathcal{F}_*)$, we set

$$\phi X = \psi X + \omega X, \tag{19}$$

where $\psi X \in \Gamma(\ker\mathcal{F}_*)$ and $\omega X \in \Gamma(\omega\mathcal{D}_1 \oplus \omega\mathcal{D}_2)$.
From (18) and (19), we get

$$\phi X = \psi(PX) + \omega(PX) + \psi(QX) + \omega(QX) + \psi(RX) + \omega(RX).$$

Since $\phi\mathcal{D} = \mathcal{D}$, therefore $\omega PX = 0$. Hence we obtain

$$\phi X = \psi(PX) + \psi QX + \omega QX + \psi RX + \omega RX. \tag{20}$$

Thus, we have

$$\phi(\ker\mathcal{F}_*) = \mathcal{D} \oplus (\psi\mathcal{D}_1 \oplus \psi\mathcal{D}_2) \oplus (\omega\mathcal{D}_1 \oplus \omega\mathcal{D}_2), \tag{21}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$, then $g_{\mathcal{M}}(V, W) = 0$. Now from the Definition 2.2, we obtain $g_{\mathcal{M}}(\phi V, W) = g_{\mathcal{M}}(V, \phi W) = 0$.
Let $Z \in \Gamma(D)$, $Y \in \Gamma(D_1)$ and $X \in \Gamma(D_2)$. Then, we have

$$g_{\mathcal{M}}(\psi Y, Z) = 0,$$
$$g_{\mathcal{M}}(\psi X, Z) = 0.$$

So, we can write $\psi D_1 \cap \psi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}$.

If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and D_2 is anti-invariant, i.e., $\phi(D_2) \subseteq (\ker \mathcal{F}_*)^{\perp}$. In this case we denote D_2 by $D^{\perp}$.

We also have

$$\phi(\ker \mathcal{F}_*) = D \oplus \psi D_1 \oplus \omega D_1 \oplus \phi D^{\perp}. \tag{22}$$

Since $\omega D_1 \subseteq (\ker \mathcal{F}_*)^{\perp}$, $\omega D_2 \subseteq (\ker \mathcal{F}_*)^{\perp}$, so we can write

$$(\ker \mathcal{F}_*)^{\perp} = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V},$$

where $\mathcal{V}$ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker \mathcal{F}_*)^{\perp}$.

Also for any $V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$, we have

$$\phi V = CV + BV, \tag{23}$$

where $CV \in \Gamma(\mathcal{V})$ and $BV \in \Gamma(\ker \mathcal{F}_*)$.

Lemma 3.3. *Let $\mathcal{F}$ be a quasi bi-slant Riemannian map. Then we have*

$$\psi^2 V + B\omega V = V + \eta(V)\xi,$$
$$\omega\psi V + C\omega V = 0,$$
$$\omega BW + C^2 W = W,$$
$$\psi BW + BCW = 0$$

for all $V \in \Gamma(\ker \mathcal{F}_*)$ and $W \in \Gamma(\ker \mathcal{F}_*)^{\perp}$.

Proof. By making use of the equations (10), (19) and (23), Lemma 3.3 follows. □

Lemma 3.4. *Let $\mathcal{F}$ be a quasi bi-slant Riemannian map. Then, we have*

(i) $\psi^2 V_i = (\cos^2 \theta_i) V_i$,
(ii) $g_{\mathcal{M}}(\psi V_i, \psi W_i) = \cos^2 \theta_i g_{\mathcal{M}}(V_i, W_i)$,
(iii) $g_{\mathcal{M}}(\omega V_i, \omega W_i) = \sin^2 \theta_i g_{\mathcal{M}}(V_i, W_i)$, *for all $V_i, W_i \in \Gamma(D_i)$ and $i = 1, 2$.*

Proof. The proof of the above Lemma is the same as Lemma 3.2 of [18], therefore, we omit its proof. □

Lemma 3.5. *Let $\mathcal{F}$ be a quasi bi-slant Riemannian map. Then, we have*

$$\mathcal{V}\nabla_X \psi Y + \mathcal{T}_X \omega Y - \psi \mathcal{V}\nabla_X Y - B\mathcal{T}_X Y = g_{\mathcal{M}}(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$
$$\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y = \omega \mathcal{V}\nabla_X Y + C\mathcal{T}_X Y, \tag{24}$$
$$\mathcal{V}\nabla_U BV + \mathcal{A}_U CV - g_{\mathcal{M}}(CU, V)\xi = \psi \mathcal{A}_U V + B\mathcal{H}\nabla_U V, \tag{25}$$
$$\mathcal{A}_U BV + \mathcal{H}\nabla_U CV = \omega \mathcal{A}_U V + C\mathcal{H}\nabla_U V, \tag{26}$$
$$\mathcal{V}\nabla_X BU + \mathcal{T}_X CU = \psi \mathcal{T}_X U + B\mathcal{H}\nabla_X U, \tag{27}$$
$$\mathcal{T}_X BU + \mathcal{H}\nabla_X CU = \omega \mathcal{T}_X U + C\mathcal{H}\nabla_X U, \tag{28}$$

$$\mathcal{V}\nabla_V \psi X + \mathcal{A}_V \omega X = B\mathcal{A}_V X + \psi\mathcal{V}\nabla_V X, \tag{29}$$

$$\mathcal{A}_V \psi X + \mathcal{H}\nabla_V \omega X - \eta(X)V = C\mathcal{A}_V X + \omega\mathcal{V}\nabla_V X \tag{30}$$

for any $X, Y \in \Gamma(\ker \mathcal{F}_*)$ *and* $U, V \in \Gamma(\ker \mathcal{F}_*)^\perp$.

Proof. By using the equations (5) – (8), (10), (11) and (14), we can easily get the equations (24) – (30). □

Now, we define

$$(\nabla_V \psi)W = \mathcal{V}\nabla_V \psi W - \psi\mathcal{V}\nabla_V W, \tag{31}$$

$$(\nabla_V \omega)W = \mathcal{H}\nabla_V \omega W - \omega\mathcal{V}\nabla_V W, \tag{32}$$

$$(\nabla_X C)Y = \mathcal{H}\nabla_X CY - C\mathcal{H}\nabla_X Y, \tag{33}$$

$$(\nabla_X B)Y = \mathcal{V}\nabla_X BY - B\mathcal{H}\nabla_X Y \tag{34}$$

for any $V, W \in \Gamma(\ker \mathcal{F}_*)$ and $X, Y \in \Gamma(\ker \mathcal{F}_*)^\perp$.

Lemma 3.6. *Let $\mathcal{F}$ be a quasi bi-slant Riemannian map. Then, we have*

$$(\nabla_V \phi)W = B\mathcal{T}_V W - \mathcal{T}_V \omega W + g_{\mathcal{M}}(V, W)\xi + 2\eta(V)\eta(W)\xi + \eta(W)V,$$
$$(\nabla_V \omega)W = C\mathcal{T}_V W - \mathcal{T}_V \psi W,$$
$$(\nabla_X C)Y = \omega\mathcal{A}_X Y - \mathcal{A}_X BY,$$
$$(\nabla_X B)Y = \psi\mathcal{A}_X Y - \mathcal{A}_X CY + g_{\mathcal{M}}(X, Y)\xi,$$

for any $V, W \in \Gamma(\ker \mathcal{F}_*)$ *and* $X, Y \in \Gamma(\ker \mathcal{F}_*)^\perp$.

Proof. By using the equations (25) – (28) and (31) – (34) Lemma 3.6 follows. □

Now, if the tensors ϕ and ω are parallel with respect to ∇ on $\mathcal{M}$, then

$$B\mathcal{T}_V W = \mathcal{T}_V \omega W - g_{\mathcal{M}}(V, W)\xi - 2\eta(V)\eta(W)\xi - \eta(W)V,$$

$$C\mathcal{T}_V W = \mathcal{T}_V \psi W, \text{ for all } V, W \in \Gamma(T\mathcal{M}) .$$

4. Integrability conditions

In this section, we obtain necessary and sufficient conditions for quasi bi-slant Riemannian maps to be integrable:

Theorem 4.1. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the invariant distribution D is integrable if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X, \omega QZ + \omega RZ) = -g_{\mathcal{M}}(\mathcal{V}\nabla_X \phi Y - \mathcal{V}\nabla_Y \phi X, \psi QZ + \psi RZ),$$

for all $X, Y \in \Gamma(D)$ *and* $Z \in \Gamma(D_1 \oplus D_2 \oplus < \xi >)$.

Proof. For $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_1 \oplus D_2 \oplus <\xi>)$, using equations (5), (10) − (14), (18) and (19), we have

$$\begin{aligned} g_{\mathcal{M}}([X,Y],Z) &= g_{\mathcal{M}}(\nabla_X \phi Y, \phi Z) - g_{\mathcal{M}}(\nabla_Y \phi X, \phi Z) - \eta(Z)\eta(\nabla_X Y) + \eta(Z)\eta(\nabla_Y X), \\ &= g_{\mathcal{M}}(\nabla_X \phi Y, \phi Z) - g_{\mathcal{M}}(\nabla_Y \phi X, \phi Z), \\ &= g_{\mathcal{M}}(\mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X, \omega RZ + \omega QZ) \\ &\quad + g_{\mathcal{M}}(-\mathcal{V}\nabla_Y \phi X + \mathcal{V}\nabla_X \phi Y, \psi QZ + \psi RZ), \end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the slant distribution D_1 is integrable if and only if*

$$\begin{aligned} g_{\mathcal{M}}(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, U) &= g_{\mathcal{M}}(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z, \phi PU + \psi RU) \\ &\quad + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z, \omega RU) \end{aligned} \tag{35}$$

for all $Z, W \in \Gamma(D_1)$ and $U \in \Gamma(D \oplus D_2 \oplus <\xi>)$.

Proof. For all $Z, W \in \Gamma(D_1)$ and $U \in \Gamma(D \oplus D_2 \oplus <\xi>)$, we have

$$g_{\mathcal{M}}([Z,W],U) = g_{\mathcal{M}}(\nabla_Z W, U) - g_{\mathcal{M}}(\nabla_W Z, U).$$

By using the equations (5), (6), (10) − (14), (18) and (19) and Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}}([Z,W],U) &= g_{\mathcal{M}}(\phi\nabla_Z W, \phi U) - g_{\mathcal{M}}(\phi\nabla_W Z, \phi U), \\ &= g_{\mathcal{M}}(\nabla_Z \phi W, \phi U) - g_{\mathcal{M}}(\nabla_W \phi Z, \phi U), \\ &= g_{\mathcal{M}}(\nabla_Z \psi W, \phi U) + g_{\mathcal{M}}(\nabla_Z \omega W, \phi U) - g_{\mathcal{M}}(\nabla_W \psi Z, \phi U) \\ &\quad - g_{\mathcal{M}}(\nabla_Z \omega W, \phi U), \\ &= \cos^2\theta_1 g_{\mathcal{M}}(\nabla_Z W, U) - \cos^2\theta_1 g_{\mathcal{M}}(\nabla_W Z, U) \\ &\quad + g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W - \mathcal{T}_W \omega\psi Z, U) \\ &\quad + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W + \mathcal{T}_Z \omega W, \phi PU + \psi RU + \omega RU) \\ &\quad - g_{\mathcal{M}}(\mathcal{H}\nabla_W \omega Z + \mathcal{T}_W \omega Z, \phi PU + \psi RU + \omega RU). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2\theta_1 g_{\mathcal{M}}([Z,W],U) &= g_{\mathcal{M}}(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z, \phi PU + \psi RU) \\ &\quad + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z, \omega RU) + g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W - \mathcal{T}_W \omega\psi Z, U), \end{aligned}$$

which completes the proof. □

In a similar way, we have the following:

Theorem 4.3. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the slant distribution D_2 is integrable if and only if*

$$\begin{aligned} g_{\mathcal{M}}(\mathcal{T}_Y \omega\psi X - \mathcal{T}_X \omega\psi Y, Z) &= g_{\mathcal{M}}(\mathcal{H}\nabla_X \omega Y - \mathcal{H}\nabla_Y \omega X, \omega QZ) \\ &\quad + g_{\mathcal{M}}(\mathcal{T}_X \omega Y - \mathcal{T}_Y \omega X, \phi PZ + \psi QZ), \end{aligned} \tag{36}$$

for all $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D \oplus D_1 \oplus <\xi>)$.

5. Totally Geodesic conditions

In the present section, we obtain necessary and sufficient conditions for quasi bi-slant Riemannian maps to be totally geodesic:

Proposition 5.1. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the vertical distribution $(\ker \mathcal{F}_*)$ does not define a totally geodesic foliation on $\mathcal{M}$.*

Proof. Let $X \in \Gamma(\ker \mathcal{F}_*)$ and $Z \in \Gamma(\ker \mathcal{F}_*)^{\perp}$, by using equation (13) we have

$$g_{\mathcal{M}}(\nabla_X \xi, Z) = g_{\mathcal{M}}(\phi X, Z),$$

as $g_{\mathcal{M}}(\phi X, Z) \neq 0$, so $g_{\mathcal{M}}(\nabla_X \xi, Z) \neq 0$. Hence, $(\ker \mathcal{F}_*)$ does not define a totally geodesic foliation on $\mathcal{M}$. □

Theorem 5.2. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the distribution $(\ker \mathcal{F}_*) - < \xi >$ defines a totally geodesic foliation on $\mathcal{M}$ if and only if*

$$\begin{aligned} &g_{\mathcal{M}}(\mathcal{T}_U PV + \cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV, X) \\ &\quad = -g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi RV, X) \\ &\quad - g_{\mathcal{M}}(\mathcal{T}_U \omega V, BX) - g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega V, CX) \end{aligned} \tag{37}$$

for all $U, V \in \Gamma(\ker \mathcal{F}_*) - < \xi >$ and $X \in \Gamma(\ker \mathcal{F}_*)^{\perp}$.

Proof. For all $U, V \in \Gamma(\ker \mathcal{F}_*) - < \xi >$ and $X \in \Gamma(\ker \mathcal{F}_*)^{\perp}$, using the equations (11), (12) and (18) we have

$$g_{\mathcal{M}}(\nabla_U V, X) = g_{\mathcal{M}}(\nabla_U \phi PV, \phi X) + g_{\mathcal{M}}(\nabla_U \phi QV, \phi X) + g_{\mathcal{M}}(\nabla_U \phi RV, \phi X).$$

Now, using equations (7), (8), (12), (18) and (19), Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_U V, X) &= g_{\mathcal{M}}(\mathcal{T}_U PV, X) + \cos^2\theta_1 g_{\mathcal{M}}(\mathcal{T}_U QV, X) + \cos^2\theta_2 g_{\mathcal{M}}(\mathcal{T}_U RV, X) \\ &+ g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV, X) \\ &+ g_{\mathcal{M}}(\nabla_U(\omega PV + \omega QV + \omega RV), \phi X). \end{aligned}$$

Now, since $\omega PV + \omega QV + \omega RV = \omega V$ and $\omega PV = 0$, thus we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_U V, X) &= g_{\mathcal{M}}(\mathcal{T}_U PV + \cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV, X) \\ &+ g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV, X) \\ &+ g_{\mathcal{M}}(\mathcal{T}_U \omega V, BX) + g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega V, CX), \end{aligned}$$

which completes the proof. □

Theorem 5.3. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the horizontal distribution $(\ker \mathcal{F}_*)^{\perp}$ does not define a totally geodesic foliation on $\mathcal{M}$.*

Proof. Let $Z, V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$, using equation (13), we have

$$g_{\mathcal{M}}(\nabla_Z V, \xi) = -g_{\mathcal{M}}(V, \nabla_Z \xi) = -g_{\mathcal{M}}(V, \phi Z),$$

as $g_{\mathcal{M}}(V, \phi Z) \neq 0$, therefore $g_{\mathcal{M}}(\nabla_Z V, \xi) \neq 0$. Hence, $(\ker \mathcal{F}_*)^{\perp}$ does not define a totally geodesic foliation on $\mathcal{M}$. □

Proposition 5.4. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the distribution D does not define a totally geodesic foliation on $\mathcal{M}$.*

Proof. For all $U, V \in \Gamma(D)$, using equation (13), we have

$$g_{\mathcal{M}}(\nabla_U V, \xi) = -g_{\mathcal{M}}(V, \phi U),$$

since $g_{\mathcal{M}}(V, \phi U) \neq 0$, so $g_{\mathcal{M}}(\nabla_U V, \xi) \neq 0$. Hence D does not define a totally geodesic foliation on $\mathcal{M}$. □

Theorem 5.5. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the distribution $D\oplus <\xi>$ define a totally geodesic foliation if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_X \phi PY, \omega RZ + \omega QZ) = -g_{\mathcal{M}}(\mathcal{V}\nabla_X \phi PY, \psi QZ + \psi RZ), \tag{38}$$

and

$$g_{\mathcal{M}}(\mathcal{T}_X \phi PY, CV) = -g_{\mathcal{M}}(\mathcal{V}\nabla_X \phi PY, BV), \tag{39}$$

for all $X, Y \in \Gamma(D\oplus <\xi>), Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$ and $V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$.

Proof. For all $X, Y \in \Gamma(D\oplus <\xi>), Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$ and $V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$, using equations (5), (10) – (14), (18) and (19), we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_X Y, Z) &= g_{\mathcal{M}}(\nabla_X \phi Y, \phi Z), \\ &= g_{\mathcal{M}}(\nabla_X \phi PY, \phi QZ + \phi RZ), \\ &= g_{\mathcal{M}}(\mathcal{T}_X \phi PY, \omega RZ + \omega QZ) + g_{\mathcal{M}}(\mathcal{V}\nabla_X \phi PY, \psi QZ + \psi RZ). \end{aligned}$$

Now, again using equations (5), (10) – (14), (18) and (23), we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_X Y, V) &= g_{\mathcal{M}}(\nabla_X \phi Y, \phi V), \\ &= g_{\mathcal{M}}(\nabla_X \phi PY, BV + CV), \\ &= g_{\mathcal{M}}(\mathcal{V}\nabla_X \phi PY, BV) + g_{\mathcal{M}}(\mathcal{T}_X \phi PY, CV), \end{aligned}$$

which completes the proof. □

Proposition 5.6. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the distribution D_i does not define a totally geodesic foliation on $\mathcal{M}$, where $i = 1, 2$.*

Proof. For all $Z, V \in \Gamma(D_i)$, using equation (13) we have

$$g_{\mathcal{M}}(\nabla_Z V, \xi) = -g_{\mathcal{M}}(Z, \phi V),$$

since $g_{\mathcal{M}}(Z, \phi V) \neq 0$, so $g_{\mathcal{M}}(\nabla_Z V, \xi) \neq 0$. Hence D_i does not define a totally geodesic foliation on $\mathcal{M}$, where $i = 1, 2$. □

Theorem 5.7. *Let $\mathcal{F}$ be a proper quasi bi-slant Riemannian map. Then the distribution $D_1\oplus <\xi>$ define a totally geodesic foliation if and only if*

$$\begin{aligned} g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W, X) = &-g_{\mathcal{M}}(\mathcal{T}_Z \omega W, \phi PX + \psi RX) - g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, \omega RX) \\ &+ \eta(W) g_{\mathcal{M}}(Z, \phi PX + \psi RX), \end{aligned} \tag{40}$$

and

$$\begin{aligned} g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega\psi W, V) = &-g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, CV) - g_{\mathcal{M}}(\mathcal{T}_Z \omega W, BV) \\ &+ \eta(W) g_{\mathcal{M}}(Z, BV), \end{aligned} \tag{41}$$

for all $Z, W \in \Gamma(D_1 \oplus < \xi >), X \in \Gamma(D \oplus D_2)$ *and* $V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$.

Proof. For all $Z, W \in \Gamma(D_1 \oplus < \xi >), X \in \Gamma(D \oplus D_2)$ and $V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$, using equations (6), (10) – (14), (18), (19), and Lemma 3.4 we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_Z W, X) &= g_{\mathcal{M}}(\nabla_Z \phi W, \phi X) - \eta(W) g_{\mathcal{M}}(Z, \phi X) \\ &= g_{\mathcal{M}}(\nabla_Z \psi W, \phi X) + g_{\mathcal{M}}(\nabla_Z \omega W, \phi X) - \eta(W) g_{\mathcal{M}}(Z, \phi PX + \psi RX), \\ &= \cos^2 \theta_1 g_{\mathcal{M}}(\nabla_Z W, X) + g_{\mathcal{M}}(\mathcal{T}_Z \omega \psi W, X) \\ &+ g_{\mathcal{M}}(\mathcal{T}_Z \omega W, \phi PX + \psi RX) + g_{\mathcal{M}}(\mathcal{H} \nabla_Z \omega W, \omega RX) \\ &- \eta(W) g_{\mathcal{M}}(Z, \phi PX + \psi RX). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2 \theta_1 g_{\mathcal{M}}(\nabla_Z W, X) = g_{\mathcal{M}}(\mathcal{T}_Z \omega \psi W, X) + g_{\mathcal{M}}(\mathcal{T}_Z \omega W, \phi PX + \psi RX) \\ + g_{\mathcal{M}}(\mathcal{H} \nabla_Z \omega W, \omega RX) - \eta(W) g_{\mathcal{M}}(Z, \phi PX + \psi RX) \end{aligned}$$

Next, from equations (6), (10) – (14), (18), (19), and Lemma 3.4 we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_Z W, V) &= g_{\mathcal{M}}(\nabla_Z \phi W, \phi V) - \eta(W) g_{\mathcal{M}}(Z, \phi V), \\ &= g_{\mathcal{M}}(\nabla_Z \psi W, \phi V) + g_{\mathcal{M}}(\nabla_Z \omega W, \phi V) - \eta(W) g_{\mathcal{M}}(Z, \phi V), \\ &= \cos^2 \theta_1 g_{\mathcal{M}}(\nabla_Z W, V) + g_{\mathcal{M}}(\mathcal{H} \nabla_Z \omega \psi W, V) \\ &+ g_{\mathcal{M}}(\mathcal{H} \nabla_Z \omega W, CV) + g_{\mathcal{M}}(\mathcal{T}_Z \omega W, BV) - \eta(W) g_{\mathcal{M}}(Z, BV). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2 \theta_1 g_{\mathcal{M}}(\nabla_Z W, V) = g_{\mathcal{M}}(\mathcal{H} \nabla_Z \omega \psi W, V) + g_{\mathcal{M}}(\mathcal{H} \nabla_Z \omega W, CV) + g_{\mathcal{M}}(\mathcal{T}_Z \omega W, BV) \\ - \eta(W) g_{\mathcal{M}}(Z, BV), \end{aligned}$$

which completes the proof. □

In a similar way, we can easily prove the following:

Theorem 5.8. *Let* $\mathcal{F}$ *be a proper quasi bi-slant Riemannian map. Then the distribution* $D_2 \oplus < \xi >$ *define a totally geodesic foliation if and only if*

$$\begin{aligned} g_{\mathcal{M}}(\mathcal{T}_X \omega \psi Y, Z) = g_{\mathcal{M}}(\mathcal{T}_X \omega QY, \phi PZ + \phi RZ) + g_{\mathcal{M}}(\mathcal{H} \nabla_X \omega QY, \omega RZ) \\ + \eta(Y) g_{\mathcal{M}}(X, \phi PZ + \psi RZ), \\ g_{\mathcal{M}}(\mathcal{H} \nabla_X \omega \psi Y, V) = -g_{\mathcal{M}}(\mathcal{H} \nabla_X \omega Y, CV) - g_{\mathcal{M}}(\mathcal{T}_X \omega Y, BV) + \eta(Y) g_{\mathcal{M}}(X, BV), \end{aligned}$$

for all $X, Y \in \Gamma(D_2 \oplus < \xi >), Z \in \Gamma(D \oplus D_1)$ *and* $V \in \Gamma(\ker \mathcal{F}_*)^{\perp}$.

By using Proposition 5.1 and Theorem 5.3, one can give the following theorem:

Theorem 5.9. *Let* $\mathcal{F}$ *be a proper quasi bi-slant Riemannian map. Then the map* $\mathcal{F}$ *is not a totally geodesic map.*

6. Example

Example 6.1. *Let R^9 be a LP-Sasakian structure (as in Example 3.2) and $\mathcal{F} : R^9 \to R^4$ be a map defined by*

$$\mathcal{F}(x_1, ..., x_4, y_1...., y_4, z) = (a, \cos\theta_1 x_3 + \sin\theta_1 x_4, \sin\theta_2 y_2 - \cos\theta_2 y_3, b)$$

where $a, b \in R$. Then $\mathcal{F}$ is quasi bi-slant Riemannian map such that

$$D =< 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial Z}\right), 2\frac{\partial}{\partial y_1} >,$$

$$D_1 =< 2\left[\sin\theta_1\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial Z}\right) - \cos\theta_1\left(\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial Z}\right)\right], 2\frac{\partial}{\partial y_4} >,$$

$$D_2 =< 2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial Z}\right), 2\left[\cos\theta_2\frac{\partial}{\partial y_2} + \sin\theta_2\frac{\partial}{\partial y_3}\right] >,$$

$$< \xi > =< 2\frac{\partial}{\partial z} >,$$

$$(\ker\mathcal{F}_*)^\perp =< V_1 = 2\left[\cos\theta_1\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) + \sin\theta_1\left(\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z}\right)\right],$$

$$V_2 = 2\left[\sin\theta_2\frac{\partial}{\partial y_1} - \cos\theta_2\frac{\partial}{\partial y_2}\right] >,$$

with bi-slant angles θ_1 and θ_2. Also by direct computations, we obtain

$$\mathcal{F}_*V_1 = 2\frac{\partial}{\partial v_2},\ \mathcal{F}_*V_2 = 2\frac{\partial}{\partial v_3}.$$

Hence, we get

$$g_{R^9}(V_1, V_1) = g_{R^4}(\mathcal{F}_*V_1, \mathcal{F}_*V_1), g_{R^9}(V_2, V_2) = g_{R^4}(\mathcal{F}_*V_2, \mathcal{F}_*V_2).$$

Example 6.2. *Let R^9 be a LP-Sasakian structure (as in Example 3.2.) and $\mathcal{F} : R^9 \to R^4$ be a map defined by*

$$\mathcal{F}(x_1, \ldots, x_4, y_1, \ldots, y_4, z) = \left(a, \frac{\sqrt{3}x_2 + x_3}{2}, b, \frac{y_2 - y_4}{\sqrt{2}}\right)$$

where $a, b \in R$. Then $\mathcal{F}$ is quasi bi-slant Riemannian map such that

$$D =< \frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1} >,$$

$$D_1 =< \left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z} - \sqrt{3}\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right)\right), \frac{\partial}{\partial y3} >,$$

$$D_2 =< \frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z}, \left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_4}\right) >,$$

$$< \xi > =< 2\frac{\partial}{\partial z} >,$$

with bi-slant angles $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{\pi}{4}$.

It can be easily seen that Theorem 5.3 is satisfied by the Examples 6.1 and 6.2.

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