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SUPER EDGE CONNECTIVITY NUMBER OF AN ARITHMETICS GRAPH

Sujitha, S.[1], Mary Jenitha, L.[2]

[1]Department of Mathematics, Holy Cross College (Autonomous), Nagercoil, India.
sujitha.s@holycrossngl.edu.in
[2]Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, India.
jeni.mathematics@gmail.com

Abstract. An edge subset F of a connected graph G is a super edge cut if $G-F$ is disconnected and every component of $G-F$ has atleast two vertices. The minimum cardinality of super edge cut is called super edge connectivity number and it is denoted by $\lambda^{'}(G)$. Every arithmetic graph $G=V_n,\ n \neq p_1 \times p_2$ has super edge cut. In this paper, the authors study super edge connectivity number of an arithmetic graphs $G=V_n, n=p_1^{a_1}\times p_2^{a_2}, a_1>1, a_2\geq 1$ and $G=V_n,\ n=p_1^{a_1}\times p_2^{a_2}\times\cdots\times p_r^{a_r},\ r>2, a_i\geq 1, 1\leq i\leq r$.

Key words and Phrases: arithmetic graph, super edge cut, super edge connectivity number.

1. INTRODUCTION

Theorem 1.1. [5]*For an arithmetic graph* G=V_n, n =$p_1^{a_1}\times p_2^{a_2}$ *where* p_1 *and* p_2 *are distinct primes,* $a_1, a_2 \geq 1$ *then* $\epsilon = 4a_1a_2 - a_1 - a_2$, *where* ϵ *is the size of the graph* G.

Theorem 1.2. [5]*For an arithmetic graph* G=V_n, n =$p_1^{a_1}\times p_2^{a_2}$ *where* p_1 *and* p_2 *are distinct primes,* $a_1, a_2 \geq 1$ *then* G *is a bipartite graph.*

Theorem 1.3. [5] *Let* $G = V_n$ *an arithmetic graph* $n = p_1^{a_1}\times p_2^{a_2}\times\cdots\times p_r^{a_r}$, *for any vertex* $u = \prod\lim_{i\in B} p_i^{\alpha_i}$ *where* $B \subseteq 1,2,3,\dots r, 1\leq \alpha_i \leq a_i \forall i\in B$.

(1) *If* $u = p_j$ *where* $j \in \{1,2,3,\dots,r\}$, *then*

$$deg(u) = \left[a_j \prod \lim_{i=1,i\neq j}^{r} (a_i+1) - 1\right] - |a_j - 1|\,.$$

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(2) *If* $u = p_i^{\alpha_i}$ $1 < \alpha_i \le a_i \forall i \in B$, *then* $deg(u) = [\prod \lim_{i=1, i \notin B}^{r} (a_i + 1)] - 1$

(3) *If* $u = \prod \lim_{i \in B} p_i^{\alpha_i}, |B| \ge 2, 1 < \alpha_i \le a_i, \forall i \in B$ *then*

$$deg(u) = |B| \prod \lim_{i=1, i \notin B}^{r} (a_i + 1) .$$

(4) *If* $u = \prod \lim_{i \in B} p_i^{\alpha_i}, \alpha_i = 1$ *for some* $i \in B' \subseteq B$, *then* $\deg(u) = |B - B'| + \sum lim_{i \in B'} a_i \prod \lim_{i=1, i \notin B}^{r} (a_i + 1)$ *where* B *is the number of distinct prime factors in a chosen vertex* u, B' *is the number of prime factors having power 1 in chosen vertex* u.

2. SUPER EDGE CONNECTIVITY NUMBER OF AN ARITHMETIC GRAPH $G = V_n$

In this section, the super edge connectivity number $\lambda^{'}(G)$ of an arithmetic graph $G = V_n$, where $n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ is determined.

Theorem 2.1. *For an arithmetic graph* $G = V_n, n = p_1^{a_1} \times p_2^{a_2}$ *where* $a_1 = a_2 = 1$ *has no super edge cut.*

Proof. Consider an arithmetic graph $G = V_n$, where n is the product of two distinct primes. The vertex set of V_n contains three vertices namely $p_1, p_2, p_1 \times p_2$. By the definition of an arithmetic graph G is a path with 3 vertices. The removal of any one of the edge results the graph disconnected containing an isolated vertex and an edge. Hence proved. □

Theorem 2.2. *For an arithmetic graph* $G = V_n, n = p_1^{a_1} \times p_2^{a_2}$ *where* $a_1 > 1, a_2 = 1$ *then* $\lambda^{'}(G) = 2$.

Proof. Given arithmetic graph $G = V_n$ has the vertex set$V(G) = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2, p_1 \times p_2, p_1^2 \times p_2, p_1^3 \times p_2, \ldots, p_1^{a_1} \times p_2\}$. By Theorem 1.2, G is a bipartite graph with partitions $A = \{p_1, p_1^2, \ldots, p_1^{a_1}, p_2\}$ and $B = \{p_1 \times p_2, p_1^2 \times p_2, p_1^3 \times p_2, \ldots, p_1^{a_1} \times p_2\}$. Also, the graph G has $a_1 - 1$ pendant vertices say $p_1^2, p_1^3, \ldots, p_1^{a_1}$ and all these pendant vertices have a common neighbour $p_1 \times p_2$. The removal of two edge say $p_1 \times p_2\ p_1$ and $p_1 \times p_2\ p_2$, the graph G gets disconnected. Since $d(p_1 \times p_2) = a_1 + 1$, the resultant graph has exactly two components G_1 and G_2 where $G_1 = k_{1, a_1 - 1}$ and G_2 is a connected graph. Hence $F = \{p_1 \times p_2\ p_1, p_1 \times p_2\ p_2\}$ is a super edge cut. Since G is not a tree and it does not have bridges, F is a minimum cardinality set. Thus $\lambda^{'}(G) = 2$. □

Theorem 2.3. *For an arithmetic graph* $G = V_n, n = p_1^{a_1} \times p_2^{a_2}$ *where* $a_1 \ge a_2 > 1$ *then* $\lambda^{'}(G) = a_1 + a_2 - 1$.

Proof. By Theorem 1.2, G is a bipartite graph. Since $a_1 \ge a_2 > 1$ we have $d(p_1^m) \le d(p_2^n)$ for $1 < m \le a_1$, $1 \le n \le a_2$. Choose a vertex of the form p_1^m ; $1 < m \le a_1$,from the first partition. Let it be $p_1^{a_1}$ the vertices which are adjacent to $p_1^{a_1}$ are $\{p_1 \times p_2, p_1 \times p_2^n\ ;\ 1 < n \le a_2\}$. Since the vertices $\{p_1 \times p_2^n\ ;\ 1 < n \le a_2\}$ have less degree compared to $p_1 \times p_2$, choose any one of the vertex of the form

$p_1 \times p_2^n$; $1 < n \leq a_2$, let it be $p_1 \times p_2^{a_2}$. Now, remove all the edges incident on $p_1^{a_1}$ and $p_1 \times p_2^{a_2}$ other than the edge $p_1^{a_1}\ p_1 \times p_2^{a_2}$. The resultant graph is disconnected having two components, in which one of the component is an edge $p_1^{a_1}\ p_1 \times p_2^{a_2}$ and the other is a connected graph. Since the degree of these two vertices say $p_1^{a_1}$ and $p_1 \times p_2^{a_2}$ is minimum we have $|F| = d(p_1^{a_1}) + d(p_1 \times p_2^{a_2}) - 2$. Hence by the proof of Theorem1.1, $\lambda^{'}(G) = a_1 + a_2 - 1$. □

Theorem 2.4. *For an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}, r > 2$ and $a_i \geq 1, i \in \{1, 2, \ldots, r\}$. Then $\lambda^{'}(G) = 2^{r-1} + r - 3$.*

Proof. Consider an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}, r > 2$ and $a_i \geq 1, i \in \{1, 2, \ldots, r\}$. Following steps are used to find the super edge connectivity number of an arithmetic graph.
(i) Arrange all $a_i's$ in such a way that $a_1 \geq a_2 \geq \cdots \geq a_r$.
(ii)Choose an edge $e = uv$ such that $d(u) + d(v) = min\{d(v_i) + d(v_j)/v_iv_j \in E(G); i \neq j$ and for all $i, j \in \{1, 2, \ldots r\}\}$.
(iii) Remove all the edges incident on u and v other than the edge $e = uv$.(i.e) we remove $d(u) + d(v) - 2$ edges. Now, the resultant graph is disconnected and it has exactly two components one of the component is an edge $e = uv$ and the other one is a connected graph. Since $d(e = uv)$ is minimum, $|d(u) + d(v) - 2|$ is the super edge connectivity number.
case(i) If $a_i = 1$ for all i then choose the edge $e = uv$ where u can be any one of $p_i; i \in \{1, 2, \ldots, r\}$, let it be p_1 and $v = p_1 \times p_2 \times \cdots \times p_r$. Since the removal of the edges incident on u and v other than the edge uv results the graph disconnected. Also, $d(u) + d(v)$ is minimum, $\lambda^{'}(G) = |F| = d(p_1) + d(p_1 \times p_2 \times \cdots \times p_r) - 2$.
By Theorem1.3, we have $\lambda^{'}(G) = 2^{r-1} + r - 3$.
case(ii) If $a_i > 1$ for exactly one i, then choose the edge $e = uv$ where $u = p_1^{a_1}$ and $v = p_1 \times p_2 \times \cdots \times p_r$. Now similar as previous case we have $\lambda^{'}(G) = d(p_1^{a_1}) + d(p_1 \times p_2 \times \cdots \times p_r) - 2$.
By Theorem1.3, we have $\lambda^{'}(G) = [\prod \lim_{i=1, i \notin B}^{r} (a_i + 1)] - 1 + r - 2 = 2^{r-1} + r - 3$. □

Example 2.5. *Consider an arithmetic graph $G = V_{210}, 210 = 2 \times 3 \times 5 \times 7$ here the super edge cut $F = \{2\ 2 \times 3, 2\ 2 \times 5, 2\ 2 \times 7, 2\ 2 \times 3 \times 5, 2\ 2 \times 3 \times 7, 2\ 2 \times 5 \times 7, 2 \times 3 \times 5 \times 7\ 3, 2 \times 3 \times 5 \times 7\ 5, 2 \times 3 \times 5 \times 7\ 7\}$. Hence $\lambda^{'}(G) = 9$.By Theoreom2.4, $\lambda^{'}(G) = 2^3 + 4 - 3 = 9$*

Theorem 2.6. *For an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}, r > 2$ and $a_i > 1$, for at least two a_i, $i \in \{1, 2, \ldots, r\}$. Then $\lambda^{'}(G) = \prod \lim_{i=2}^{m} (a_i + 1) 2^{r-m} + a_1 + r - 4$.*

Proof. **case(i)** If $a_i > 1$ for exactly two i, without loss of generality let $a_1 \geq a_2 > 1$, then choose the edge $e = uv$ where $u = p_1^{a_1}$ and $v = p_1 \times p_2^{a_2} \times p_3 \cdots \times p_r$. Similar as above we have $\lambda^{'}(G) = d(p_1^{a_1}) + d(p_1 \times p_2^{a_2} \times p_3 \times \cdots \times p_r) - 2$. By Theorem1.3, we have $\lambda^{'}(G) = [\prod \lim_{i=1, i \notin B}^{r} (a_i + 1)] - 1 + [|1| + (a_1 + r - 2)] - 2$
$= (a_2 + 1)2^{r-2} + a_1 + r - 4$.
case(ii) If $a_1 \geq a_2 \geq \ldots a_m > 1$, then choose the edge $e = uv$ where $u = p_1^{a_1}$ and

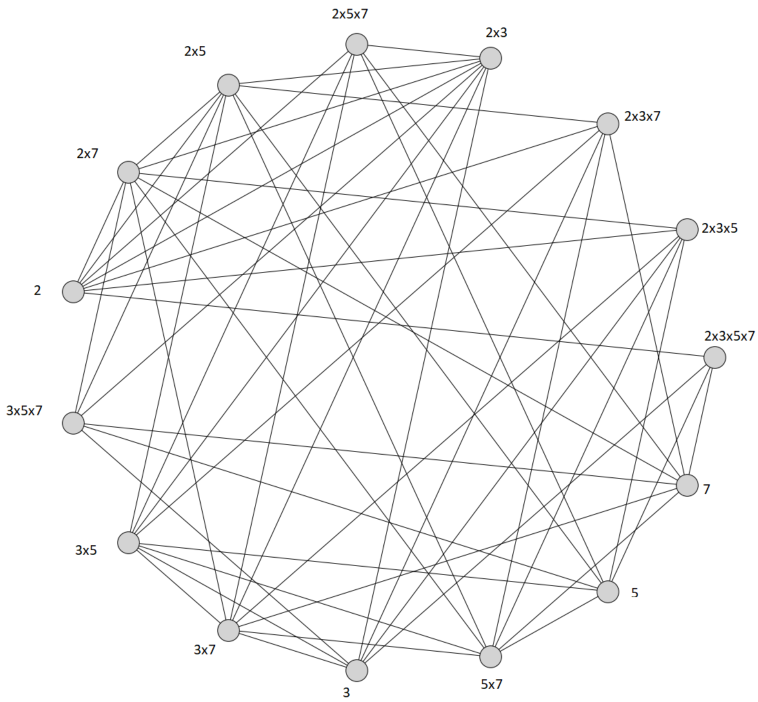

FIGURE 1. Arithmetic Graph $G = V_{210}$

$v = p_1 \times p_2^{a_2} \times p_3^{a_3} \times \dots p_m^{a_m} \times p_{m+1} \cdots \times p_r$. Similar as above we have $\lambda^{'}(G) = d(p_1^{a_1}) + d(p_1 \times p_2^{a_2} \times p_3^{a_3} \times \dots p_m^{a_m} \times p_{m+1} \cdots \times p_r) - 2$. By Theorem 1.3, we have $\lambda^{'}(G) = [\prod \lim_{i=1, i \notin B}^{r} (a_i + 1)] - 1 + [m - 1 + a_1 + r - m] - 2 = [(a_2 + 1)(a_3 + 1) \dots (a_m + 1)2^{r-m}] + a_1 + r - 4$.

case(iii) If $a_i > 1$ for all i, then choose the edge $e = uv$ where $u = p_1^{a_1}$ and $v = p_1 \times p_2^{a_2} \times p_3^{a_3} \times \cdots \times p_r^{a_r}$. Similar as above we have $\lambda^{'}(G) = d(p_1^{a_1}) + d(p_1 \times p_2^{a_2} \times p_3^{a_3} \times \cdots \times p_r^{a_r}) - 2$
we have $\lambda^{'}(G) = [\prod \lim_{i=1, i \notin B}^{r} (a_i + 1)] - 1 + [r - 1 + a_1] - 2 = (a_2 + 1)(a_3 + 1) \dots (a_r + 1) + a_1 + r - 4$. □

3. SUPER $\lambda^{'}$ OPTIMALITY OF AN ARITHMETIC GRAPH $G = V_n$

Let $G = (V, E)$ be a graph for $e = uv \in E(G)$, let $\xi_G(e) = d_G(u) + d_G(v) - 2$ and $\xi_(G) = min\{\xi_G(e) : e \in E(G)\}$. The parameter $\xi(G)$ is called minimum edge degree of G. If $\lambda^{'}(G) = \xi(G)$ then G is called optimal; otherwise G is non-optimal. For two disjoint non empty subsets X and Y of V, let $(X, Y) = \{e = uv \in E; u \in X, v \in Y\}$. If $Y = \overline{X} = V - X$ then we write $\partial(X)$ for $(X, \overline{X})$ and $d(X)$ for $|\partial(X)|$. A super edge cut F of G is called λ'-cut if $|F| = \lambda'(G)$. It is clear that for any λ'-cut F that $G - F$ has two connected components.

Let X be a proper subset of V. If $\partial(X)$ is a $\lambda^{'}$-cut of G, then X is called a fragment of G. It is clear that if X is a fragment of G, then so is $\overline{X}$. Let $r(G)$=min$\{|X|\,; X$ is a fragment of $G\}$. Obviously $2 \leq r(G) \leq \frac{|V|}{2}$. A fragment X is called an atom if $|X| = r(G)$.

Theorem 3.1. *For an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2}$ where $a_1, a_2 \geq 1$ then the minimum edge degree* $\xi(G) = \begin{cases} 1 & if\ a_1 = a_2 = 1 \\ a_1 & if\ a_1 > 1, a_2 = 1 \\ a_1 + a_2 - 1 & if a_1 \geq a_2 > 1 \end{cases}$

Proof. The proof is obivious from the proof of Theorem 2.3. □

Theorem 3.2. *For an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}, r > 2$ where $a_i \geq 1$ for $i \in \{1, 2, \ldots, m, \ldots, r\}$ then the minimum edge degree*
(i)$\xi(G) = 2^{r-1} + r - 3$ if $a_1 \geq 1$ and $a_j = 1$ for $j \in \{2, 3, \ldots, r\}$.
(ii)$\xi(G) = [\prod \lim_{i=2}^{m}(a_i + 1)2^{r-m}] + (m - 1) + a_1 + (r - m) - 3$ if $a_i > 1$ for more than m $i's$, $m \geq 2, i \in \{1, 2, \ldots, r\}$

Proof. The proof follows from Theorem 2.4 and 2.6. □

Theorem 3.3. *For every arithmetic graph other than $G = V_n, n = p_1^{a_1} \times p_2, a_1 > 2$ are optimal and the atom $r(G) = 2$.*

Proof. Let $G = V_n$ be an arithmetic graph,
Case (i) If $n = p_1^{a_1} \times p_2, a_1 > 2$ then by Theorem 2.2 we have the super edge connectivity number $\lambda^{'}(G) = 2$. By Theorem 3.1, the minimum edge degree $\xi(G) = a_1$. Clearly $\lambda^{'}(G) \neq \xi(G)$, hence it is non optimal.
Case (ii) Consider $G = V_n$ where $n \neq p_1^{a_1} \times p_2, a_1 > 2$, then by using the theorems in section 2 and by Theorem 3.1 it is clear that $\lambda^{'}(G) = \xi(G)$. Hence $G = V_n$ is optimal. Also since $G - F$ contains exactly two component such as k_2 and a connected component containing more than 2 vertices. Clearly, by the definition of fragment $X = K_2$ and the atom of G is $r(G) = |X| = 2$. □

Conclusion

From the above theorems, it is identified that all arithmetic graph other than $G = V_n, n = p_1 \times p_2$ has super edge cut. In addition to that, for every arithmetic graphs $G = V_n, n \neq p_1^{a_1} \times p_2, a_1 > 2$ are optimal and the atom $r(G)$ is 2.

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