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EQUIVALENCE OF LEBESGUE'S THEOREM AND BAIRE CHARACTERIZATION THEOREM

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Abstract. Let X be a complete separable metric space and Y be a separable Banach space. We provide a proof of equivalence by linking explicitly the following statements: For every $\epsilon > 0$ there exists a countable collection of closed sets $\{C_n\}$ of X such that $X = \bigcup_{n=1}^{\infty} C_n$ and $\omega_f(C_n) < \epsilon$ for each n (Lebesgue's Theorem) and; For every nonempty perfect set $K \subset X$, the function $f|_K$ has at least one point of continuity in K. In fact, $C(f|_K)$ is dense in K (Baire Characterization Theorem). Moreover, replacing "closed" by "open" in the Lebesgue's Theorem, we obtain a characterization of continuous functions on space X.

Key words and Phrases: Baire class one, Lebesgue's Theorem, Baire Characterization Theorem

1. INTRODUCTION

Let X and Y be metric spaces. Rene Baire in his 1899 dissertation defined a function $f : X \to Y$ to be of the first class or Baire class one or Baire-1 as pointwise limit of a sequence of continuous functions $\{f_n : X \to Y\}$[1]. It was further shown that for any complete separable metric space X and separable Banach space Y, the following statements are equivalent: (See for instance [4] or [5])

(i) $f : X \to Y$ is Baire class one function;
(ii) for every open set $G \subseteq Y$, the inverse image of G under f is an $\mathcal{F}_\sigma$ set in X;

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(iii) (Lebesgue's Theorem) For every $\epsilon > 0$ there exists a countable collection of closed sets $\{C_n\}$ of X such that

$$X = \bigcup_{n=1}^{\infty} C_n \text{ and } \omega_f(C_n) < \epsilon \text{ for each } n.$$

(iv) (Baire Characterization Theorem) For every nonempty perfect set K in X, the function $f|_K$ has a point of continuity in K.

The equivalence of the four statements is generally established by showing that statement (i) is equivalent to statement (ii); statement (ii) is equivalent to statement (iii); and statement (i) is equivalent to statement (iv). As historical note, it was H. Lebesgue who first demonstrated on the real number line that statement (iii) is a necessary and sufficient condition for the function f to be of Baire class one while it was B. Gagaeff who proved the equivalence of statement (ii) and statement (iii) in these general spaces. Please see references [2] and [4]. R. Baire proved the equivalence of statement (i) to statement (ii) and statement (iv) [1].

As far as the authors know, there is no known proof that directly connects Lebesgue's Theorem and the Baire Characterization Theorem. In this paper, we provide a proof of the equivalence of Lebesgue's Theorem and the Baire Characterization Theorem. As an aside result, we obtain a characterization of continuous functions by simply replacing "closed" by "open" in the Lebesgue's Theorem, demonstrating once again the intimate connection between the first Baire class and the class of continuous functions.

2. EQUIVALENCE OF LEBESGUE'S THEOREM AND BAIRE CHARACTERIZATION THEOREM

In our discussion, (X, ρ) is any complete separable metric space and $(Y, \|\cdot\|)$ a separable Banach space, unless otherwise indicated. An open ball in X centered at x_0 with radius $\delta > 0$ denoted by $B(x_0, \delta)$ is the set $\{y \in X : \rho(x_0, y) < \delta\}$. We denote the oscillation of a function $f : X \to Y$ on a set $A \subset X$ as

$$\omega_f(A) = \sup\{\|f(x) - f(y)\| : x, y \in A\}.$$

Finally, the closure, interior and boundary of A are denoted by $\overline{A}$, A° and ∂A, respectively.

Before we proceed further, let us introduce first the tools necessary to prove the main result. More specifically, the Baire Category Theorem as well as its corollary plays a significant role in establishing the equivalence of Lebesgue's Theorem and the Baire Characterization Theorem. For the sake of completeness, we will state the Baire Category Theorem and its the relevant corollary with the corresponding proof. For the detailed proof of the Baire Category Theorem, one may refer to [3, pp. 69-70].

Theorem 2.1 (Baire Category Theorem)**.** *If E is a nonempty closed set and*

$$E = \bigcup_{k=1}^{\infty} E_k$$

then there exist an open ball B and a natural number p such that $E \cap B \neq \emptyset$ and $E \cap B \subseteq \overline{E_p}$.

Corollary 2.2. *If E is a nonempty closed set and $\{E_k\}_{k=1}^{\infty}$ is a sequence of closed sets such that*

$$E = \bigcup_{k=1}^{\infty} E_k$$

then there exist an open ball B and a natural number p such that $E \cap B \neq \emptyset$ and $E \cap B \subseteq E_p$.

Proof. This collorary follows from the fact that the closure of a closed set is the closed set itself. $\square$

We are now ready to prove the equivalence of Lebesgue's Theorem and the Baire Characterization as rewritten below. The proof of the necessity part is based on a proof of a lemma found in reference [3, p.77-78]. For notational purposes, we let $C(f)$ to denote the set of points of continuity of the function f.

Theorem 2.3. *Let $f : X \to Y$ be a function. The following statements are equivalent:*

(i) For every $\epsilon > 0$ there exists a countable collection of closed sets $\{C_n\}$ of X such that

$$X = \bigcup_{n=1}^{\infty} C_n \text{ and } \omega_f(C_n) < \epsilon \text{ for each } n.$$

(ii) For every nonempty perfect set $K \subset X$, the function $f|_K$ has at least one point of continuity in K. In fact, $C(f|_K)$ is dense in K.

Proof. Suppose statement (i) holds. Let K be any nonempty perfect set in X. We will show that $f|_K$ has at least a point of continuity in K. Suppose $f|_K$ is not continuous at any point of K. It follows that for each $x \in K$ there exists a real number $\epsilon_x > 0$ such that for any open ball I containing x there exist $y, y' \in K \cap I$ with $\|f(y) - f(y')\| \geq \epsilon_x$. For each natural number n, let K_n be the set containing all $x \in K$ such that for all open ball I containing x there exist $y, y' \in K \cap I$ with $\|f(y) - f(y')\| \geq \frac{1}{n}$. Notice that $K = \bigcup_{n=1}^{\infty} K_n$. We claim that K_n is closed for each n. Let p be any accumulation point of K_n and let B be any open ball containing p. To show that $p \in K_n$, one must find $y, y' \in B \cap K$ such that $\|f(y) - f(y')\| \geq \frac{1}{n}$. Since p is an accumulation of K_n then the set $B \cap K_n$ contains at least one element say q different from p. By definition of K_n, for every open ball $B' \subseteq B$ containing q there exist $s, t \in B' \cap K$ such that $\|f(s) - f(t)\| \geq \frac{1}{n}$. Since $B' \cap K \subseteq B \cap K$

then $p \in K_n$. Thus, K_n is closed for each natural number n as claimed.
By the corollary of the Baire Category Theorem, there exists an open ball I with $K \cap I \neq \emptyset$ and a positive integer n_0 such that $K \cap I \subseteq K_{n_0}$. Observe that we may choose I so that $K \cap \overline{I} \subseteq K_{n_0}$. By statement (i), there exists a countable collection of closed sets $\{C_n\}$ such that

$$X = \bigcup_{n=1}^{\infty} C_n \text{ and } \omega_f(C_n) < \frac{1}{n_0} \text{ for each } n.$$

Now, $K \cap \overline{I} = \bigcup_{n=1}^{\infty} \left(K \cap \overline{I} \cap C_n\right)$ and since $K \cap \overline{I}$ is a nonempty closed set then by using again the corollary of the Baire Category Theorem there exists an open ball U with $K \cap \overline{I} \cap U \neq \emptyset$ and a positive integer p such that

$$K \cap \overline{I} \cap U \subseteq K \cap \overline{I} \cap C_p.$$

Observe that $\omega_f(K \cap \overline{I} \cap C_p) < \frac{1}{n_0}$ since $K \cap \overline{I} \cap C_p \subseteq C_p$. It remains to show that $\omega_f(K \cap \overline{I} \cap U) \geq \frac{1}{n_0}$ to arrive at a contradiction. Notice that one can find $x_0 \in K \cap U \cap I$ and open ball U' containing x_0 such that $U' \subseteq I \cap U$. Since $x_0 \in K_{n_0}$ then there exist $y, y' \in K \cap U' \subseteq K \cap \overline{I} \cap U$ such that $\|f(y) - f(y')\| \geq \frac{1}{n_0}$. Consequently, $\omega_f(K \cap \overline{I} \cap U) \geq \frac{1}{n_0}$. All these show that $f|_K$ is continuous at some point of K. Finally, let us show that $C(f|_K)$ is dense in K. Let $x \in K$. We are done if we can show that x is a limit point of $C(f|_K)$. Let B be any open ball containing x and A be any nonempty closed ball such that $A \subset B$ and $K \cap A \neq \emptyset$. Since $K \cap A$ is a closed set then the restricted function $f|_{K \cap A}$ has at least one point of continuity in $K \cap A^{\circ}$. Clearly, this is also a point of continuity of the restricted function $f|_K$. All these tell us that $B \cap C(f|_K)$ contains a point other than x and so $C(f|_K)$ is dense in K.

Suppose statement (i) does not hold. Then there exists a real number $\epsilon_0 > 0$ such that for any countable collection of closed sets $\{K_n\}$ in X with $X = \bigcup_{n=1}^{\infty} K_n$ there is always some set K_j such that $\omega_f(K_j) \geq \epsilon_0$. We claim that there exists a countable collection of open balls $\{B_i\}$ of X such that $K_j = \bigcup_{i=1}^{\infty} (B_i \cap K_j)$ and $\omega_f(B_i \cap K_j) < \epsilon_0$ for all natural number i. Contradiction is achieved by noting that the set $B_i \cap K_j$ is of type $\mathcal{F}_\sigma$ for each natural number i. By statement (ii), the restricted function $f|_{K_j}$ has a dense set of points of continuity in K_j. Observe that $C\left(f|_{K_j}\right) \nsubseteq \partial K_j$. Otherwise, $K_j \subseteq \overline{C\left(f|_{K_j}\right)} \subseteq \partial K_j$. It follows that there is at least one point of continuity $x' \in K_j$ of $f|_{K_j}$ such that $x' \notin \partial K_j$. Consequently, there exists an open ball B' in X containing x' such that $\omega_f(B' \cap K_j) < \epsilon_0$. Next, consider the set $K_j - B'$ and restrict the function f to this set. Notice that $K_j - B'$ is closed. By assumption, the restricted function $f|_{K_j - B'}$ has a dense set of points of continuity in $K_j - B'$. Moreover, $C(f|_{K_j - B'}) \nsubseteq \partial B' \cap K_j$. It follows that the restricted function $f|_{K_j - B'}$ has at least one point of continuity $x'' \in K_j - B'$

such that $x'' \notin \partial B' \cap K_j$. Thus, there exists an open ball B'' containing x'' such that $B'' \cap \left(\overline{B'} \cap K_j\right) = \emptyset$ and $\omega_f\left(B'' \cap (K_j - B')\right) < \epsilon_0$. However, notice that $B'' \cap (K_j - B') = B'' \cap K_j$. Next, consider the closed set $(K_j - B') - B''$ and restrict the function f to this set. By the same argument there exists an open ball B''' such that $B''' \cap \left[\overline{B''} \cap (K_j - B')\right] = \emptyset$ and $\omega_f\left(B''' \cap [(K_j - B') - B'']\right) = \omega_f\left(B''' \cap K_j\right) < \epsilon_0$. Continue the process until one obtains a collection of open balls $\{B^\alpha\}_\alpha$ possibly uncountable such that $K_j = \bigcup_\alpha (B^\alpha \cap K_j)$ and $\omega_f\left(B^\alpha \cap K_j\right) < \epsilon_0$ for any α. Since X is a separable space and so K_j is a separable subspace of X, one can find a countable collection of open balls $\{B_i \cap K_j\}_{i=1}^\infty$ relative to K_j from the collection $\{B^\alpha \cap K_j\}_\alpha$ such that $K_j = \bigcup_{i=1}^\infty (B_i \cap K_j)$ and $\omega_f(B_i \cap K_j) < \epsilon_0$ for all i. Clearly, this is a contradiction. $\square$

At this point, it would be instructive to provide example that illustrates the extent of use of the main theorem.

Example 2.4. *Consider the following function on* $[0,1]$

$$f(x) = \begin{cases} \frac{1}{q} & \textit{if } x = \frac{p}{q} (x \textit{ is rational }), \textit{with } p \in \mathbb{Z} \textit{ and } q \in \mathbb{N} \textit{ coprime} \\ 0 & \textit{if } x \textit{ is irrational.} \end{cases}$$

It is known that the set of points of points of discontinuity of this function is the set $D(f) = [0,1] \cap \mathbb{Q}$. *Let* $\epsilon > 0$. *There exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $\frac{1}{n} < \epsilon$. *Notice that the set* $D_\epsilon = \{x : |f(x)| \geq \epsilon\}$ *is a finite set, say* $D_\epsilon = \{r_1, r_2, \ldots, r_k\}$, *for some* $k \in \mathbb{N}$. *Further, we have*

$$f(x) < \epsilon, \textit{ for all } x \in [0,1] \setminus D_\epsilon. \tag{1}$$

If we assume that $r_1, r_2, \ldots, r_k$ *is in increasing order, then*

$$[0,1] \setminus D_\epsilon = [0, r_1) \cup (r_1, r_2) \cup \cdots \cup (r_k, b].$$

Observe that each interval in the union above can be expressed as a union of countable closed intervals. Furthermore, in view of (1) on each of these intervals the oscillation of f is less than ϵ satisfying statement (i) of Theorem 2.3. On the other hand, statement (ii) of Theorem 2.3 is likewise satisfied. It is enough to observe that the set of of points of discontinuity of f is countable and any nonempty perfect set is uncountable. It follows that for any nonempty perfect set K, the restricted function $f|_K$ has always at least one point of continuity in K.

Finally, let us end by stating a result that demonstrates the intimate connection between the class of Baire one functions and the class of continuous functions. As mentioned earlier, replacing "closed" by "open" in the Lebesgue's Theorem, one readily obtains a characterization of continuous functions on space X. Let (X, ρ_1) be a separable metric space and (Y, ρ_2) be any metric space. It is a standard exercise to show that if $f : X \to Y$ is continuous on the space X then for each real number $\epsilon > 0$ there exists a countable collection of open sets $\{U_n\}$ in X such that

$X = \bigcup_{n=1}^{\infty} U_n$ and $\omega_f(U_n) < \epsilon$ for each n. However, it seems that no one points out that the converse of the previous statement also holds true. That is, the condition that for every $\epsilon > 0$ there corresponds a countable collection of open sets $\{U_n\}$ in X such that $X = \bigcup_{n=1}^{\infty} U_n$ and $\omega_f(U_n) < \epsilon$ for each n is a necessary and sufficient condition for the function f to be continuous on the space X. As much as we would like to imagine that this is an old result, we are unable to find a reference. For clarity and for the sake of completeness, we will write it as a theorem below and provide a proof of the necessity part of the theorem.

Theorem 2.5. *The function $f : X \to Y$ is continuous if and only if for every $\epsilon > 0$ there exists a countable collection of open sets $\{U_n\}$ in X such that*

$$X = \bigcup_{n=1}^{\infty} U_n \ \textit{ and } \ \omega_f(U_n) < \epsilon \ \textit{ for each } \ n.$$

PROOF. Let $x \in X$. By the assumption, there exists a countable collection of open sets $\{U_n\}$ of X such that

$$X = \bigcup_{n=1}^{\infty} U_n \ \text{ and } \ \omega_f(U_n) < \epsilon \ \text{ for each } \ n.$$

There exists a least index $n(x)$ such that $x \in U_{n(x)}$. Since $U_{n(x)}$ is open and $x \in U_{n(x)}$ there is a positive number δ_x such that $B(x, \delta_x) \subset U_{n_x}$. Take $\delta = \delta_x$. Suppose that $\rho_1(x, y) < \delta_x$. It follows that $y \in B(x, \delta_x) \subset U_{n_x}$. Hence,

$$\rho_2(f(x), f(y)) \leq \omega_f(B(x, \delta_x)) \leq \omega_f(U_{n_x}) < \epsilon.$$

Hence, f is continuous on X. $\square$

REFERENCES

[1] Baire, R., Sur les fonctions des variables reeles, *Ann. Mat. Pura ed Appl.*, **3** (1899), 1122.

[2] Bressoud, D. M., *A Radical Approach To Lebesgues Theory of Integration*, Cambridge University Press, United States of America, 2008.

[3] Gordon, R. A., *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, 4, American Mathematical Society, 1994.

[4] Kuratowski, K., *Topology*, Academic Press, London, 1966.

[5] Lee, P.Y., Tang, W.K., and Zhao, D., An equivalent Definition of Functions of the First Baire Class, *Proc.Amer. Math. Soc.*, **129:8** (2001), 2273-2275.